Connection Formulae and Classification of Scattering Regions for Low-Frequency Shelf Waves

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ABSTRACT

The scattering of shelf waves at simultaneous changes in depth, direction and width is considered. In the low-frequency limit the scattering is shown to be determined by the connection of f/H contours. The description "simple" is introduced for regions in which no incident f/H contour terminates and then restarts. An explicit connection formula is derived for simple regions. It is shown that energy is transported without loss across a scattering region if no incident f/H contours terminate there. This subclass of simple regions is described as conservative. Particular examples are given for exponential shelves joined by both simple and nonsimple, conservative and nonconservative scattering regions, and for both incident shelf waves and irrotational flows driven across the region. In the latter case, energy is scattered out of the flow into a transmitted wave field. Finally it is noted that if the irrotational flow determined by a particular shelf geometry is geostrophic then even at arbitrary frequencies no scattering of energy occurs from the flow or among shelf waves.

1. Introduction

The effects of obstructions to long barotropic shelf waves on shelves with parallel isobaths have been discussed for flows of vanishingly small but nonzero viscosity in Johnson (1989a, called **I** herein). By confining attention to the low-frequency limit simple direct results can be obtained for scattering by obstacles of quite general shape. It is the purpose of the present paper to extend these results to general changes in shelf depth, direction and width. The geometry considered is that of two fixed profile shelves joined by a scattering region of extent of order the shelf width. Outside the scattering region the dynamics are governed by the long wave low-frequency form of the topographic wave equation. It is shown in section 2 that in the scattering region at low frequencies the flow is oscillatory and geostrophic, constrained to follow contours of f/H (here as elsewhere the parameter definitions are those of **I**). The required value of the streamfunction immediately after scattering follows from the incident field by tracing backwards along f/H contours. The transmitted field in the low-frequency limit for a general scattering region is then completely determined. The concept is introduced of a "simple" scattering region, one in which no f/H contour terminates and then restarts. For a simple region, the transmitted field is given explicitly in terms of the incident field by a connection formula without detailed consideration of the exact geometry. It is shown that a scattering region is conservative, incident shelf-wave energy is transmitted without loss, if all incident f/H contours extend through the scattering region but is dissipative, with energy destroyed in sidewall boundary layers, if incident f/H contours terminate within the region.

Section 3 gives the form of these results for the particular example of exponential shelves. This geometry includes that considered in Middleton and Wright (1988, denoted MW herein). They, however, consider changes in shelf depth that are abrupt even on the scale of the shortest waves and matching across the scattering region is less direct. Section 4 discusses the difficulties associated with wall-step junctions noting that the analysis in Johnson (1985) suggests the presence of singular regions of infinite energy, not well treated by mode-matching techniques. Section 5 presents a simple geometry giving a change in shelf depth that can be analyzed by mode-matching techniques without requiring the low-frequency limit. The qualitative form of the solutions is obtained by comparison with solutions for the related geometry of a shelf with a terminating bay zone considered in Stocker and Johnson (1989). Section 6 briefly discusses the results, extending to arbitrary frequencies the result obtained for low frequencies in section 2, that many joining regions cause no scattering. Even at discontinuous changes in shelf width, isobaths will be such that no scattering occurs at any frequency if irrotational flow through the region is geostrophic.

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2. Governing equations

The inviscid topographic wave equation is (Rhines 1969)

$$\nabla\cdot(H^{-1}\nabla\Psi_t)+\hat{\mathbf{z}}\cdot[\nabla\Psi\times\nabla(f/H)]=0. \quad (2.1)$$

Consider an infinite shelf consisting of two semi-infinite shelves whose directions, widths l_1 and l_2, and cross-shelf profiles are fixed but arbitrary. Let the shelves be joined by a scattering region of size of order the shelf width and bounded by impervious walls at C_0 and C_1. Let $0'x'y'$ be arbitrary axes and $0'_1, x'_1, y'_1$, and $0'_2x'_2y'_2$ be axes aligned with the shelves as illustrated in Fig. 1a. Consider the scattering of a shelf wave of frequency ωf, writing $\Psi = \mathrm{Re}\{e^{-i\omega ft}\psi\}$. Introduce nondimensional variables

$$x_1 = x'_1/l_1, \quad y_1 = y'_1/l_1, \quad x_2 = x'_2/l_2, \quad y_2 = y'_2/l_2 \quad (2.2)$$

and the long scale (for ω small)

$$X = \begin{cases} \omega x_1, & \text{if} \quad x_1 < 0 \\ \omega x_2, & \text{if} \quad x_2 > 0. \end{cases} \quad (2.3)$$

In the long-wavelength low-frequency limit ($\omega \to 0$), (2.1) reduces to

$$\psi_{yy} - G(\psi_y + i\psi_X) = 0, \quad X \neq 0, \quad (2.4)$$

where the logarithmic slope is

$$G = \begin{cases} (\log H_1)_{y_1}, & X < 0 \\ (\log H_2)_{y_2}, & X > 0 \end{cases} \quad (2.5)$$

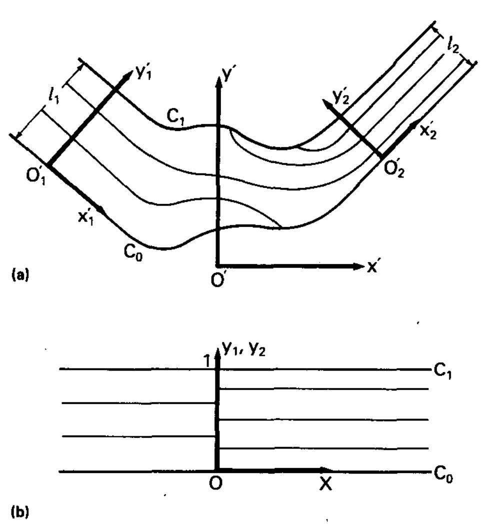

FIG. 1. A plan view of a shelf with an arbitrary change in depth, direction and width. The impervious boundaries at C_0 and C_1 are thickened and typical isobaths included. $0'x'y'$ is an arbitrary Cartesian frame and $0'_1x'_1y'_1$ and $0'_2x'_2y'_2$ Cartesian frames relevant to the rectilinear shelves (a) on the scale of the shelf width and (b) on the long-wave scale. The scattering occurs across $X = 0$ and the origins $0'_1$, $0'_2$ are chosen to coincide.

for shelf profiles $H_1(y_1)$ and $H_2(y_2)$. For definiteness G is taken to be strictly positive so long waves are incident from $X < 0$. The variable y represents y_1 in $X < 0$ and y_2 in $X > 0$. The origins of (y_1, X) and (y_2, X) are arbitrary and so can be chosen to coincide (Fig. 1b). The low-frequency scattering problem for arbitrary changes in direction, width and depth has been reduced to scattering at a depth change in a constant width rectilinear channel.

