Force systems from an ideal arch — large deflection considerations

By Herbert A. Koenig, BME, MSME, PhD and Charles J. Burstone, DDS, MS

A rational approach to the analysis and design of orthodontic appliances was previously presented and used to demonstrate important features of a straight wire appliance applied to malaligned brackets.[1] The mathematical formulation for the approach was presented by Koenig and Burstone[2] and DeFranco, and Koenig and Burstone.[3]

This algorithm, which was subsequently cast into the form of a computer code, was used in several fruitful attempts to demonstrate the design and prediction aspects of the simulation. Loop design,[4,5] lingual arches[6] and headgear[7] are but a few of the clinical applications which have been studied in depth. Non-planar activations, similar to actual clinical situations were also studied and documented.

In all the studies heretofore, the analysis has been confined to small deflection theory, nonrotated geometries and rigid attachments. The small deflection theory was, in the main, accurate to provide the researchers with qualitative evaluations of the appliances. The quantitative aspects of the simulations were less accurate since most appliances undergo large activations during clinical practice. Furthermore, a "zero slip condition" was imposed upon the brackets and the wires so that they could not move or rotate during activation.

The analysis has now been further enhanced so as to overcome all of the aforementioned limitations and to produce a simulation which will now yield quantitatively accurate results for clinical situations in which the previous model was deficient. The analysis is now capable of simulating activations in three planes of space which are large and are affected by the manner in which the bracket interacts with the wire. The effects of bracket-wire sliding with rotated boundary conditions are also available.

Method

Theoretical Development

At this point, one may ask what is the distinc-

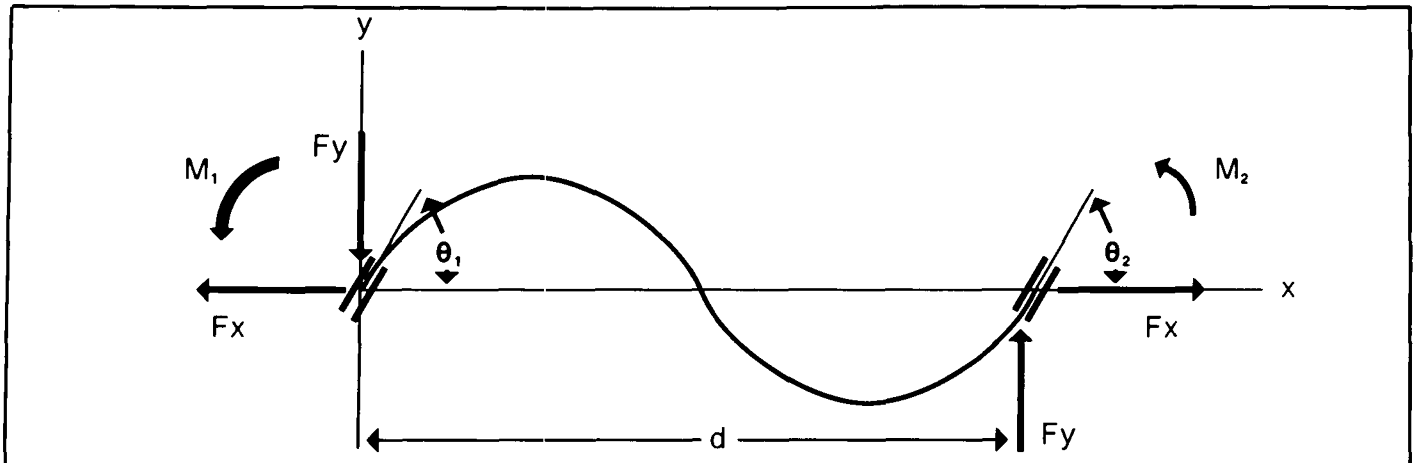
Abstract

A sophisticated mathematical simulation is presented which allows for the consideration of large activations in orthodontic appliances and their effect upon the resulting force systems which are delivered to teeth. Effects of bracket/wire interaction are studied using this new tool.

