

Solution of the system of differential equations related to Marangoni convections in one fluid layer

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Abstract. A linear stability analysis applied to a system consisting of a horizontal layers which contains conducting fluid, affected by uniform vertical magnetic field and solute, the fluid with uniform heating from below. The flow in a fluid layer is assumed to be governed by the Navier-Stokes equation. Numerical solutions were obtained for stationary convection case using the method of expansion of Chebyshev polynomials (the spectral method). We conclude that the presence of the magnetic field always has a stabilizing effect on the flow, and that the increasing of the solute Marangoni number leads to the increasing of instability in the fluid.

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1 Introduction

The aim of this paper is to investigate the effect of a uniform vertical magnetic field and solute on the Marangoni instability of horizontal three-dimensional planer layer of quiescent electrically conducting fluid, with a free surface whose surface tension depends on temperature and solute, subject to a uniform vertical temperature gradient, within steady stationary instability only. The earliest work on the Marangoni instability of a fluid layer heated from below was performed by Pearson [9], who demonstrated that if the surface tension of the free surface is linearly dependent on temperature then instability can occur in the form of steady convection cells analogous to those described by Raleigh [10] in purely bouncy-driven flows. In particular, Pearson's [9] linear analysis showed that there is a critical value of a certain non-dimensional group of parameters below which all disturbances are stable and above which unstable disturbances exist. This non-dimensional group is now called the Marangoni number, denoted by is defined to be

$$Ma = \frac{\gamma(T_1 - T_2)d}{\rho\nu\kappa},$$

where the constant $-\gamma$ is the rate of change of surface tension with respect to temperature, T_1 is the temperature at the solid lower boundary of the layer, T_2 is the temperature of the undisturbed upper free surface of the layer, d is the thickness of the layer, ρ is the density, ν is the kinematic viscosity and κ is the thermal diffusivity of the fluid. Nield [7] included both the effects of thermocapillary and buoyancy in the bulk of the fluid and found that the two destabilizing mechanisms are tightly coupled and reinforce one another. One significant limitation of the early work was that both Person [9] and Nield [7] considered only the problem with a non-deformable free surface corresponding to the limit of large surface tension. This restriction was relaxed by Scriven and Sterling [18] who included capillary but not gravity waves at the free surface. They showed that in this case the liquid layer is always unstable to sufficiently long wavelength disturbances however small the temperature difference across the layer is. This result was clarified by Smith [19] who showed that the inclusion of gravity waves at the free surface as a stabilizing effect, and results in the reinstatement of critical Marangoni number below which all disturbances are stable. Similar results were obtained by Takashima [20] who also showed that when the lower boundary of the layer is free surface and the upper boundary is solid then the presence of a vertical temperature gradient can stabilize the layer. Sarma [11] investigated a layer of fluid in uniform rotation about a transverse axis and showed that rotation has a stabilizing effect on the onset of steady convection.

The effect of a uniform vertical magnetic field on the thermocapillary instability of a layer of electrically conducting fluid with a non-deformable free surface was first considered by Nield [8] who found that the presence of the field has the effect of increasing the critical Marangoni number for the onset of steady convection and is therefore always a stabilizing influence. Maekawa & Tanasawa [5] considered the same problem with an inclined magnetic field and found that, in this case, steady convection always sets in the form of longitudinal rolls whose axes are aligned with the horizontal component of the magnetic field, and furthermore that only the vertical component of the field has any effect on the critical Marangoni number. Both Nield [8] and Maekawa and Tanasawa [5] calculated asymptotic expressions for the critical Marangoni number and wave number in the limit of large magnetic-field strength. Subsequently Maekawa and Tanasawa [6] extended their analysis to include the effect of buoyancy in the bulk of the fluid. Wilson [24] was generalized in a series of paper by Sarma [12, 17] who considered the effect of a magnetic field on both the conducting and insulating problems with a deformable free surface. Sarma [12, 17] used an incorrect boundary condition at the free surface in his analysis and so all his results for systems with a nonzero magnetic field and a deformable free surface need to be reexamined. The correct description of onset of steady Marangoni convection in the conducting case was given by Wilson [21, 23]. In the present study, we shall consider the onset of Marangoni convection in a horizontal fluid layer affected by a uniform vertical magnetic field and solute. The linear stability equations will be solved using expansion of Chebyshev polynomials. This method has been used by Abdullah [1] in the study of the Benard problem in the presence of a non-linear magnetic field and by Lindsay and Ogden [4] in the implementation of spectral methods resistant to the generation of spurious Eigenvalue.

