On α_i-distance in $n-$dimensional space

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Abstract. In this study, we introduce the concept of α_i-distance, which is a generalization of Taxicab, Chinese Checker, and $\alpha-$distances from $\mathbb{R}^2$ to $\mathbb{R}^n$.

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H. Minkowski [8] has introduced a family of metrics including the taxicab metric at the beginning of the last century. Later, Menger [7] developed the taxicab plane geometry using the taxicab metric instead of the well-known Euclidean metric for the distance between any two points in the analytical plane. E. F. Krause [6] asked the question of how to develop a metric which would be similar to the movement made by playing Chinese Checkers. An answer to this question was given by G. Chen[2]. S. Tian [9] developed a family of metrics, which is a generalization of both the Taxicab and Chinese Checker distances on the plane, and named it the $\alpha-$distance. In [4] and [5], the authors extended the concept of $\alpha-$distance to three and $n-$dimensional space, respectively. In this paper, we introduce the α_i-distance in $n-$dimensional space as a generalization of the work in [4] and [5].

The following definition introduce a family of distances in $\mathbb{R}^n$, which include Taxicab, Chinese Checker and $\alpha-$distances as special cases.

Definition 1 . Let $P_1 = (x_1, x_2, \ldots, x_n)$ and $P_2 = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$. Denote $A=\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$, $\Delta_1=\max A$ and $\Delta_i=\max\left\{A \setminus \bigcup_{k=1}^{i-1} \Delta_k\right\}$,$i = \{2, \ldots, n\}$. For each $i = 1, 2, \ldots, n$, let $\alpha_i \in [0, \pi/4]$ and $\alpha_i \leq \alpha_{i+1}$. Then the function $d_{\alpha_i} : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ α_i-*distance* between points P_1 and P_2 is defined by

$$\begin{aligned} d_{\alpha_i}(P_1, P_2) &= \Delta_1 + (\sec\alpha_1 - \tan\alpha_1)\Delta_2 + \cdots + (\sec\alpha_{n-1} - \tan\alpha_{n-1})\Delta_n \\ &= \Delta_1 + \sum_{j=1}^{n-1} (\sec\alpha_j - \tan\alpha_j)\Delta_{j+1}. \end{aligned}$$

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The α_i-*distance* in $\mathbb{R}^n$ generalize the Taxicab, Chinese Checker, and Alpha distances between points P_1 and P_2 in $\mathbb{R}^n$ as follows:

$$d_T(P_1,P_2) = \Delta_1 + \Delta_2 + \cdots + \Delta_n \quad , \quad d_c(P_1,P_2) = \Delta_1 + (\sqrt{2}-1)\sum_{i=2}^{n}\Delta_i$$

and

$$d_\alpha(P_1,P_2) = \Delta_1 + (\sec\alpha - \tan\alpha)\sum_{i=2}^{n}\Delta_i,$$

respectively. Notice that

1. if all of $\alpha_i = 0$, $\alpha_i = \pi/4$, $\alpha_i = \alpha \in [0,\pi/4]$, then $d_0(P_1,P_2) = d_T(P_1,P_2)$, $d_{\pi/4}(P_1,P_2) = d_c(P_1,P_2)$ and $d_\alpha(P_1,P_2) = d_\alpha(P_1,P_2)$.
2. if $\Delta_2 > 0$ and $\alpha > \alpha_i$, then
$$d_E(P_1,P_2) < d_c(P_1,P_2) < d_\alpha(P_1,P_2) < d_{\alpha_i}(P_1,P_2) < d_T(P_1,P_2)$$
for all $\alpha_i \in (0,\pi/4)$ where $P_1,P_2 \in \mathbb{R}^n$ and d_E stands for the Euclidean distance.
3. if $\Delta_2 = 0$, then P_1 and P_2 lie on a line which is parallel to one of the coordinate axes, and
$$d_c(P_1,P_2) = d_\alpha(P_1,P_2) = d_{\alpha_i}(P_1,P_2) = d_T(P_1,P_2) = d_E(P_1,P_2)$$
for all $\alpha_i \in [0,\pi/4]$.

