Self-equivalence 3-rd order ODEs by time-fixed transformations

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Abstract. Let $y''' = f(x, y, y', y'')$ be a 3-rd order ODE. By Cartan equivalence method, we will study the local equivalence problem under the transformations group of time-fixed coordinates.

1 Introduction

Cartan's method of equivalence (see [1], [2] and [4]) is acknowledged to be a powerful tool for studying differential invariants. The main goal of this method is to find necessary and sufficient conditions in order that two geometric structures be equivalent, by a class of given diffeomorphisms. By introducing the invariance of the differential equation under a continuous group of symmetries, *Sophus Lie* rose to the challenge of finding a general method to uncover such invariants, but his approach had some serious defects. Roughly speaking a symmetry group of a system of differential equations is a group which transforms solutions of the system to another solutions. In the classical framework of Lie, these groups consist of geometrical transformations on the space of independent and dependent variables for the system, and act on the solutions by transforming their graphs. Constructing the compatible coframes were the main part of this method, and was done by *Élie Cartan*.

In the first step of his attempt, E. Cartan introduced the structure equations, which leads him to the differential invariants. In this paper, we study Cartan's equivalence problem $y''' = f(x, y, y', y'')$ under the transformations group

$$(1.1) \qquad\qquad X = x, \quad Y = \varphi(y).$$

This is called *time-fixed geometry of* the 3-rd order ODEs.

2 Cartan's equivalence problem

Let $\omega_U = (\omega_U^i)$ and $\Omega_V = (\Omega_V^i)$ be two coframes on an open sets $V \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ respectively, and $G \subseteq \mathrm{GL}(n; \mathbb{R})$ be a prescribed linear group, then find necessary and

sufficient conditions that there exist a diffeomorphism $\Phi : U \to V$ such that for each $u \in U$

$$(2.1) \qquad \Phi^*\Omega_V\Big|_{\Phi(u)} = \gamma_{VU}(u).\omega_U\Big|_u,$$

where $\gamma : U \to G$. (In the future we will always omit the base point notation and write the last relation as $\Phi^*\Omega_V = \gamma_{VU}.\omega_U$.)

3 Time-fixed problem

Let (U, x, y, y', y'') and (V, X, Y, Y', Y'') be open sets with standard coordinates on the $2-$jet bundle $J^2(\mathbb{R}; \mathbb{R})$ of mappings $\mathbb{R} \to \mathbb{R}$, and let there be given 3-rd order ODEs

$$(3.1) \qquad y''' = f(x, y, y', y''), \qquad \text{and} \qquad Y''' = F(X, Y, Y', Y'').$$

The usual symmetries of these equations are the diffeomorphisms $\Phi(x, y, y', y'')$ which map the integral curves into integral curves, that is

$$(3.2) \qquad \Phi^* \begin{pmatrix} dY - Y' \, dX \\ dY' - Y'' \, dX \\ dY'' - F \, dX \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ n & p & 0 \\ q & r & s \end{pmatrix} \begin{pmatrix} dy - y' \, dx \\ dy' - y'' \, dx \\ dy'' - f \, dx \end{pmatrix},$$

where $mps \neq 0$. By the transformation (1.1), we also have the following Jacobian condition on the diffeomorphisms:

$$(3.3) \qquad \Phi^* \begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where $u = \varphi'(y)$. Since $\dim J^2(\mathbb{R}; \mathbb{R}) = 4$ and there exist five relations, this is an over-determined problem on the generators:

$$(3.4) \qquad dX, \ dY, \ dY - Y' \, dX, \ dY' - Y'' \, dX, \ dY'' - F \, dX$$

4 Solving the problem

The relation

$$(4.1) \qquad (dY - Y' \, dX) - dY + Y' \, dX = 0,$$

would seem to suggest modifying the forms, hence define:

$$(4.2) \qquad \Omega_V^1 := dX, \ \ \Omega_V^2 := \frac{dY}{Y'}, \ \ \Omega_V^3 := \frac{dY' - Y'' \, dX}{Y'},$$