Lamb [3] used this method to investigate an Eigenvalue problem arising from a model discussing the instability in the earth's core. Also Bukhari [2] has used this method to solve multi layers region. The method possesses good convergence characteristics and effectively exhibits exponential convergence rather than finite power convergence.

2 Problem formulation

We consider an incompressible, Navier-Stokes occupies the horizontal layer $0 \leq x_3 \leq d$. The fluid layer containing two diffusing components, such as heat and solute, and are subject to constant gravitational acceleration $-g\mathbf{e}_3$ and uniform vertical magnetic field $H\mathbf{e}_3$. The fluid motion is constrained by a rigid lower boundary maintained at constant temperature T_0 and solutal S_0 . The upper free boundary whose temperature T_u and solutal S_u are maintained by the radiative transfer of heat into an impinging passive inviscid fluid at constant temperature T_∞ and constant pressure P_∞ . Surface-tension effects at the upper surface are allowed for, where the surface tension τ is dependent on temperature T and solute S according to the simple linear law

$$\tau = \tau_0 - \gamma(T - T_0) - \gamma'(S - S_0),$$

where τ_0 is constant of the surface tension, the depth of the layer is d . A right handed system of Cartesian coordinates x_i with associated base unit vectors \mathbf{e}_i where it will be understood in all subsequent analysis that roman indices take value 1, 2 and 3 whereas Greek indices take value 1 and 2 only. (See figure 1.)

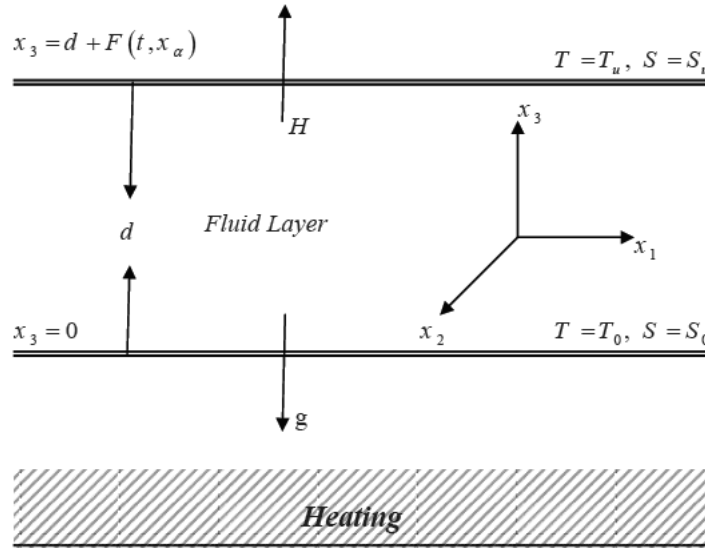


Fig.1. Schematic diagram of the problem

Subject to the Boussinesq approximation and neglecting the effect of buoyancy in the bulk of the fluid the governing equations for an incompressible, electrically

conducting fluid in the presence of a magnetic field and solute are:

$$(2.1) \quad \frac{\partial V_i}{\partial t} + V_j V_{i,j} = -\frac{1}{\rho_0} P_{,i} + \nu V_{i,jj} - g\delta_{i3} + \frac{\mu_m}{4\pi\rho_0} H_j H_{h,j},$$

$$(2.2) \quad \frac{\partial T}{\partial t} + V_j T_{,j} = \kappa T_{,jj},$$

$$(2.3) \quad \frac{\partial S}{\partial t} + V_j S_{,j} = M S_{,jj},$$

$$(2.4) \quad \frac{\partial H_i}{\partial t} + V_j H_{i,j} = H_j V_{i,j} + \eta H_{i,jj},$$