Let l be a line through P_1 and parallel to the jth-coordinate axis and $l_1,\ldots,l_n$ denote lines each of which is parallel to a coordinate axis distinct from the jth-axis. Geometrically, the shortest way of obtaining a path between the points P_1 and P_2 is taking the union of a line segment parallel to l_j and line segments each making α_i $(i = 1,\ldots,n-1)$ angle with one of $l_1,\ldots,l_n$, as shown in Figure 1. Thus, the shortest distance d_α from P_1 to P_2 is the sum of the Euclidean lengths of these n line segments.

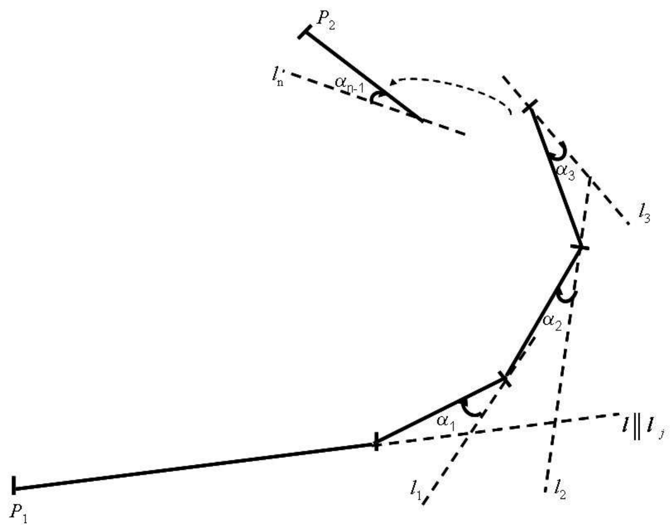

Figure 1

The next two propositions follow directly from the definition of the α_i-distance.

Proposition 1. *The α_i-distance is invariant under all translation in $\mathbb{R}^n$. That is, $T:\mathbb{R}^n\to\mathbb{R}^n\ni T(x_1,x_2,\ldots,x_n)=(x_1+a_1,x_2+a_2,\ldots,x_n+a_n)$, $a_1,a_2,\ldots,a_n\in\mathbb{R}$ does not change the distance between two any points in $\mathbb{R}^n$.*

Proposition 2. *Let $P_1=(x_1,x_2,\ldots,x_n)$ and $P_2=(y_1,y_2,\ldots,y_n)$ be two points in $\mathbb{R}^n$. If $q_j=\sec\alpha_j-\tan\alpha_j$, $j\in\{1,2,\ldots,n-1\}$, then*

$$\Delta_1+\sum_{j=1}^{n-1}q_j\Delta_{j+1}\geq\Delta_i+\sum_{k\in K,j=1}^{n-1}q_j\Delta_k,\quad \text{provided that each of } i,\ j,\ k \text{ appears only once in the last sum,}$$

where $i\in I=\{1,\ldots,n\}$, $k\in K=I\setminus\{i\}$.

Proof. Let $P_1=(x_1,x_2,\ldots,x_n)$ and $P_2=(y_1,y_2,\ldots,y_n)$. First observe that $\Delta_1+\sum_{j=1}^{n-1}q_j\Delta_{j+1}\geq\Delta_i+\sum_{j=1}^{n-1}q_j\Delta_k$, where $i\in I=\{1,2,\ldots,n\}$, $k\in K=I\setminus\{i\}$ is self-evident when dimension is 1. We assume that

$$\Delta_1+\sum_{j=1}^{n-2}q_j\Delta_{j+1}\geq\Delta_{i'}+\sum_{j'=1}^{n-2}q_{j'}\Delta_{k'},$$

where $i'\in I'=\{1,2,\ldots,n-1\}$, $k'\in K'=I'\setminus\{i'\}$. That is, we suppose that the inequality holds when dimension is $n-1$. So we have to show that the inequality holds if the dimension is n. One can easily get

$$\begin{aligned}&\Delta_1+\sum_{j=1}^{n-2}q_j\Delta_{j+1}\geq\Delta_{i'}+\sum_{j'=1}^{n-2}q_{j'}\Delta_{k'}\\&\qquad\Rightarrow\Delta_1+\sum_{j=1}^{n-2}q_j\Delta_{j+1}+q_{n-1}\Delta_n\geq\Delta_{i'}+\sum_{j'=1}^{n-2}q_{j'}\Delta_{k'}+q_{n-1}\Delta_n\\&\qquad\Rightarrow\Delta_1+\sum_{j=1}^{n-1}q_j\Delta_{j+1}\geq\Delta_{i'}+\sum_{j'=1}^{n-2}q_{j'}\Delta_{k'}+q_{n-1}\Delta_n,\end{aligned}$$

where $k'\in K'=I'\setminus\{i'\}$.