$$\Omega_V^4 := dY'' - F \, dX, \ \ \Omega_V^5 := \frac{dY - Y' \, dX}{Y'}$$

Now we have the following relation between forms:

$$(4.3) \qquad \Omega_V^5 - \Omega_V^2 + \Omega_V^1 = 0.$$

By the Jacobian conditions, we can find the diffeomorphisms which satisfy in the above ones; now by (1.1), we have

$$(4.4) \qquad \Phi^* \begin{pmatrix} \Omega_V^1 \\ \Omega_V^2 \\ \Omega_V^3 \\ \Omega_V^4 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} \omega_U^1 \\ \omega_U^2 \\ \omega_U^3 \\ \omega_U^4 \end{pmatrix}.$$

Then, $a_{11} = 1$ since

$$(4.5) \qquad \Phi^* \Omega_V^1 = \Phi^* dX = dx = \omega_U^1,$$

in the same manner, $a_{21} = 0$ and $a_{22} = 1$, since

$$\begin{aligned} \Phi^* \Omega_V^2 &= \Phi^* \left(\frac{dY}{Y'} \right) = \frac{d(\varphi(y))}{((\varphi(y))_x} \\ (4.6) &= \frac{\varphi'(y)\, dy}{\varphi'(y)\, y'} = \omega_U^2 \end{aligned}$$

Moreover, $a_{32} = -a_{31} = \varphi'(y) = u$, because

$$\begin{aligned} \Phi^*(\Omega_V^3) &= \Phi^* \left(\frac{dY' - Y''\, dX}{Y'} \right) = \frac{\varphi'(y)\, dy' + y'\, \varphi''(y)\, dy}{\varphi'(y).y'} \\ (4.7) &= \frac{dy' - y''\, dx}{y'} + \frac{y'\, \varphi''(y)}{\varphi'(y)} . \frac{dy}{y'} - \frac{y'\, \varphi''(y)}{\varphi'(y)} . dx \\ &= \omega_U^3 + u.\omega_U^2 - u.\omega_U^1. \end{aligned}$$

Now by assuming $v = \ell y'$, we have

$$\begin{aligned} \Phi^* \Omega_V^4 &= \Phi^* \left(dY'' - F\, dX \right) \\ (4.8) &= \ell.(dy - y'\, dx) + a.(dy' - y''\, dx) + b.(dy'' - f\, dx) \\ &= -v.\omega_U^1 + v.\omega_U^2 + a.\omega_U^3 + b.\omega_U^4. \end{aligned}$$

thus, $a_{42} = -a_{41} = v$, $a_{43} = a$, $a_{44} = b$; so, the group structure $G \subset \mathrm{GL}(4, \mathbb{R})$ is the set of elements in the following form:

$$(4.9) \qquad g(a, b, u, v) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -u & u & 1 & 0 \\ -v & v & a & b \end{pmatrix}, \quad (a, b, u, v \in \mathbb{R}\ ,\ b \neq 0)$$

Theorem 1. *G is a $4-dimensional$ Lie subgroup of $\mathrm{GL}(4, \mathbb{R})$ with multiplication:*

$$(4.10) \qquad g(a, b, u, v).g(a', b', u', v') = g(a + ba', bb', u + u', v + au' + bv')$$

and inversion,

$$(4.11) \qquad g(a, b, u, v)^{-1} = g\left(-\frac{a}{b}, \frac{1}{b}, -u, \frac{a\, u - v}{b} \right)$$

and its Lie algebra is the set of all matrices in the form:

$$(4.12) \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -u & u & 0 & 0 \\ -v & v & a & b \end{pmatrix} \in \text{Mat}(4 \times 4; \mathbb{R})$$

Proof: The first two statements are trivial. The last part is due to defining relations on the Maurer-Cartan matrix form and in fact the defining relations on the Lie algebra of G, so it is necessary to compute $dg.g^{-1}$. □

5 Prolongation

Now we lift the problem to the associated spaces $U \times G$ and $V \times G$ with the natural left action, that is

$$(5.1) \qquad g \cdot (p, h) = (p, gh), \qquad g, h \in G, \ \ p \in U \ \text{ or } \ p \in V.$$