$$(2.5) \quad V_{i,i} = 0,$$

$$(2.6) \quad H_{i,i} = 0,$$

here \mathbf{V} is the fluid velocity, \underline{H} is the magnetic field, T is the temperature, \mathbf{g} is the external gravity field, S is the solutal, and P is the magnetic pressure, which is defined to be $P = \bar{P} + \frac{\mu_m}{8\pi} H^2$, where \bar{P} is the fluid pressure. The properties of the fluid are represented by the fluid density ρ , the kinematic viscosity ν , the magnetic permeability μ_m , the electrical resistivity $\eta = \frac{1}{4\pi\sigma\mu_m}$, the thermal diffusivity κ and the mass diffusivity M . Once \mathbf{V} and magnetic induction \underline{B} have been determined the current density \underline{J} and the electric field \underline{E} can be obtained from the additional Maxwell equations

$$\underline{J} = \bar{\sigma}(\underline{E} + \mathbf{V} \times \underline{B}), \quad \underline{B} = \mu_m \underline{H}.$$

In any non-conducting region, the magnetic field satisfies equation (2.6) together with

$$(2.7) \quad e_{ijk} H_{k,j} = 0.$$

We neglect buoyancy forces in the bulk of the fluid (equivalent to taking the coefficient of the thermal expansion and solute expansion of the fluid to be zero) but include the effect of gravity in the momentum equation (2.1), thus allowing for the presence of gravity-driven surface wave.

Suppose from the outset that the region exterior to the fluid layer is filled with non-conducting material so that no currents can flow there. At the boundaries between conducting and non-conducting materials, the component of the current density normal to the interface is zero. At any interface, normal components magnetic induction are always continuous and so the natural way to guarantee the current condition is to extend continuity to all components of the magnetic induction. Within an insulating material, equation (2.6) indicates that \underline{H} is irrotational so that \underline{H} is the gradient of $\phi(t, x_i)$ where ϕ is a solution of the Laplace equation since $div \underline{H} = 0$, that is

$$H_i = H\delta_{i3} + \phi_{,i}, \quad \phi_{,jj} = 0.$$

Since $x_3 = 0$ is a rigid boundary at a both fixed temperature T_0 and salinity S_0 then the appropriate boundary conditions there are

$$V_i = 0, \quad T = T_0, \quad S = S_0, \quad B_i = \mu_m H_i \quad \text{continuous}$$

With irrotational magnetic field in $x_3 < 0$. The treatment of the upper boundary $x_3 = d$ is more involved since it can move. Suppose that it has equation $x_3 = d + F(t, x_\alpha)$ at time t with unit normal $\underline{n} = n_i e_i$ directed from the viscous fluid in to the passive inviscid fluid. The boundary conditions come from four sources. General radiation conditions heat and mass transfer are

$$(2.8) \quad T_{,i} n_i + L(T - T_\infty) = 0,$$

$$(2.9) \quad S_{,i} n_i + C(S - S_\infty) = 0,$$

where L and C are constants. Material surface fluid particles on the surface $x_3 = d + F(t, x_\alpha)$ remain there and so

$$(2.10) \quad V_3 - \frac{\partial F}{\partial x_\alpha} V_\alpha = \frac{\partial F}{\partial t}$$

Magnetic condition since the region $x_3 > d$ is electrically insulating then $B_i = \mu_m H_i$ is continuous across $x_3 = d$ and the magnetic field in $x_3 > d$ is irrotational, that is, derived from a potential function.

The stress conditions have the components given by

$$\begin{aligned} \tau_{,\alpha} a^{\alpha\beta} x_{i,\beta} + \tau b_\alpha^\alpha n_i &= -P_\infty n_i - \left(P + \frac{\mu_m}{8\pi} H^2\right) n_i \\ &+ \rho_0 \nu (V_{i,j} + V_{j,i}) n_j + \frac{\mu_m}{4\pi} (H_j n_j) H_i, \end{aligned}$$

where $a^{\alpha\beta}$ is the surface metric tensor, τ is the surface tension and b_α^α is the mean curvature of the interface.