The boundary conditions on the impervious walls can be written

$$\psi = 0 \quad (\text{on } C_0), \quad \psi = \alpha \quad (\text{on } C_1), \quad (2.6)$$

where a nonzero α gives an oscillatory instantaneous flux. To complete determination of the scattered field in $X > 0$ it is sufficient to know $\psi|_{X=0^+}$. This follows by considering (2.1) in the low-frequency limit on the length scale l_1. The sole surviving term gives

$$\nabla\psi \times \nabla(f/H) = 0, \quad (2.7)$$

i.e., ψ is constant along contours of f/H. On length scales of order of the shelf width, the flow is entirely geostrophic at low frequencies. The required value of $\psi|_{X=0^+} = \psi|_{x_2=\infty}$ at a given y_2 follows by tracing backwards along the f/H contour until the contour intersects C_0 or C_1 (and so ψ takes the value 0 or α, respectively) or until the contour reaches $x_1 = -\infty$ and the value of ψ is given by the incident wave. At low frequencies it is the connection of the isobaths that determine the flow and not their precise shape. If contours originating at $x_1 = -\infty$ terminate on either C_0 or C_1 there will generally be a conflict in the value of ψ given by the incident wave and that from (2.6). It is shown in I that the presence of vanishingly small viscosity resolves this conflict in a sidewall boundary layer of thickness ωl_1. Short waves satisfying (2.1) with wavelengths of order ωl_1 occur only within this layer. It is their absence on the scale l_1 that leads to the simple form (2.7). The boundary layer turns the volume flux carried by the incident wave and dissipates incident wave energy. It is only if all incident f/H contours emerge from the far side of the scattering region that dissipative boundary layers are absent and energy flux is conserved. Such a region is denoted "conservative."

It is helpful to also distinguish a larger class of scattering regions, denoted herein as "simple," in which no incident f/H contours terminate and then restart. This precludes neither closed contours, incident contours terminating, contours originating nor contours which both originate and terminate entirely within the region (Fig. 2a). All conservative scattering regions are simple as are some dissipative regions. Isobaths for a nonsimple region containing a valley are given in Fig. 2b showing incident contours of f/H terminating and restarting. This region is equivalent to the headland treated in I. Volume flux incident on AB is turned in a boundary layer of thickness ωl_1 on AB′ to emerge

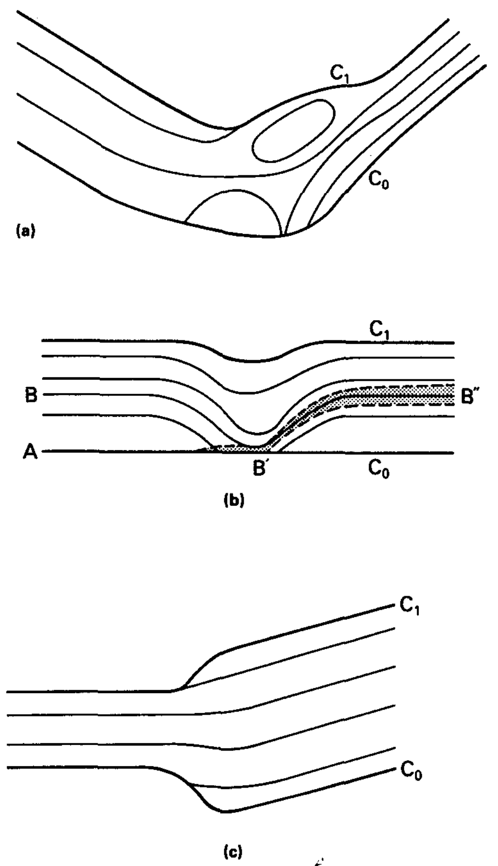

FIG. 2. (a) A simple scattering region. Although there are present isolated closed contours, contours starting in the scattering region and terminating there or extending downstream, and incident contours terminating in the region, no incident contour of f/H terminates and restarts in the region. (b) An example of a nonsimple scattering region. A valley is present in the region and contours of incident f/H terminate and restart. The matching conditions for such regions must be derived individually. The discussion of I shows that volume flux incident on AB is turned in a boundary layer on AB′ to emerge from a source at B′ as a spreading current centered on the isobath B′B″. (c) An example of a scattering region across which energy is conserved. The region is simple and all incident contours emerge on the far side.

from a source at B′ as a spreading boundary layer of thickness $\omega^{1/2}l_1$ centered on the isobath B′B″. Dissipative layers are present in any nonsimple connecting region and no nonsimple region is conservative.

The connections of f/H contours for different nonsimple regions must be considered individually and no general formula is possible. It is sufficient, however, for a region to be simple to determine an explicit, unique connection formula. The formula is unaffected by the precise form of the bathymetry. Introduce the functional inverse $y_1(H_1)$ of the profile $H_1(y_1)$ (which exists as $G > 0$), so y_1 is the value of y corresponding in $X < 0$ to a depth H_1. Then, for a simple scattering region,

$$\psi(0^+, y_2) = \begin{cases} \alpha, & \text{if } H_2(y_2) > H_1(1) \\ \psi(0^-, y_1(H_2(y_2))), & \text{if } H_1(0) \leqslant H_2(y_2) \leqslant H_2(1) \\ 0, & \text{if } H_2(y_2) < H_1(0). \end{cases} \tag{2.8}$$

Equations (2.4), (2.6) and (2.8) uniquely determine the transmitted wave field and general H_2, ψ can be found by separating variables and expanding in cross-stream modes. The required normalizations and orthogonality relations are given in Hsieh and Buchwald (1985).