Given $\Omega_V = (\Omega_V^i)$ and $\omega_U = (\omega_U^i)$ are adapted coframes on open sets $V, U \subseteq \mathbb{R}^n$ respectively, and diffeomorphism $\Phi : U \to V$ satisfying

$$(5.2) \qquad \Phi^* \Omega_V = \gamma_{VU}.\omega_U, \qquad \gamma_{VU} : U \to G.$$

We define new column vectors of $1-$forms on $V \times G$ and $U \times G$ by

$$(5.3) \qquad \Omega|_{(V,g)} = g \, \Pi_V^* \, \Omega_V, \qquad \omega|_{(U,h)} = h \, \Pi_U^* \, \omega_U$$

respectively, where $\Pi_V : V \times G \to V$ and $\Pi_U : U \times G \to U$ are natural projections.

Theorem 2. *There exists a diffeomorphism $\Phi : U \to V$ satisfying (5.2) if and only if there exists a diffeomorphism $\Phi^1 : U \times G \to V \times G$ such that $\Phi^{1*} \, \Omega = \omega$. (See [4].)*

The above theorem is the key to the usefulness of the lifting procedure. Moreover this diffeomorphism Φ^1 covers Φ, i.e. the diagram with the natural projections

$$(5.4) \qquad \begin{array}{ccc} U \times G & \xrightarrow{\ \Phi^1\ } & V \times G \\ {\scriptstyle \Pi_U}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_V} \\ U & \xrightarrow{\ \Phi\ } & V \end{array}$$

commutes. Further, Φ^1 is uniquely determined and automatically satisfies

$$(5.5) \qquad \Phi^1(u, gh) = g.\Phi^1(u, h), \qquad g, h \in G, \ \ u \in U.$$

Definition. (*Right invariant Maurer-Cartan 1−forms.*) Assume G be a Lie group and let R_c denote right multiplication by $c \in G$, if we choose a basis $\{\omega^i\big|_e\}$ of $T_e^* G$, the cotangent space of G at the identity point $e \in G$, then we may define global differential forms by

$$(5.6) \qquad \omega^i|_A \; = \; R_{A^{-1}}^*(w^i|_e), \qquad \forall A \in G.$$

Since

$$(5.7) \qquad \begin{aligned} R_C^* \left(\omega^i|_{AC} \right) &= \; R_C^* \circ R_{(AC)^{-1}}^* \left(\omega^i|_e \right) \\ &= \; R_C^* \circ R_{C^{-1}}^* \circ R_{A^{-1}}^* \left(\omega^i|_e \right) \\ &= \; \omega^i|_A \end{aligned}$$

there are a basis for the right invariant Maurer-Cartan 1−forms . Matters being so, a set of right invariant Maurer-Cartan 1−forms $\{\omega^i|_e\}$ defines functions C_{jk}^i via the equations

$$(5.8) \qquad dw^i = \frac{1}{2} \sum_{j<k} C_{jk}^i . \omega^j \wedge \omega^k$$

where $C_{jk}^i = -C_{kj}^i$.

The right translational invariance immediately implies that the functions C_{jk}^i are in fact constants. These constants are called the *structure constants of G*, relative to the choice of Maurer-Cartan 1−forms.

6   Absorption first step

Define

$$(6.1) \qquad \begin{aligned} \omega_U^1 &= \; dx \\ \omega_U^2 &= \; \frac{dy}{y'} \\ \omega_U^3 &= \; -u\,dx + u\,\frac{dy}{y'} + \big(\frac{dy' - y''\,dx}{y'}\big) \\ \omega_U^4 &= \; -v\,dx + v\,\frac{dy}{y'} + a\,\big(\frac{dy' - y''\,dx}{y'}\big) + b\,(dy'' - f\,dx). \end{aligned}$$

we drop the index U and differentiate the (ω^i)'s, giving

$$(6.2) \qquad d\omega^i = \sum A_{jk}^i . \Pi^k \wedge \omega^j + \sum T_{jk}^i(u,g) . \omega^j \wedge \omega^k$$

and we called them *structure equations*. The matrix $A_{jk}^i \Pi^k$ are now Lie algebra valued differential form. The terms involving the coefficients T_{jk}^i are called *torsion terms*, and the coefficients themselves are called the *torsion coefficients*.