This can be decomposed further into the normal and tangential components,

$$(2.11) \quad \tau(T, S) b_\alpha^\alpha = -P_\infty - \left(P + \frac{\mu_m}{8\pi} H^2\right) + 2\rho_0 \nu V_{i,j} n_i n_j + \frac{\mu_m}{8\pi} (H_j n_j)^2,$$

$$(2.12) \quad \tau_{,\alpha} = \rho_0 \nu (V_{i,j} + V_{j,i}) n_j x_{i,\alpha} + \frac{\mu_m}{4\pi} (H_j n_j) (H_i x_{i,\alpha}).$$

It is easily verified that equations (2.1)-(2.4) have a steady state conduction solution in which the viscous fluid is stationary, the top surface is flat, the magnetic field is constant at the imposed value and the fluid interior is permeated by temperature, mass concentration and pressure fields which are functions of x_3 only. The actual solution satisfying all the boundary conditions on $x_3 = 0$ and $x_3 = d$ is

$$V_i = 0, \quad H_i = H \delta_{i3}, \quad F(x_\alpha, t) = 0.$$

$$(2.13) \quad T_E(x_3) = T_0 + \beta x_3, \quad S_E(x_3) = S_0 + \beta^\lambda x_3.$$

$$P_E = P_\infty + \rho_0 g (d - x_3),$$

where β, β^λ denote temperature gradient and concentration gradient respectively and are determined from the thermal and solute boundary conditions at $x_3 = d$, the thermal and solute conditions at $x_3 = 0$ being satisfied trivially. In the equilibrium temperature profile in (2.13) and the conditions (2.8), (2.9) thus lead

$$T_u = \frac{T_0 + Nu T_\infty}{1 + Nu}, \quad S_u = \frac{S_0 + Nu_s S_\infty}{1 + Nu_s},$$

$$\beta d = \frac{Nu}{1 + Nu} (T_\infty - T_0), \quad \beta^\lambda d = \frac{Nu_s}{1 + Nu_s} (S_\infty - S_0),$$

where Nu, Nu_s are the Nusselt numbers such that

$$Nu = L d, \quad Nu_s C d,$$

the Nusselt numbers Nu and Nu_s may take values between 0 and ∞ . $Nu \rightarrow 0$ corresponds to an insulating boundary, while $Nu \rightarrow \infty$ corresponds to an conducting boundary, also $Nu_s \rightarrow 0$ corresponds to an impermeable boundary, whereas $Nu_s \rightarrow \infty$ to a permeable boundary, suppose small perturbation of velocity, pressure, magnetic field, temperature and solute concentration about their equilibrium value $P_E(x_3), H_{e3}, T_E(x_3)$ and $S_E(x_3)$ so that

$$V_i = 0 + \nu_i, \quad P = P_E(x_3) + p, \quad H_i = H \delta_{i3} + h_i,$$

$$T = T_E(x_3) + \theta, \quad S = S_E(x_3) + s.$$

It can be established easily from (2.1)-(2.4) that p, h_i, θ and s satisfy the field equations

$$(2.14) \quad \frac{\partial \nu_i}{\partial t} + \nu_j \nu_{i,j} = \frac{-1}{\rho_0} p_{,i} + \nu \nu_{i,jj} + \frac{\mu_m}{4\pi\rho_0} (H h_{i,3} + h_j h_{i,j}),$$

$$(2.15) \quad \frac{\partial \theta}{\partial t} + \beta \nu_3 + \nu_j \theta_{,j} = \kappa \theta_{,jj},$$

$$(2.16) \quad \frac{\partial s}{\partial t} + \beta' \nu_3 + \nu_j s_{,j} = M s_{,jj}$$

$$(2.17) \quad \frac{\partial h_i}{\partial t} + \nu_j h_{i,j} = h_j \nu_{i,j} + H \nu_{i,3} + \eta h_{i,jj},$$

Where ν and h are solenoidal vector fields. The boundary conditions on the lower boundary $x_3 = 0$ become $\theta = 0, s = 0, \nu_i = 0, \mu_m h_i$ is continuous, while, the conditions on the upper boundary $x_3 = d + F(t, x_\alpha)$ corresponding to (2.8)-(2.10) are modified respectively to

$$(2.18) \quad (n_3 - 1)(T_u - T_0) + d n_i \theta_{,i} + Nu \theta + Nu (T_u - T_0) \frac{F}{d} = 0,$$

$$(2.19) \quad (n_3 - 1)(S_u - S_0) + d n_i s_{,i} + Nu_s s + Nu_s (S_u - S_0) \frac{F}{d} = 0,$$

$$(2.20) \quad \nu_3 - \frac{\partial F}{\partial x_\alpha} \nu_\alpha = \frac{\partial F}{\partial t},$$