As shown in I, the time-averaged flux of wave energy over a cross-channel plane is a constant in each of the separate regions $X < 0$ and $X > 0$. The flux may, however, alter across $X = 0$. If an oscillatory flux is present ($\alpha \neq 0$) energy is scattered into the shelf wave field. If the incident field comprises only shelf waves ($\alpha = 0$) then the relation between the fluxes follows from (2.8) as

$$\int_{y_2=0}^{1} \frac{1}{2}(H_2^{-1})_{y_2}\psi^2|_{X>0}\,dy_2 = \int \frac{1}{2}(H_1^{-1})_{y_1}\psi^2|_{X<0}\,dy_1, \tag{2.9}$$

where the second integral is evaluated over those values of y_1 corresponding to isobaths continuing through the scattering region. Energy is conserved if and only if all incident isobaths emerge from the far side of the scattering region (Fig. 2c).

If $H_2(1) \leqslant H_1(0)$, i.e., the maximum depth in $X > 0$ is smaller than the minimum in $X < 0$, no f/H contours extend through the scattering region and (2.8) gives

$$\psi(0^+, y_2) = 0, \quad 0 \leqslant y_2 < 1. \tag{2.10}$$

In the absence of an oscillatory current ($\alpha = 0$) all incident shelf wave energy is dissipated in the scattering region and the transmitted wave field is absent. If an oscillatory current is present ($\alpha \neq 0$) this crosses $X = 0$ in a singularity on C_1, acting as an oscillatory source as discussed in Johnson (1985). Energy from the current is scattered into transmitted shelf waves as in Johnson (1989a,b). A specific example is given in the following section.

If $H_2(0) \geqslant H_1(1)$ so the minimum depth in $X > 0$ is greater than the maximum in $X < 0$, then (2.8) gives

$$\psi(0^+, y_2) = \alpha, \quad 0 \leqslant y_2 \leqslant 1. \tag{2.11}$$

Once again in the absence of a current ($\alpha = 0$) there is no transmitted field and if a current is present ($\alpha \neq 0$) flow crosses $X = 0$ in a singularity on the boundary, in this case on C_0.

If $H_2(0) \leqslant H_1(0)$ and $H_2(1) \geqslant H_1(1)$ then all f/H contours from $X < 0$ extend into $X > 0$, all incident information contributes to the transmitted field, and if $\alpha = 0$ energy flux is conserved.

Any incident field can be expressed as a set of incident shelf waves with, for $\alpha \neq 0$, a superposed oscillatory current given by

$$\psi^0(y, X) = \begin{cases} \alpha \int_0^y H_1(\eta)d\eta \Big/ \int_0^1 H_1(\eta)d\eta, & X < 0 \\ \alpha \int_0^y H_2(\eta)d\eta \Big/ \int_0^1 H_2(\eta)d\eta, & X > 0. \end{cases} \tag{2.12}$$

The current is geostrophic, running parallel to isobaths and uniform with constant speed

$$U = -H^{-1}\psi_y = \alpha/A, \tag{2.13}$$

where A is the cross-sectional area of the shelf. Since the velocity is constant the flow is trivially irrotational. As ψ^0 satisfies the inhomogeneous conditions on C_1, writing $\psi = \psi^0 + \psi^w$ leaves a standard scattering problem for the wavefield ψ^w governed by the connection formulae for ψ.

This decomposition is also possible at arbitrary frequencies. A useful form for the oscillatory current, since it remains valid in the absence of either rotation or topography, is the irrotational flow satisfying the boundary conditions (2.6) with vanishing vorticity, i.e., with

$$\nabla \cdot (H^{-1}\nabla\psi^0) = 0. \tag{2.14}$$

Again the decomposition $\psi = \psi^0 + \psi^w$ leaves a standard scattering problem for the wave field ψ^w, but with a forcing term in any region where ψ^0 violates (2.7), i.e., anywhere irrotational flow is not geostrophic. If ψ^0 is not geostrophic then energy is scattered from the current into shelf waves, even in the absence of an incident shelf wave. Section 3 gives an example in the low-frequency limit.

If ψ^0 is geostrophic then it takes no part in the scattering and the problems for ψ^0 and ψ^w decouple completely. No energy is scattered into the wave field from the irrotational flow. Further, (2.7) implies that $H^{-1} = R'(\psi^0)$ for some differentiable function R. From (2.14) $R(\psi^0)$ is harmonic in (x, y) and from (2.6) it is constant on C_0 and C_1. Thus $R(\psi^0)$ defines a conformal mapping of the shelf onto a rectilinear channel. Since (2.1) is invariant under conformal mappings (Davis 1983; Johnson 1987) and no energy scatters between shelf wave modes in a rectilinear channel there is no scattering on the shelf. This argument shows that if irrotational flow through a scattering region is geostrophic then there is no scattering of either the flow or incident shelf waves at any frequency.

3. Exponential topography

As an illustration of the results of the previous section, consider the exponential shelves given by

$$H_1(y_1) = h_1 \exp(2b_1y_1), \quad H_2(y_2) = h_2 \exp(2b_2y_2), \tag{3.1}$$

so

$$y_1(H_1) = (1/2b_1)\log(H_1/h_1)$$
$$y_2(H_2) = (1/2b_2)\log(H_2/h_2), \tag{3.2}$$

with f constant and the scattering region simple. Consider first the case $H_2(0) \leq H_1(0)$ and $H_2(1) \geq H_1(1)$ so that all isobaths from $X < 0$ extend into $X > 0$ and the region is conservative (Fig. 2c). Introduce

$$d = y_2(H_1(0)) = (1/2b_2)\log(h_1/h_2), \tag{3.3}$$

so $d \geq 0$ measures the displacement from C_0 of incident information in passing from $X < 0$ to $X > 0$. Introduce also $\beta = b_2/b_1$, the ratio of the logarithmic slopes. Then $d + \beta^{-1} \leq 1$ and the connection formula (2.8) becomes

$$\psi(0^+, y_2) = \begin{cases} \alpha, \quad d + \beta^{-1} \leq y_2 \leq 1 & (3.4a) \\ \psi(0^-, \beta(y - d)), \quad d \leq y_2 \leq d + \beta^{-1} & (3.4b) \\ 0, \quad 0 \leq y_2 \leq d. & (3.4c) \end{cases}$$

Incident information is displaced by d and contracted by a factor $\beta \geq 1$. For an incident mode m shelf wave, $\alpha = 0$ and

$$\psi = h_1{}^{1/2} \exp(b_1y_1) \sin m\pi y_1 \exp\{i(b_1{}^2 + m^2\pi^2)(X/2b_1)\}, \quad X < 0. \tag{3.5}$$