Equations (6.2) do not define the torsion coefficients nor $1-$forms Π^k uniquely, so it is necessary to simplify, even eliminate if possible, this process is called *Lie algebra valued compatible absorption.* So we have

$$(6.3) \qquad d\omega^1 = d\,(dx) = 0, \qquad d\omega^2 = \sum_{i<j} T_{ij}^2\,\omega^i \wedge \omega^j$$

where

$$(6.4) \qquad T_{12}^2 = -u - \frac{y''}{y'}, \qquad T_{23}^2 = 1,$$

and the rest are zero. With respect to this reality that,the elements of the group, which are only in the last two rows, are essential torsion coefficients and thus they are used for reducing the parameters. Now, by absorption of u, we will have $u = -y''/y'$, and the structure group, reduced to the following subgroup,

$$(6.5) \qquad g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v & v & a & b \end{pmatrix}$$

We substitute the acquired value for group parameter u, and repeat the procedure in the sequel, so we have

$$
(6.6) \qquad \begin{aligned}
\omega^1 &= dx, & \omega^2 &= \frac{dy}{y'}, \\
\omega^3 &= -\frac{y''\,dy}{y'^2} + \frac{dy'}{y'}, \\
\omega^4 &= \left(\frac{a\,y''}{y'} - v - b\,f\right)dx + \frac{v\,dy}{y'} + \frac{a\,dy'}{y'} + b\,dy''.
\end{aligned}
$$

7 Absorption second step

By the last step, and using the group parameter u ,we can compute the torsion coefficients, in this step some parameters eliminate,

$$
(7.1) \qquad \begin{aligned}
T_{12}^3 &= -\frac{a\,y'' + v\,y' + b\,f\,y'}{b\,y'^2}, & T_{13}^3 &= T_{14}^3 = T_{34}^3 = 0, \\
T_{23}^3 &= \frac{2\,y''\,b + a}{b\,y'}, & T_{24}^3 &= \frac{1}{b\,y'},
\end{aligned}
$$

and the rest are zero. Since the elements of the group are in the last row, the coefficients are essential; and thus they could be absorbed. In the same manner, we can eliminate three other parameters, so we have,

$$(7.2) \qquad v = -\frac{a\,y''}{y'} - b\,f, \qquad a = -2\,y''\,b, \qquad b = -\frac{1}{y'}.$$

Iterating the procedure, so we have

$$
\begin{aligned}
d\omega^1 &= 0, & d\omega^2 &= \omega^2 \wedge \omega^3, \\
d\omega^3 &= \omega^2 \wedge \omega^4, & d\omega^4 &= \sum_{i<j} T_{ij}^4 \, \omega^i \wedge \omega^j.
\end{aligned}
$$

(7.3)

It is clear that the invariants of this problem are non-zero coefficients on the fourth line of (34), in other words;

$$
\begin{aligned}
I_1 &:= T_{12}^4 = -\frac{1}{y'} \, f_x, \\
I_2 &:= T_{23}^4 = \frac{1}{y'} \left(-3\,f + y'\,f_{y'} + 2\,y''\,f_{y''} \right) \\
I_3 &:= T_{24}^4 = \frac{1}{y'} \left(-3\,y'' + y'\,f_{y''} \right).
\end{aligned}
$$

(7.4)

Theorem *A necessary condition that the equations (3) are equivalent under the time-fixed transformations is that, there exist a time-fixed transformation $(X,Y) = \Phi(x,y) = (x, \varphi(y))$ such that $I_1(f) \circ \Phi^{(1)} = I_1(F)$, $I_2(f) \circ \Phi^{(2)} = I_2(F)$ and $I_3(f) \circ \Phi^{(2)} = I_3(F)$, where $\Phi^{(i)} := J^{(i)}\Phi$ is the $i-$jet prolongation of Φ; in another words*

$$
\begin{aligned}
\frac{f_x}{y'} \circ \Phi^{(1)} &= \frac{F_X}{Y'}, \\
\frac{-3f + y'\,f_{y'} + 2y''\,f_{y''}}{y'} \circ \Phi^{(2)} &= \frac{-3\,F + Y'\,F_{Y'} + 2Y''F_{Y''}}{Y'}, \\
\frac{-3y' + y'\,f_{y''}}{y'} \circ \Phi^{(2)} &= \frac{-3Y'' + Y'F_{Y''}}{Y'}.
\end{aligned}
$$