The modified form of normal component of (11) is

$$(2.21) \quad \begin{aligned} \tau(T, S) b_\alpha^\alpha &= -p + \frac{\mu_m}{8\pi} ((h_j n_j)^2 + 2H n_3 h_j n_j) + \rho_0 g F \\ &- \frac{\mu_m}{8\pi} H^2 (1 - n_3^2) + 2 \rho_0 \nu \nu_{i,j} n_i n_j, \end{aligned}$$

whereas the tangential surface stress condition (12) yields

$$(2.22) \quad \frac{\partial \tau}{\partial T} T_{,\alpha} + \frac{\partial \tau}{\partial S} S_{,\alpha} = \rho_0 \nu (\nu_{i,j} + \nu_{j,i}) n_j x_{i,\alpha} + \frac{\mu_m}{4\pi} (H n_3 + h_j n_j) (h_i x_{i,\alpha}).$$

In (22) it is assumed that the derivative of the surface tension with respect to temperature and concentration are evaluated at

$$T = T_u + (T_u - T_0) d^{-1} F + \theta, \quad S = S_u + (S_u - S_0) d^{-1} F + s.$$

3 Non-dimensionalization

To simplify the analysis we introduce non-dimensional variables. Spatial coordinates x_i are scaled with respect to d , time t with respect to $\frac{d^2}{\kappa}$ so that, the non-dimensional form of the upper surface becomes $x_3 = 1 + f(t, x_\alpha)$. Similarly, perturbed velocity components are scaled with respect to $\frac{\nu}{d}$, magnetic field components with respect to $\frac{H\kappa}{\eta}$, pressures with respect to $\frac{\rho_0 \nu^2}{d^2}$, temperatures with respect to $\frac{|T_0 - T_u| \nu}{\kappa}$, and salinity $\frac{|S_0 - S_u| \nu}{M}$. Non-dimensionalizing the equations and boundary conditions give rise to nine non-dimensional groups, namely the Prandtl number, P_r , defined as $P_r = \frac{\nu}{\kappa}$, the magnetic Prandtl number, P_m , defined as $P_m = \frac{\eta}{\kappa}$, the Chandrasekhar number, Q defined as $Q = \frac{\mu_m H^2 d^2}{4\pi \rho_0 \nu \eta}$, the Schmidt number, S_c , defined as $S_c = \frac{\nu}{M}$, the Lewis number, Le , defined as $Le = \frac{M}{\kappa}$, the Marangoni numbers, Ma , Ma_s , defined as $Ma = \frac{-\partial \tau}{\partial T} \frac{d |T_0 - T_u|}{\rho_0 \nu \kappa}$, $Ma_s = \frac{-\partial \tau}{\partial S} \frac{d |S_0 - S_u|}{\rho_0 \nu M}$, the Crispation number, Cr , defined as $Cr = \frac{\rho_0 \nu \kappa}{d \tau(T_u, S_u)}$, and the Bond number, B_0 , defined as $B_0 = \frac{\rho_0 g d^2}{d \tau(T_u, S_u)}$.

4 Linearized problem

Until now the analysis has been exact. Henceforth suppose that perturbations in ν , h , θ , s and p are small that their products can be ignored whenever they occur. Equations become

$$(4.1) \quad P_r^{-1} \frac{\partial \nu_i}{\partial t} = -p_{,i} + \nu_{i,jj} + Q P_r^{-1} h_{i,3},$$

$$(4.2) \quad \frac{\partial \theta}{\partial t} = \xi_T \nu_3 + \theta_{,jj},$$

$$(4.3) \quad \frac{1}{Le} \frac{\partial s}{\partial t} = \xi_s \nu_3 + s_{,jj},$$

$$(4.4) \quad P_m^{-1} \frac{\partial h_i}{\partial t} = \nu_{i,3} + h_{i,jj},$$

Where $\xi_T = \text{sign}(T_0 - T_u)$ and $\xi_s = \text{sign}(S_0 - S_u)$ indicates the boundary at which heat and solute are supplied. Such that

$$\xi_T = \text{sign}(T_0 - T_u) = \begin{cases} +1 & \text{when heating from below} \\ -1 & \text{when heating from above} \end{cases}$$

$$\xi_s = \text{sign}(S_0 - S_u) = \begin{cases} +1 & \text{when soluting from below} \\ -1 & \text{when soluting from above} \end{cases}$$