Therefore, using this result one can show the required inequality holds as follows: Let $j''\in I''=I'\setminus\{s\}$, $k''\in K''=I\setminus\{i,m\}$.

$$\begin{aligned}\Delta_i+\sum_{j=1}^{n-1}q_j\Delta_k&=\Delta_i+\sum_{j''\in I''}q_{j''}\Delta_{k''}+q_s\Delta_n+q_{n-1}\Delta_m\\&=\Delta_i+\sum_{j''\in I''}q_{j''}\Delta_{k''}+q_s\Delta_n+q_{n-1}\Delta_m+q_s\Delta_m+q_{n-1}\Delta_n-q_s\Delta_m-q_{n-1}\Delta_n\\&=\Delta_i+\sum_{j'=1}^{n-2}q_{j'}\Delta_{k'}+q_{n-1}\Delta_n+(q_s-q_{n-1})(\Delta_n-\Delta_m)\\&\leq\Delta_i+\sum_{j'=1}^{n-2}q_{j'}\Delta_{k'}+q_{n-1}\Delta_n\leq\Delta_1+\sum_{j=1}^{n-1}q_j\Delta_{j+1},\end{aligned}$$

where $k'\in K'=I'\setminus\{i'\}$. □

The following theorem shows that α_i-distance is a metric.

Theorem 3. *For each $\alpha_i \in [0, \pi/4]$, the α_i-distance is a metric for $\mathbb{R}^n$.*

Proof. We have to show that d_{α_i} is positive definite and symmetric, and triangle inequality holds for d_{α_i}. Let $P_1 = (x_1, x_2, \ldots, x_n)$, $P_2 = (y_1, y_2, \ldots, y_n)$ and $P_3 = (z_1, z_2, \ldots, z_n)$ be three points in $\mathbb{R}^n$. The α_i-distance between the points P_1 and P_2 is

$$d_{\alpha_i}(P_1, P_2) = \Delta_1 + \sum_{i=1}^{n-1} (\sec\alpha_i - \tan\alpha_i)\Delta_{i+1},\ \alpha_i \le \alpha_{i+1},\ \alpha_i \in [0, \pi/4]\ .$$

Because $|x_i - y_i| \ge 0$ for $i = 1, 2, \ldots, n$ and $\sec\alpha_i - \tan\alpha_i \ge 0$ for each $\alpha_i \in [0, \pi/4]$, $d_{\alpha_i}(P_1, P_2) \ge 0$. Clearly, $d_{\alpha_i}(P_1, P_2) = 0$ iff $P_1 = P_2$. So d_{α_i} is positive definite.

Since $|x_i - y_i| = |y_i - x_i|$, $i = 1, 2, \ldots, n$, obviously $d_{\alpha_i}(P_1, P_2) = d_{\alpha_i}(P_2, P_1)$. That is, d_{α_i} is symmetric.

Now, we shall prove that $d_{\alpha_i}(P_1, P_2) \le d_{\alpha_i}(P_1, P_3) + d_{\alpha_i}(P_3, P_2)$ for all $P_1, P_2, P_3 \in \mathbb{R}^n$ and $\alpha_i \in [0, \pi/4]$.