Previous studies of force systems from an ideal arch were redone with the new analysis in which the wire was either rigidly restrained or free to slide. The restraint of the wire produced large mesio-distal forces and increased the magnitude of the moments on each bracket. If the wire is free to slide, both large deflection and small deflection solutions give similar results. The relative force system M_1 / M_2 fundamentally held true with large deflections and restraint; however, some differences were noted. The significance of allowing wire to slide in the bracket is discussed.

Key Words

Forces • Side effects • Straight wire • Beam theory • Friction



Class I $\frac{\theta_1}{\theta_2} = 1.00$

Class II $\frac{\theta_1}{\theta_2} = 0.50$

Class III $\frac{\theta_1}{\theta_2} = 0.00$

Class IV $\frac{\theta_1}{\theta_2} = -0.50$

Class V $\frac{\theta_1}{\theta_2} = -0.75$

Class VI $\frac{\theta_1}{\theta_2} = -1.00$

Figure 1

Force system produced by a straight wire in maligned brackets. Θ_1, Θ_2 angle of bracket rotation. M_1, M_2, moments at each bracket. F_y, F_x, vertical and horizontal forces. Θ_1 / Θ_2 ratios are given for each class of bracket geometry. All forces are activation forces acting on the wire. Forces acting on the teeth are equal and opposite.

tion between a large (non-linear) analysis and a linear (small deflection) analysis and why doesn't a small deflection analysis yield the proper force system.

A small deflection analysis determines the force system based upon the original wire configuration. No notice is made concerning the deflection which results from an appliance. For example, if a straight wire is activated by a malocclusion between two brackets, the small deflection analysis uses the zero curvature and zero twist of the original shape to calculate the resulting force system. However, as the wire is deforming to its final shape, the curvature and/or the twist is constantly changing. Thus, an analysis which considers only small deflections will be in error in a manner in which the resulting force system will not be in equilibrium with the final shape. This is the anomaly which is to be addressed by a large (non-linear) prediction theory.

Initially, the small deflection analysis was reconverted from a transfer matrix approach to a stiffness matrix approach. In this manner, the resulting analysis closely approximated a Finite Element Analysis. Originally, the force system at one end of the beam F_1 was related to the force system at the other end of the beam F_n by a transfer matrix.

$$\left\{\frac{F_1}{\delta_1}\right\} = \Omega \left\{\frac{F_n}{\delta_n}\right\} \qquad [1]$$

By mathematical techniques (unpublished), this formulation was converted to a stiffness matrix approach

$$[F] = [K]\ [\delta] \qquad [2]$$

where the forces in the entire beam are related to the activations in the beam by a large stiffness matrix. In this approach, multiple beams may be attached to one another by adding to the force vector F and the displacement vector δ and expanding the size of the stiffness matrix K. With the change in formulation (and the resultant computer analysis), the prediction was converted from a small deflection single beamed analysis to a large deflection multiple beamed analysis. Multiple bracket attachments as well as multiple force activation points can be treated easily. The interaction of bracket locations upon the force systems can be determined. Beams of varying cross-sections which are attached to one another can now be treated. In addition, out-of-plane activations, in which one of the wires is activated in another plane from the main appliance may be studied. This was done in deactivation studies[6] and predicted interesting characteristics of special appliances design.

An effective engineering approach to the nonlinear construction of the stiffness matrix K for the purpose of studying large deflection is found in the Newton-Raphson Jacobean Iteration technique. In this approach, the error which occurs in the incompatibility between the force system and the deflected shape is measured and used as the means to direct the analysis toward the proper calculation of curvatures, twists and forces which are mathematically consistent. When the error, called the large deflection criterion (LDC) is reduced below an arbitrarily chosen value, the analysis is said to have converged and an equilibrium of forces and geometric shape is established. The error, called the residual r_i, is calculated from

$$r_i = K_{ij}\ \delta_j - F_i \qquad [3]$$

the Jacobean matrix

$$J_{ik} = K_{ik} + X_j \frac{\partial K_{ij}}{\partial X_k} - \frac{\partial F_i}{\partial X_k} \qquad [4]$$

is then formulated based upon the last iteration in the solution process. This Jacobean J is then used to calculate the new deflected shape

$$\delta X_i = J_{ik}^{-1}\ r_k \qquad [5]$$

$$X_i' = X_i + \delta X_i \qquad [6]$$

This new shape is then used to calculate a new error r_i. The iteration process is continued until r_i is less than an established error value. This process is then converged and the final force/deflection system is in equilibrium with the final appliance geometry.