The transmitted wave field can then be written

$$\psi^w = h_2{}^{1/2} \exp(b_2y_2) \sum_{n=1}^{\infty} a_n \sin n\pi y_2 \times \exp\{i(b_2{}^2 + n^2\pi^2)(X/2b_2)\}, \quad X > 0, \tag{3.6}$$

where the mode amplitudes depend on the topography solely through the parameters β and d and are given by

$$a_n = \begin{cases} \dfrac{2}{\pi} m\beta\{(-1)^m \sin[n\pi(d + \beta^{-1})] - \sin n\pi d\}/(n^2 - \beta^2m^2), & \text{if} \quad n \neq \beta m \\ \beta^{-1}\cos(m\pi\beta d), & \text{if} \quad n = \beta m. \end{cases} \tag{3.7}$$

The time-averaged energy flux associated with any mode, normalized on the flux of the incident wave, is given by $\beta a_n{}^2$, and the sum over all modes is unity. The conditions (3.4) are precisely those for a shelf of width $\beta^{-1} \leq 1$ widening to width unity by simultaneous steps of $-d$ on C_0 and $1 - d - \beta^{-1}$ on C_1. The present analysis includes I as the special case $d + \beta^{-1} = 1$ with L of I given by

$$L = -d/(1-d) = 1-\beta, \quad (d+\beta^{-1}=1), \quad (3.8)$$

as can be verified by direct substitution in (3.7). This special case corresponds also to the "nearshore jump" in MW. Figure 3a and 3b of **I** thus gives the distribution of energy flux among the transmitted modes normalized on the incident flux as a function of the fractional change in slope β where the ordinate $L = 1 - \beta \leqslant 0$. For $\beta = 1$, $L = 0$ and no scattering occurs. Thus, provided a region is simple and the profiles of the shelves before and after the region are the same, low frequency waves are transmitted without scattering. The extension of this result to arbitrary frequencies is discussed in section 6. As β increases, the shelf in $X > 0$ becomes steeper, L becomes more negative, and transmitted energy is initially confined closer to C_1 and carried in progressively higher modes. The total normalized transmitted flux is unity; incident energy is transmitted without loss. The relationship of this result to that in MW is discussed in more detail in the following section.

The region is no longer conservative if any isobaths terminated in the scattering region. For topography (3.1) this occurs if $H_2(0) > H_1(0)$ $(d < 0)$ or $H_2(1) < H_1(1)$ $(d + \beta^{-1} > 1)$. In the latter case (3.4a) is absent and (3.4b) holds for $d \leqslant y_2 < 1$. There is a dissipation layer of length of order of the channel width along C_1 where incident wave energy is destroyed and volume flux turned to emerge at $X = 0^+$, $y_2 = 1$. In the former case (3.4c) is absent, (3.4b) holds for $0 < y_2 \leqslant d + \beta^{-1}$, there is a dissipation layer along C_0 and volume flux emerges at $X = 0^+$, $y_2 = 0$. If both $d < 0$ and $d + \beta^{-1} > 1$, there are dissipation layers and volume flux sources on both C_0 and C_1 and (3.4b) applies for $0 < y_2 < 1$. The transmitted wave field for an incident mode m shelf wave $(\alpha = 0)$ is given by (3.6) with

$$a_n = \begin{cases} \frac{2}{\pi}\{(-1)^m m\beta \sin[n\pi(d+\beta^{-1})] \\ \qquad - n \sin m\pi\beta d\}/(n^2-\beta^2 m^2), \\ \qquad\qquad \text{if} \quad n \neq \beta m \\ (d+\beta^{-1})\cos m\pi\beta d - \sin m\pi\beta d/m\pi\beta, \\ \qquad\qquad \text{if} \quad n = \beta m \\ \qquad \text{if} \quad d \leqslant 0 \quad \text{and} \quad d+\beta^{-1} \leqslant 1. \end{cases} \quad (3.9a)$$

$$a_n = \begin{cases} \frac{2}{\pi}\{(-1)^{n+1} n \sin[m\pi\beta(1-d)] \\ \qquad - m\beta \sin n\pi d\}/(n^2-\beta^2 m^2), \\ \qquad\qquad \text{if} \quad n \neq \beta m \\ (1-d)\cos m\pi\beta d - \sin m\pi\beta \\ \qquad \times (1-d)\cos m\pi\beta/m\pi\beta, \\ \qquad\qquad \text{if} \quad n = \beta m; \\ \qquad \text{if} \quad d \geqslant 0 \quad \text{and} \quad d+\beta^{-1} \geqslant 1. \end{cases} \quad (3.9b)$$

$$a_n = \begin{cases} \frac{2}{\pi}(-1)^{n+1} n\{\sin[m\pi\beta(1-d)] \\ \qquad + \sin m\pi\beta d\}/(n^2-\beta^2 m^2), \\ \qquad\qquad \text{if} \quad n \neq \beta m \\ \cos m\pi\beta d - \cos m\pi\beta(1-d)\sin m\pi\beta/m\pi\beta, \\ \qquad\qquad \text{if} \quad n = \beta m; \\ \qquad \text{if} \quad d \leqslant 0 \quad \text{and} \quad d+\beta^{-1} \geqslant 1. \end{cases} \quad (3.9c)$$

Case (3.9a) corresponds to a shelf of width β^{-1} being offset by a simultaneous narrowing of $-d > 0$ on C_0 and widening of $d + \beta^{-1} - 1$ on C_1; (3.9b) corresponds to an offset by a simultaneous narrowing of $d + \beta^{-1} < 0$ on C_1 and widening of d on C_0; and (3.9c) corresponds to a narrowing shelf with simultaneous inward steps of $d + \beta^{-1} - 1$ on C_1 and $-d$ on C_0. Once again the present analysis includes **I** as a special case of (3.9a) or (3.9c) with $d + \beta^{-1} = 1$ and L given by (3.8), a "nearshore drop." Figure 3a and 3b of **I** gives the distribution of transmitted energy flux, for incident mode 1 and mode 2 waves, as a function of the fractional change in slope β with ordinate $L - 1 = \beta \geqslant 0$. As expected energy is dissipated in the scattering region with the transmitted field vanishing for $\beta \leqslant 0$. For general d and β^{-1}, the fractions of incident flux dissipated in the scattering region corresponding to the three cases (3.9) are

$$\sin 2m\pi\beta d/2m\pi - \beta d, \quad (3.10a)$$

$$\sin 2m\pi\beta(1-d)/2m\pi - \beta + \beta d + 1, \quad (3.10b)$$

$$\sin m\pi\beta(1-d)\cos m\pi\beta/m\pi\beta - \beta + 1, \quad (3.10c)$$

becoming unity in any case when no incident isobaths extend into $X > 0$[1].