Let x_k and y_k are components about Δ_{i+1} and $k \in K = I \setminus \{j\}$.

$$\begin{aligned}
d_{\alpha_i}(P_1, P_2) &= \Delta_1 + \sum_{i=1}^{n-1}(\sec\alpha_i - \tan\alpha_i)\Delta_{i+1} = |x_j - y_j| + \sum_{i=1}^{n-1}(\sec\alpha_i - \tan\alpha_i)\,|x_k - y_k| \\
&= |x_j - z_j + z_j - y_j| + \sum_{i=1}^{n-1}(\sec\alpha_i - \tan\alpha_i)\,|x_k - z_k + z_k - y_k| \\
&\le |x_j - z_j| + |z_j - y_j| + \sum_{i=1}^{n-1}(\sec\alpha_i - \tan\alpha_i)(|x_k - z_k| + |z_k - y_k|) = l\ .
\end{aligned}$$

One can easily see that d_{α_i} satisfies the triangle inequality by examining the following cases:

Case I: If $|x_j - z_j| \ge |x_k - z_k|$ and $|z_j - y_j| \ge |z_k - y_k|$, $j \ne k = \{1, 2, \ldots, n\}$, then for each $\alpha_i \le \alpha_{i+1}$, $\alpha_i \in [0, \pi/4]$, $k \in K$, and $I = \{1, 2, \ldots, n\} \setminus \{j\}$,

$$\begin{aligned}
d_{\alpha_i}(P_1, P_2) &\le l = |x_j - z_j| + |z_j - y_j| + \sum_{i=1}^{n-1}(\sec\alpha_i - \tan\alpha_i)(|x_k - z_k| + |z_k - y_k|) \\
&= |x_j - z_j| + \sum_{i=1}^{n-1}(\sec\alpha_i - \tan\alpha_i)\,|x_k - z_k| + |z_j - y_j| + \sum_{i=1}^{n-1}(\sec\alpha_i - \tan\alpha_i)\,|z_k - y_k| \\
&= d_{\alpha_i}(P_1, P_3) + d_{\alpha_i}(P_3, P_2).
\end{aligned}$$

Case II: If $|x_j - z_j| \ge |x_k - z_k|$ and $|z_j - y_j| \le |z_k - y_k|$, $j \ne k = \{1, 2, \ldots, n\}$, then we have the following two possible situations:

(i) Let $|x_j - z_j| + |z_j - y_j| \ge |x_k - z_k| + |z_k - y_k|$. Then for each

$\alpha_i \in [0, \pi/4]$, $k \in K$, and $I = \{1, 2, \ldots, n\} \setminus \{j\}$,

$$\begin{aligned}
d_\alpha(P_1, P_2) &\leq l = |x_j - z_j| + |z_j - y_j| + \sum_{i=1}^{n-1} (\sec\alpha_i - \tan\alpha_i)(|x_k - z_k| + |z_k - y_k|) \\
&= |x_j - z_j| + \sum_{i=1}^{n-1} (\sec\alpha_i - \tan\alpha_i)\,|x_k - z_k| + |z_j - y_j| + \sum_{i=1}^{n-1} (\sec\alpha_i - \tan\alpha_i)\,|z_k - y_k| \\
&= d_{\alpha_i}(P_1, P_3) + |z_j - y_j| + \sum_{i=1}^{n-1} (\sec\alpha_i - \tan\alpha_i)\,|z_k - y_k| \\
&\leq d_{\alpha_i}(P_1, P_3) + d_{\alpha_i}(P_3, P_2),
\end{aligned}$$

where $|z_j - y_j| + \sum_{i=1}^{n-1} (\sec\alpha_i - \tan\alpha_i)\,|z_k - y_k| \leq d_{\alpha_i}(P_3, P_2)$ because of Proposition 2.

(ii) Let $|x_j - z_j| + |z_j - y_j| \leq |x_k - z_k| + |z_k - y_k|$. One can easily give a proof for the situation (ii) as in situation (i).

Case III: If $|x_j - z_j| \leq |x_k - z_k|$ and $|z_j - y_j| \geq |z_k - y_k|$, $j \neq k = \{1, 2, \ldots, n\}$, then we have the following two possible situations:

(i) Let $|x_j - z_j| + |z_j - y_j| \geq |x_k - z_k| + |z_k - y_k|$.

(ii) Let $|x_j - z_j| + |z_j - y_j| \leq |x_k - z_k| + |z_k - y_k|$.

One can easily give a proof for the Case III as in the Case II. □

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