This formulation is cast into a computer formulation with an iterative process to obtain a convergence of r_i. This new nonlinear method was then applied to the problem of determining the force systems from straight wires in malaligned brackets.

Previously the force systems delivered by straight orthodontic wires with several classes of malocclusions were studied using a small deflection, linear theory. The six cases which were presented were redone using the large deflection formulation which is described above. The classes of bracket geometry are based on the ratio Θ_1 / Θ_2 which is shown in Figure 1. If Θ_1 is not equal to Θ_2, then Θ_2 is the larger angle by convention.

Specific Studies

In our previous research of the effects of straight stainless steel wires using small deflection theory mesio-distal forces were ignored. In this study mesio-distal forces were calculated under two conditions: 1) Freedom of the wire to slide in the left attachment parallel to the bracket and 2) No freedom for the wire to slide; i.e. rigid attachment. In addition, small deflection theory was used to re-calculate the force system found in the previous study allowing freedom of the wire to slide in the left attachment. This enabled comparison of the utility and accuracy of applying small deflection theory to the study of orthodontic appliances. The bracket angulations used are given in Figure 1 and are based on the angulation required to produce a maximum bending moment of 1860 gm-mm in a .016″ stainless steel wire at some section along the wire above which yield would occur at a stress of 400,000 PSI. (Small deflection theory not considering horizontal forces.) 400,000 PSI is an arbitrary yield stress that can be found in high springback stainless steel wires.[8] Nevertheless, many steel orthodontic wires may have lower yield strengths. This yield strength was selected for ease of comparison to our former study and to allow as large as possible wire deflection to study the effects of using large deflection theory.

Results

The varying bracket geometrics using a straight wire are given in Table I. M_1 and M_2 are the respective moments acting at each bracket when a straight wire is inserted. Force F_y is the

Table I — Force Systems From A Straight Wire

BRACKET GEOMETRY BY CLASS	METHOD	FY gm	M1 gm-mm	M2 gm-mm	FX gm	Θ_2 degrees	$\frac{M_1}{M_2}$
I	LFr	532	1860.0	1860.0	40.9	4.4	1.0
I	NLFr	534	1890.0	1890.0	41.9	4.4	1.0
I	NLFi	568	1990.0	1990.0	2390.0	4.4	1.0
II	LFr	476	1480.0	1850.0	22.9	5.3	0.8
II	NLFr	475	1490.0	1840.0	22.9	5.3	0.8
II	NLFi	503	1520.0	2000.0	2400.0	5.3	0.75
III	LFr	399	931.0	1860.0	0.0	6.6	0.5
III	NLFr	399	951.0	1840.0	0.0	6.6	0.5
III	NLFi	435	844.0	2200.0	3810.0	6.6	0.4
IV	LFr	266	0.0	1860.0	42.7	8.8	0.0
IV	NLFr	274	28.3	1890.0	42.7	8.8	0.0
IV	NLFi	331	-672.0	2970.0	8480.0	8.8	0.2
VI	LFr	0	-1860.0	1860.0	5.8	13.2	-1.0
VI	NLFr	-24.9	-1960.0	1760.0	5.8	13.2	-1.1
VI	NLFi	-10.4	-5900.0	6090.0	-20200.0	13.2	-1.0

*L = Linear (small deflection)
NL = Nonlinear (large deflection)
Fr = Free to slide at right bracket.
Fi = Fixed

E = 0.21744E + 08 g/mm**2
I = 0.13E-02 mm**4
A = 0.1297 mm**2
Interbracket distance = 7 mm.
Wire cross-section = 0.016 in.
Material stainless steel.

vertical force produced if the brackets are malaligned in an occluso- gingivally direction or facial-lingual force if the malalignment is in a facial-lingual direction. The horizontal forces F_x are directed to move the brackets toward each other. It should be noted that three separate solutions are given for each bracket geometry class: 1) Small deflection linear theory allowing freedom of the wire to slide through the right bracket, 2) Large deflection non-linear theory allowing freedom of the wire to slide through the right bracket and 3) Large deflection non-linear theory fixing the wire at both brackets. (Table I)