As an example of a scattering region that is not simple, consider a nearshore trough between two shelves of the same profile ($h_1 = h_2$, $b_1 = b_2$, Fig. 2b). Depth is constant along C_1 and has a maximum $h_0 > h_2$ on C_0. Isobath connections give

$$\psi(0^+, y_2) = \begin{cases} \psi(0^-, y_2), & L \leqslant y_2 \leqslant 1 \\ 0, & 0 \leqslant y_2 < L, \end{cases} \quad (3.11)$$

where $L = (\frac{1}{2}b_2)\log(h_0/h_2) > 0$. These conditions are precisely those for a headland of fractional length L treated in section 6 of **I**. Expressions for the mode amplitudes are given there and Fig. 8 of **I** gives the distribution among modes of the transmitted energy flux for incident shelf waves of modes 1 and 2. As the region is not simple it is not conservative and energy is dissipated. For $L \geqslant 1$ the trough is sufficiently deep that no incident isobath extends into $X > 0$ and no incident shelf wave effects the flow in $X > 0$, although as shown

[1] For $\beta < 0$ the direction of the long-wave propagation is reversed in $X > 0$; no mechanism exists to carry energy away from the scattering region and so it is dissipated entirely there.

below there is a scattered field if an oscillatory current is present ($\alpha \neq 0$).

If $\alpha \neq 0$ forcing fluid columns to cross isobaths scatters energy into shelf wave modes. Introducing the irrotational flow (2.12) allows the incident field to be split into an incident shelf wave field treated as above and a current with no shelf wave component, given by

$$\psi = \alpha \exp[b_2(y_1 - 1)] \sinh b_1 y_1 / \sinh b_1, \quad X < 0. \tag{3.12}$$

The field in $X > 0$ can be written

$$\psi = \alpha \exp[b_2(y_2 - 1)]\{\sinh b_2 y_2/\sinh b_2 - \sum_{n=1}^{\infty} a_n \sin n\pi y_2 \exp[i(b_2^2 + n^2\pi^2)(X/2b_2)]\}, \tag{3.13}$$

where the shelf wave amplitudes in the various cases follow from the connection formulae as for incident shelf waves. It is shown in I that the time-averaged energy flux carried over a cross-shelf plane by the transmitted waves is independent of the position of the plane, and is given by

$$\int_0^1 \frac{1}{2}(f/H_2)_{y_2} |\psi|^2 dy_2 = \frac{1}{2} b_2 |\alpha|^2 (f/h_2) e^{-2b_2} \sum_{n=1}^{\infty} a_n^2. \tag{3.14}$$

A convenient normalization that removes the parametric dependent on α and h_2 is given by f times the time-averaged kinetic energy per unit length of the associated irrotational flow

$$\frac{f}{2}\int_0^1 H_2 |U_1|^2 dy_2 = b_2 |d|^2 (f/h_2)(e^{2b_2} - 1)^{-1}. \tag{3.15}$$

The results presented in section 4 of I again correspond to the special case $d + \beta^{-1} = 1$ [with L given by (3.8)] of a nearshore jump (for $L < 0$, i.e., $\beta > 1$) or drop (for $L > 0$, i.e., $\beta < 1$), where

$$a_n = \begin{cases} 2b_2 \sin n\pi d/(n^2\pi^2 + b_2^2) \sinh b_2(1 - d), & \beta > 1 \\ 2n\pi \sinh b_2 d (n^2\pi^2 + b_2^2) \sinh b_2(1 - d), & \beta < 1. \end{cases} \tag{3.16}$$

Unlike the scattering amplitudes and fluxes when $\alpha = 0$, the amplitudes a_n and their associated fluxes depend explicitly on the logarithmic slope parameter b_2. Figure 3 gives the distribution of fluxes in (3.14) normalized on (3.15), as a function of the isobath displacement d for $b_2 = 1$, showing the wave field to be significantly more energetic, exceeding a quarter the energy associated with the irrotational flow, for a drop rather than for a jump where the wave energy approaches only 4% of the irrotational energy.

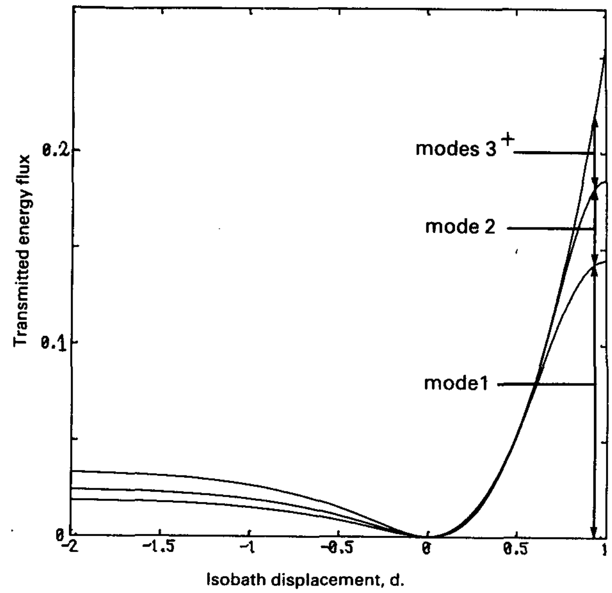

FIG. 3. The distribution of energy flux among the transmitted modes for a scattering region equivalent to a nearshore jump or drop when the incident flow is irrotational. The fluxes are given as a function of d, the displacement of the h_1 isobath in $X > 0$, and are normalized on f times the local energy per unit length of the transmitted irrotational flow. $b_2 = 1$.