(7.5)

8   Sufficient condition

Achieving the sufficient condition, we use the theory of $\{e\}$-*structures*. Let us $\mathcal{F}_0 := \mathcal{S}\{I_1, I_2, I_3\}$ be the set of all functions which made by I_1 , I_2 and I_3. We denote its rank by k_0. Since the coframe $\{\omega^1, \omega^2, \omega^3\}$ is invariant, the derivatives with respect to them are also invariants. So if $I \in \mathcal{F}_0$, we define

$$
dI = \frac{\partial I}{\partial \omega^1}\,\omega^1 + \frac{\partial I}{\partial \omega^2}\,\omega^2 + \frac{\partial I}{\partial \omega^3}\,\omega^3 + \frac{\partial I}{\partial \omega^4}\,\omega^4
$$

(8.1)

then all $\partial I/\partial \omega^i$ are also invariants. Where,

$$
\begin{aligned}
\frac{\partial}{\partial \omega^1} &= \frac{\partial}{\partial x}, & \frac{\partial}{\partial \omega^2} &= y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + f\frac{\partial}{\partial y''}, \\
\frac{\partial}{\partial \omega^3} &= y'\frac{\partial}{\partial y'} + 2\,y''\frac{\partial}{\partial y''}, & \frac{\partial}{\partial \omega^4} &= y'\frac{\partial}{\partial y''}
\end{aligned}
$$

Now we define,

$$
\mathcal{F}_1 = \mathcal{S}\left\{ I_1, I_2, I_3; \frac{\partial I_1}{\partial \omega^1}, \frac{\partial I_1}{\partial \omega^2}, \frac{\partial I_1}{\partial \omega^3}, \frac{\partial I_1}{\partial \omega^4}, ..., \frac{\partial I_3}{\partial \omega^4} \right\},
$$

(8.2)

which is the set of 15 certain functions. Iterating this procedure, we achieve $\mathcal{F}_i$ and k_i, for $i = 2, 3, ...$

By the theory of $\{e\}-$structures, if $k_i = k_{i+1}$ for some i, then $k_s = k_i$ for all $s \geq i$, moreover $k_i \leq 4$. The *order* of $\{e\}-$structure is the smallest i which $k_i = k_{i+1}$, and denoted by o, the value of k_i is also denoted by r and called the *rank* of $\{e\}-$structure.

Theorem 4. *Let $\mathcal{E} : y''' = f(x, y, y', y'')$ and $\tilde{\mathcal{E}} : Y''' = F(X, Y, Y', Y'')$ are two given 3-rd order ODEs. Compute $\mathcal{F}_i$ sets and k_i numbers corresponding to those equations. The necessary and sufficient condition that these two equations are equivalent respect to time-fixed transformations is $(\tilde{x}, \tilde{y}) = \Phi(x, y) = (x, \varphi(y))$, $\tilde{o} = o$, $r = \tilde{r}$ and $\tilde{\mathcal{F}}_{o+1} = \mathcal{F}_{o+1} \circ \Phi$. (See [4], pp. 271)*

As a result, we have

Conclusion. *Let $I_1 = \alpha$, $I_2 = \beta$, and $I_3 = \gamma$ are constants. Then $\gamma = 0$, and the ODE is in the form*

$$(8.3) \qquad \mathcal{E}_\beta \; : \; y''' = \frac{3}{2} \, y' \, y''^2 + y'^3 \, h(y) + \frac{\beta}{2} \, y',$$

where h is an arbitrary function of y. The $\mathcal{E}_\beta$ is equivalent to $\mathcal{E}_{\beta'}$ if and only if $\beta = \beta'$.

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