Let us make a comparison between the three methods, selecting a Class I bracket geometry; a geometry where brackets are parallel but stepped relative to each other. If small deflection theory is used allowing the wire to slide through the left bracket, the ratio M_1 / M_2 is 1. This relative force system is identical to what was reported in our previous study where mesio-distal forces were ignored. Here small mesio-distal forces are recorded, delivering a magnitude of approximately 41 grams. When large deflection theory was used allowing freedom of the wire to slide, the force system was approximately the same as for small deflection theory. On the other hand if the wire is fixed between the two brackets, a slight increase occurs in the magnitude of moment M_1 and moment M_2 and, more significantly, a very large increase is found in the magnitude of the horizontal forces reaching values of 2,390 grams. It is, thus, apparent that the magnitude of the horizontal forces is highly dependent upon the ability of the wire to slide in the brackets as it is being tied in place. The actual force system that is produced is dependent upon how much that wire slides and, therefore, in a clinical situation would lie somewhere between full freedom to slide at 42 grams and no freedom to slide at 2,390 grams. Nevertheless, if one looks at the ratio between M_1 and M_2 it remains the same regardless of the horizontal forces and the mathematical solution used.

As one moves from a class I to a class VI geometry, the amount of angulation of at least one of the brackets gradually increases. Thus, it would appear that large deflection theory should be more helpful in those geometries where the deflections are the greatest. We will now review the differences between the various classes in respect to three variables; 1) the ratio of M_1 / M_2 as well as the absolute moment values of each, 2) the mesio-distal forces that are produced and 3) the effect of restraint on the moment at yield.

The ratio of M_1 / M_2 generally does not change between the large deflection and small deflection solutions in all classes. However, it should be noted that as one moves to classes with larger deflections such as class III and class IV, the ratio of M_1 / M_2 tends to be slightly smaller. In class VI where a symmetrical mirror image relationship is present, ratios are identical. In all of the classes the greatest difference in the ratio M_1 / M_2, if it does occur, was found using large deflection theory where both ends are restrained. The discrepancy that is the largest in this ratio is seen in a class IV geometry where -672 gram millimeters was found at M_1 where using small deflection theory, one would expect a zero moment.

The absolute moments of M_1 and M_2 are approximately the same using small deflection theory or large deflection theory where the wire is free to slide. There is somewhat of a tendency for the magnitudes to increase with the large deflection solution and this is particularly noticeable in the solution where both ends are fixed. The increase in the magnitudes of these moments occur progressively in the higher numbered classes. For example, in a class I fixed end situation, M_1 only rises from 1,860 gm-mm (small deflection) to 1,990 gr-mm (large deflection); in a class VI geometry, M_2 increases from 1,960 gm-mm (small deflection) to a 5,900 gm-mm (large deflection).

As previously noted, the most dramatic change in the force system is found when the free ends are fixed or restrained. The horizontal forces (F_x) increase progressively from class I to class VI. At a class VI geometry they increase to such a high level that they approach 20,202 grams. On the other hand, using both large and small deflection theory where freedom is allowed for the wire, definitive horizontal forces are produced but of relatively low magnitudes in comparison to the total force system.

Discussion

The first approximation of the force system produced by straight wires and malaligned brackets was previously studied using lineal small deflection theory and ignoring mesio-distal forces. It was shown that although the absolute magnitudes of the forces and moments will change depending upon activation and inter-bracket distance, the relative force system (the ratio of M_1 / M_2) is relatively constant for each clinical geometry.

The important clinical question is, does this ratio remain constant as we use a more accurate approach to determining the force system, i.e. considering both large deflections of a non-linear nature and the effects of restraint on the wire in the bracket. The data in this paper show that if the wire is not restrained (free to slide within the brackets), the previously established M_1 / M_2 ratios hold true. There may be small deviations from this pattern as the deflections get greater and it could be that using some of the newer wires that are capable of large deflections with angles far greater than those studied, some greater deviation might occur. On the other hand, if the wire is restrained and we use our best estimate which is non-linear large deflection theory, the ratio M_1 / M_2 might deviate significantly from the unrestrained situations. This is particularly true in those geometries that have the largest deflections (the higher numbered geometries) and at their largest deflections. The exception is the class VI geometry where equal and opposite couples are produced. Equal and opposite couples are still produced in a restrained situation, although the absolute moment values become very high.