In this case no incident f/H contours penetrate into $X > 0$ only if $\beta < 0$, when no transmitted shelf-wave field is possible. The field in $X > 0$ is then simply the irrotational current with the requisite flux, matched to that in $X > 0$ by a dissipative cross-channel boundary layer. To give an example in which no f/H contours penetrate into $X > 0$ but energy is scattered into the shelf wave field consider the case when $d > 1$ so $H_1(0) > H_2(1)$, i.e., the maximum depth in $X > 0$ is less than the minimum depth in $X < 0$. The boundary conditions on the flow in $X > 0$ are (2.6) and (2.10) and so

$$a_n = (-1)^{n+1} 2n\pi/(n^2\pi^2 + b_2^2). \tag{3.17}$$

The flow in $X > 0$ is determined entirely by the logarithmic slope b_2 and the flux through the singularity at $X = 0^+$, $y_2 = 1$, and is independent of any shelf-wave component of the incident flow. Figure 4 gives, as a function of b_2, the distribution of the fluxes in (3.14) normalized on (3.15). The shelf-wave flux achieves its maximum relative to the local kinetic energy for slopes of order unity. The field given by (3.13) and (3.17) is precisely the scattered field due to a source constructed in Johnson (1989b) in discussing the oscillatory flow forced round an island by an incident shelf wave.

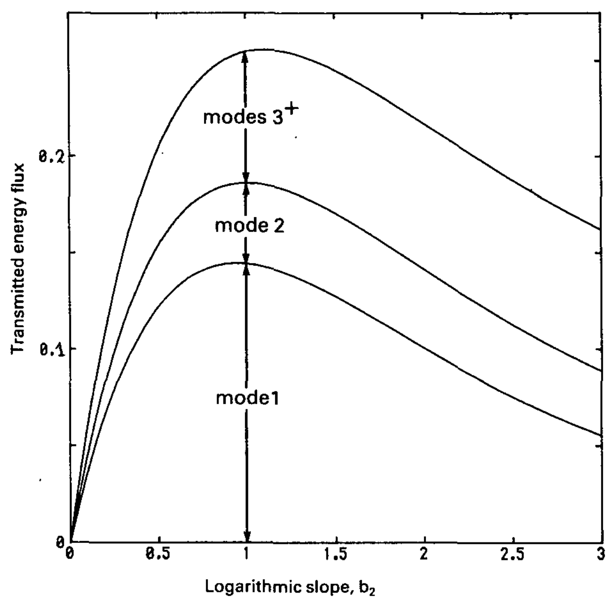

FIG. 4. The distribution of energy flux amongst the transmitted modes for a scattering region in which no incident f/H contours extend into $X > 0$ and the incident flow is irrotational with nonzero flux. The fluxes, normalized on f times the local energy per unit length of the transmitted irrotational flow, depend solely on b_2, the logarithmic slope in $X > 0$.

4. Vertical steps

The changes in shelf geometry considered in the previous sections are abrupt on the long-wave scale, taking place over scales of order of the shelf width. The analysis applies equally to even faster changes provided solely that the topography is smooth on the short-wave scale ωl_1 so the dissipative boundary layer discussion of I is valid. Steeper topography, vertical even on the scale of the shortest waves, has been considered by MW. As they note, this choice is not optimal, as precisely vertical escarpments are rare on continental shelves and difficulties can arise in mode-matching techniques. For vertical steps on shelves of constant-signed slope (i.e., $G > 0$), wall-step junctions are present at one or both walls. Take polar coordinates (r, θ) centered on a wall-step junction. Then for $r \ll 1$ the sole surviving term in (2.1) is the first. Sufficiently close to the junction, the streamfunction is harmonic and the step of constant height. This is precisely the geometry considered in Johnson (1985), where topographic waves are shown to propagate unidirectionally along the step with shallow water to their right (for $f > 0$). They have the form

$$\psi = \cos(\omega t + k \log r) \sinh\left[k|\theta| - \frac{1}{2}\pi\right], \quad (4.1)$$

where the step is taken to be at $\theta = 0$ and the wall at $\theta = \pm\frac{1}{2}\pi$. The wavenumber k is given by the dispersion relation

$$\tanh \frac{1}{2}\pi k = \left(\frac{h_1 - h_2}{h_1 + h_2}\right)\omega. \quad (4.2)$$

For each nonzero subinertial ω there is a free topographic wave with ψ bounded but infinite energy. Cross-shelf modal expansions cannot be expected to capture this behavior well. The difficulty of establishing energy conservation and matching at nonzero ω is remarked on by MW.

The particular case $\omega = 0$ is treated in detail in Johnson (1985) where it corresponds to a steady forced flow over a step. It is shown that if the geometry is such that topographic waves propagating along the step carry energy towards the wall, a singularity forms at the junction as no incident energy can be reflected. If the step is in the opposite sense, however, energy is carried away from the wall and the solution is well behaved in the neighborhood of the junction. Thus the sole case where waves of the form (4.1) are absent and energy is finite at the wall-step junction is when the low frequency limit is taken; there is no depth change along one wall, and the step is such that topographic waves travel away from the other wall. This is confirmed by the examples of MW, who mention difficulties in all other cases.

The matching condition at a cross-channel step can be written in general (Johnson 1985) as

$$[\Psi] = 0, \quad (4.3)$$

$$[H^{-1}\Psi_{xt}] + [f/H]\Psi_y = 0, \quad (4.4)$$

where brackets denote the jump in the enclosed quantity across the step. Substitution of a sine series for Ψ introduces both sine and cosine series in (4.4) leading to a matrix equation with coefficients which decrease only slowly with increasing order. This difficulty is avoided in Johnson (1985) by conformally mapping the flow domain into one in which the singularity is at infinity. If the surface is free, Kelvin waves are possible. The sole modification on shelf scales or smaller at low frequencies is to add a term representing free-surface deformation to the field equation, leaving the jump conditions unaltered. Incoming waves still carry energy towards the wall–step junction, and sufficiently close to the wall free-surface deformation becomes negligible—the solution rapidly approaching that for a rigid lid. Johnson and Davey (1989) obtain numerical solutions for the development of a singularity in low frequency free-surface flow by introducing a coordinate system stretched in the neighborhood of the wall–step junction.