Perhaps the most dramatic finding was the very large mesio-distal forces that are produced if the wire is not free to slide within the brackets during activation. The actual value of horizontal force depends on a number of factors, some of them under the control of the orthodontist. Generally, one should consider the mesio-distal forces as undesirable side effects and the orthodontist would want to eliminate them in his arch wires. If an arch wire is tied back or bent back at the posterior end of the arch, the wire is less free to move into the brackets, and one would anticipate a marked increase in the magnitude of the horizontal forces. Similarly, the factors of friction and play, the types of ligatures used and the sequence of tying, potentially can alter the force system. In this study, only the wire in the left bracket was free to slide. Although not studied, it is apparent that freedom to slide differentially between two brackets or at the greater angled bracket could give different results. The actual forces produced horizontally in a clinical situation will lie somewhere between the limits shown in this research between freedom to slide and complete fixation. Future studies on the effects of that freedom to slide including a better understanding of friction and play in appliances are imperative. Although only the horizonal forces during activation have been studied (the force system after the wire has been fully engaged in the brackets) it can be anticipated that additional horizontal forces will be operating as the teeth move to their new positions. If the wire is unable to move horizontally through the bracket, large horizontal forces would be produced and teeth would be subjected to a "round trip ride" as the horizontal forces reverse themselves as teeth assume new positions. These horizontal forces can be responsible for producing undesired rotational effects and mesio-distal contact (hence binding of crowns interproximity) minimizing tooth movement.

Although there is some difference between the magnitudes of the moments at the two brackets, using small or large deflection when the wire is free to slide the dramatic increase in moment is found when one restrains the wire so that it is not free to slide. The higher numbered classes of bracket geometry show the greatest increase in moment culminating in a class VI geometry where the moment at M_1 and M_2 are approximately 6,000 gm mm in comparison to 1,860 gm mm when the wire is free to slide. Using a .016" stainless steel wire an activation of 13 degrees can be carried out over a seven millimeter interbracket distance without permanent deformation in a class VI geometry provided that a yield strength of 400,000 PSI was present. On the other hand, if the same wire was restrained only approximately four degrees of activation would result in permanent deformation. The double edge sword of wire restraint in the bracket, could thus lead to either increased moment values which are not desired, or increased permanent deformation of the wire, as these moment values increase, or perhaps some combination of both.

Summary and conclusions

A new mathematical model for simulation of the force system produced by orthodontic bracket wire configurations, is presented, which is based upon large deflection non-linear theory. The effects of freedom of the wire to slide in the bracket and of rigid restraint of the wire were studied. Comparisons were made between the newer large deflection methods and previous methods applying small deflection theory. It was concluded that:

1. If freedom is allowed for the wire to slide within the brackets, the results using small and large deflection theory are similar in all geometry classes. Where deflections are large, small differences may be noted in the force system.
2. If the wire is restrained in the brackets so that it is not free to slide, there is a significant increase in the absolute values of the moments of each bracket.
3. The ratio of M_1 / M_2 can change decreasing slightly in most situations if large instead of small deflection theory is used.
4. Small horizontal forces are present when wires are tied into angulated brackets even if the wire is free to slide.
5. If wires are restrained in the bracket large horizontal forces are produced and are a sig-

nificant factor in the efficacy of tooth movement and are responsible for undesirable side effects.

6. Small deflection linear solutions to force systems from orthodontic appliances are not able to handle and adequately estimate these usually unwanted horizontal forces.

Author Address

Dr. Charles J. Burstone
Department of Orthodontics
University of Connecticut Health Center
Farmington, CT 06032-9984

Dr. Charles J. Burstone is the head of the Department of Orthodontics and a professor in the School of Dental Medicine at the University of Connecticut Health Center.

Dr. Herbert A. Koenig is a professor in the Department of Mechanical Engineering at the University of Connecticut, Storrs.

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