5. Arbitrary frequencies

Results in Wilkin and Chapman (1987) and MW suggest that results in the previous sections obtained in the limit $\omega \to 0$ are accurate for frequencies as high as $0.1f$. At higher frequencies reflected short waves become important in (2.1) and for inviscid flow carry (in the negative X direction) the energy dissipated in their absence in wall layers. A shelf profile that enables these effects to be investigated without the difficulties associated with a vertical step is given by

$$H(y) = \begin{cases} h\exp(2by), & x \leqslant 0 \\ h\exp(2by + 2cx), & 0 \leqslant x \leqslant a \\ h\exp(2by + 2ac), & a \leqslant x, \end{cases} \quad (5.1)$$

for constants $h > 0$, $b > 0$ and c. This depth profile has been discussed in an infinite domain by MW and in a semi-infinite shelf geometry in Stocker and Johnson (1989), where a wall is placed at $x = a$ and the problem of a shelf in $x \leqslant 0$ terminating at a bay zone at $0 \leqslant x \leqslant a$ is considered. It is shown that the boundary conditions on Ψ at the junctions of the regions are the continuity of Ψ and its normal derivative. Writing $\psi = H^{1/2}\Phi$ reduces the governing equation to a Helmholtz equation for Φ with piecewise constant wavenumber. Solutions follow directly by mode-matching. It is shown that energy is conserved at the region junctions. Moreover since the scattering region in isolation supports trapped modes, the region plus shelf supports trapped modes at isolated frequencies above the shelf cutoff and exhibits resonances at isolated frequencies below cutoff. These results and methods apply directly to the present case. Detailed results of the numerical computations will be presented in a separate paper, but for completeness the low-frequency results are given here.

The scattering region is simple, and so, matching conditions in the limit $\omega \rightarrow 0$ are given by the connection formula (2.8). Consider first the case $c \geqslant 0$. Typical isobaths are sketched in Fig. 5. Energy incident on AB is destroyed in the boundary layer AB′ on C_0 and volume flux incident on AB emerges at B′. Information incident on BD is transferred to B′D′, and the field on D′E is determined by its (constant) value on C_1. As noted in **I**, the boundary layer on AB′ is of thickness ωl_1 with velocities of order $\omega^{-1} \gg 1$. It is here that nonlinear effects and rectification first become important with increasing wave amplitude. The flux emerging at B′ spreads parabolically in a layer of thickness $\omega^{1/2} l_1$ on the scale l_1 to cover the whole channel over the long-wave scale l_1/ω. The present scattering region is the special case of that treated in section 3

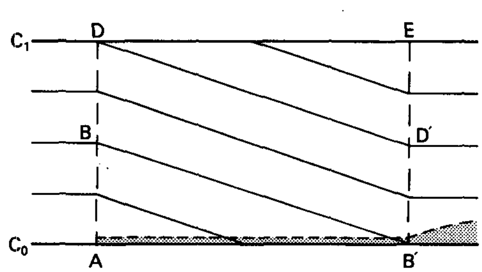

FIG. 5. Isobaths for a simple scattering region with continuous decrease in depth. Volume flux incident on AB is turned by a boundary layer, shown dotted, on AB′ to emerge through a source at B′ and form a spreading boundary current against C_0. Information on BD is carried to B′D′ without change. Information on D′E is derived from DE. This particular geometry can be analyzed simply by mode-matching techniques following Stocker and Johnson (1989).

with logarithmic slope ratio unity ($\beta = 1$) and displacement $d = -ac/b$. The transmitted mode amplitudes are thus given by setting $\beta = 1$ in (3.9a) for $c \geqslant 0$. For $c \leqslant 0$ the coefficients are given by setting $\beta = 1$ in (3.9b) and differ from those for $c \geqslant 0$ in sign alone. The energy carried by a given mode depends on $|d|$, the size of the depth changes and is independent of whether the change is up or down. This reflects the symmetry about $y_2 = 1/2$ of the problem for Φ. If $|d| > 1$ no incident isobaths extend into $X > 0$, all incident energy is dissipated in sidewall boundary layers and (for $\alpha = 0$) there is no motion in $X > 0$. Figure 6 gives the distribution of flux among the transmitted modes normalized on the incident flux as a function of the displacement d for incident wave modes $m = 1$ and $m = 2$. The total transmitted energy is

$$\sin 2m\pi d / 2m\pi + 1 - d, \quad (5.2)$$

precisely that for the problem of a headland on a shelf discussed in **I**, although the distribution among the modes is different. This follows directly from the connection formulae for the two cases.

6. Discussion

It has been shown that in the low-frequency limit a scattering region of order of the shelf width is geostrophic. Matching conditions across the region follow immediately by tracing contours of f/H. Energy is transported without loss if no f/H contour terminates in the region and if no contour terminates and restarts then the region is simple and an explicit connection formula exists.

It was noted that provided a scattering region is simple and the profiles of the shelves before and after the region are the same, low frequency waves are transmitted without scattering. Thus scattering is absent for a wide class of changes in depth, direction and width. Such a class exists also for waves at arbitrary frequencies. In fact, given any shape for C_0 and C_1 a scattering region always exists that can join the rectilinear shelves so that no scattering occurs. This is an immediate consequence of the invariance of Eq. (2.1) under conformal mappings and the Riemann mapping theorem, and is exploited in Johnson (1987) to obtain explicit solutions for shelf waves in corners and waves in elongated lakes. As noted there, the scattering region merges with the shelves exponentially fast. Thus, knowledge of the shape of the shelf boundaries is not sufficient to determine whether scattering will occur. Even at a discontinuous change in width isobaths can be such as to allow waves to pass without change of form. A physical criterion can be given. If the irrotational flow through a scattering region is geostrophic then no scattering of either the flow or incident shelf waves occurs. If irrotational flow cuts contours of f/H then energy is scattered from the flow into shelf waves and among shelf waves.

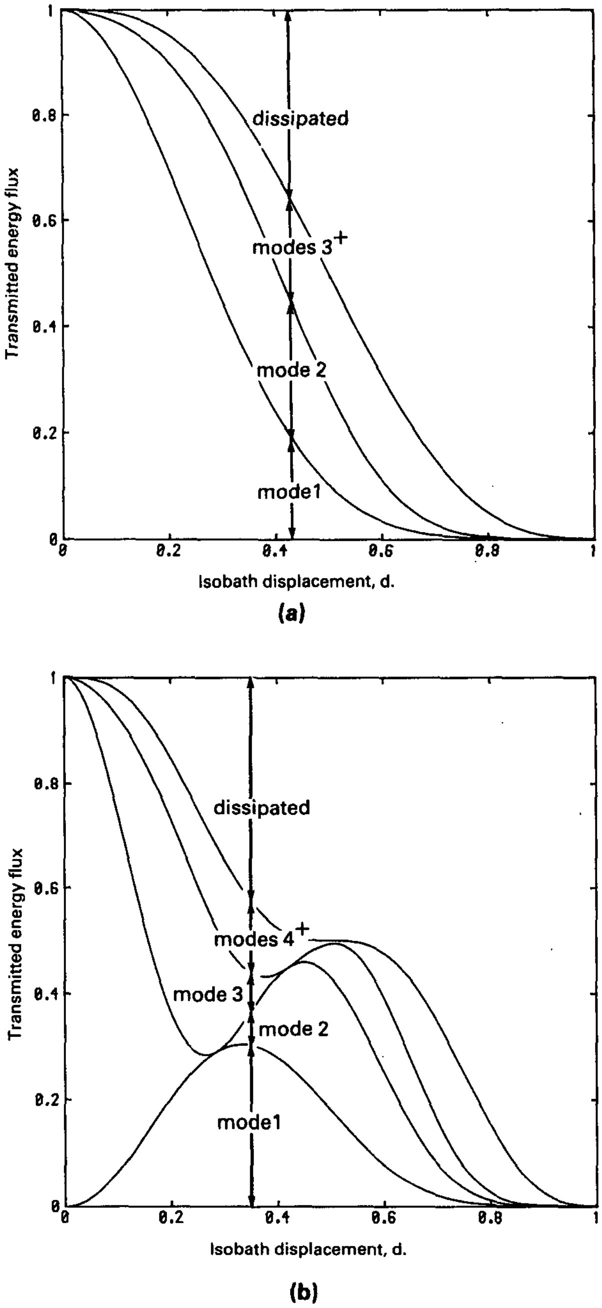

FIG. 6. The distribution among transmitted modes of energy flux for incident shelf waves of modes 1 and 2 normalized on the incident flux, as a function of the displacement d, of the h_1 isobath; (a) mode 1 incident and (b) mode 2 incident.

The results have been presented for shelves bounded by impermeable walls at C_0 and C_1. The analysis is however unaltered if either of these boundaries borders a constant depth ocean. Suppose first that C_0 remains the impermeable coast and C_1 borders the ocean. Then C_1 is an isobath and Fig. 7a gives typical isobaths for rectilinear shelves joined by a scattering region. In the ocean (2.1) reduces to Laplace's equation

$$\nabla^2\psi = 0, \qquad (6.1)$$

showing the flow there to be irrotational, but in the low-frequency limit, (2.7) and the standard connection formulae apply with the additional constraint that since C_1 is an isobath so

$$\psi = c \quad \text{on } C_1, \qquad (6.2)$$

for some complex constant c. Equation (6.1) with (6.2) and the requirement that the solution is bounded at infinity imply that ψ is identically equal to c over the whole ocean region. In the low-frequency limit the ocean is stagnant to leading order over scales of order of the shelf-width. On the long-wave scale, (6.1) reduces to

$$\psi_{yy} = 0, \quad X \neq 0, \quad y > 1. \qquad (6.3)$$

The solution of (6.3) that is continuous and has continuous velocity across $y = 1$ is given by

$$\psi(X, y) = \psi(X, 1^-), \quad y \geq 1, \qquad (6.4)$$

where $\psi(X, 1^-)$ is the solution of (2.4) with boundary conditions

$$\psi_y = 0, \quad \text{on} \quad y = 1. \qquad (6.5)$$

Note that (6.4) joins smoothly onto the shelf-scale open ocean solution with

$$c = \psi(0^-, 1^-) = \psi(0^+, 1^-). \qquad (6.6)$$

Condition (6.5) is the open ocean conditions introduced by Buchwald and Adams (1968). Since C_1 is an isobath no singular regions or boundary layers occur there although they can still occur at the coast.

For an escarpment where C_0 and C_1 are both isobaths bordering constant depth ocean regions (Fig. 7b), with C_1 deeper than C_0 in the orientation chosen here, boundary condition (6.5) applies on both $y = 0$ and $y = 1$ in the low frequency limit and the ocean regions

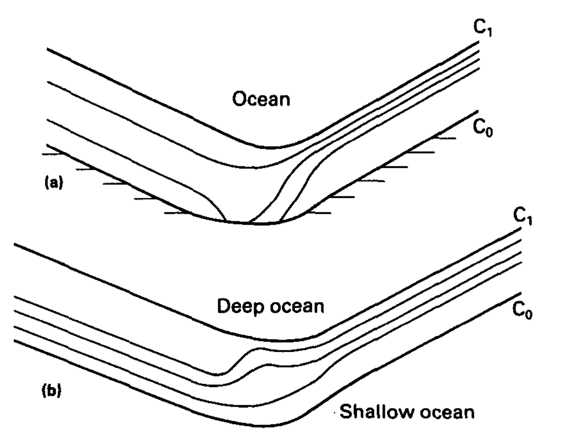

FIG. 7. (a) Isobaths for a scattering region on a shelf bounded by an impermeable coast at C_0 and a constant depth ocean at C_1. (b) Isobaths for a scattering region on an otherwise rectilinear escarpment between a constant depth shallow ocean at C_0 and a deeper constant depth ocean at C_1.

are once again stagnant on the shelf scale. The connection formulae and classifications of section 2 apply but no singular regions or boundary layers occur on C_0 or C_1 as they are isobaths. Thus for a smooth escarpment, any scattering region containing no interior vertical steps is simple and conservative. For profiles (3.1) requiring C_0 and C_1 to be isobaths forces, $h_1 = h_2$ and $b_1 = b_2$. The scaled profiles are identical and there is no scattering across $X = 0$. Such profiles have been described as shelf-similar by Hsueh (1980) in the different context of topography varying only on the long scale.

Although the results have been obtained in the context of equation (2.1) which takes the surface to be rigid, they extend straightforwardly to free-surface flows. The surface displacement is constant along isobaths in the low-frequency limit on the scale of the shelf-width and the same connection formulae apply. The oscillatory irrotational flow becomes a long Kelvin wave present even when the outer shelf boundary is taken to abut a constant-depth open ocean (Johnson, 1989c).

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