The boundary condition on the lower boundary $x_3 = 0$ are unchanged. On the upper boundary $x_3 = 1 + f(t, x_\alpha)$ the outward unit normal has components

$$n_1 = \frac{-f_{,1}}{\sqrt{1 + f_{,1}^2 + f_{,2}^2}}, \quad n_2 = \frac{-f_{,2}}{\sqrt{1 + f_{,1}^2 + f_{,2}^2}}, \quad n_3 = \frac{1}{\sqrt{1 + f_{,1}^2 + f_{,2}^2}},$$

so that the Linearized upper boundary condition are

$$P_r \frac{\partial \theta}{\partial x_3} + Nu(P_r \theta - \xi_T f) = 0,$$

$$S_c \frac{\partial s}{\partial x_3} + Nu_s(S_c s - \xi_s f) = 0,$$

$$\nu_3 = P_r^{-1} \frac{\partial f}{\partial t},$$

$$\lim_{x_3 \rightarrow 1^-} h_i = \frac{\mu_0}{\mu_m} \lim_{x_3 \rightarrow 1^+} \frac{\partial \phi_u}{\partial x_i},$$

$$Cr^{-1} P_r^{-1} b_\alpha^\alpha = -p + Q P_r^{-1} h_3 + B_0 C r^{-1} P_r^{-1} f + 2 \frac{\partial \nu_3}{\partial x_3},$$

$$-Ma P_r^{-1} (P_r \theta_{,\alpha} - \xi_T f_{,\alpha}) - Ma_s S_c^{-1} (S_c s_{,\alpha} - \xi_s f_{,\alpha}) = \nu_{3,\alpha} + \frac{\partial \nu_\alpha}{\partial x_3} + Q h_{,\alpha}.$$

Recall that the magnetic field on insulating material is irrotational and is derived from a potential function $\phi(x, t)$ which is the solution of Laplace equation. Let $\phi = \psi(t, x_3) e^{i(px_1 + qx_2)}$ then

$$\frac{\partial^2 \psi}{\partial x_3^2} - a^2 \psi = 0, \quad a^2 = p^2 + q^2, \quad \frac{\partial \psi}{\partial x_3} \rightarrow 0 \text{ as } |x_3| \rightarrow \infty.$$

Where $a = \sqrt{p^2 + q^2}$ the dimensionless wave number. Trivially ϕ_0 and ϕ_u have functional form

$$\phi_0 = C_0(t)e^{ax_3} e^{i(px_1+qx_2)}, \quad \phi_u = C_u(t)e^{-ax_3} e^{i(px_1+qx_2)}.$$

When $x_3 < 0$ then $h = C_0(t)(ip, iq, a)$ and continuity of the magnetic induction across $x_3 = 0$ requires that

$$h_{3,3} = \frac{1}{\mu_m} b_{3,3} = \frac{-1}{\mu_m} b_{\alpha,\alpha} = \frac{\mu_0}{\mu_m} a^2 C_0(t) = \frac{a}{\mu_m} b_3 = ah_3.$$

With a similar argument on $x_3 = 1+f(t, x_\alpha)$, hence the magnetic boundary conditions are

$$\begin{aligned} \frac{\partial h_3}{\partial x_3} - ah_3 &= 0, & x_3 &= 0, \\ \frac{\partial h_3}{\partial x_3} + ah_3 &= 0, & x_3 &= 1. \end{aligned}$$

We put $\omega = \nu_3$, $h = h_3$ and take the double curl of equation (4.1), to obtain

$$(4.5) \quad P_r^{-1} \frac{\partial}{\partial t} \nabla^2 \omega - Q P_r^{-1} \nabla^2 D h = \nabla^4 \omega,$$

$$(4.6) \quad \frac{\partial \theta}{\partial t} = \xi_T \omega + \nabla^2 \theta,$$

$$(4.7) \quad \frac{1}{Le} \frac{\partial s}{\partial t} = \xi_s \omega + \nabla^2 s,$$

$$(4.8) \quad P_m^{-1} \frac{\partial h}{\partial t} = \nabla^2 h + D \omega,$$

Where $D = \frac{\partial}{\partial t}$, ∇^2 the three-dimensional Laplacian operator. Now the boundary conditions on $x_3 = 0$ become

$$\omega = 0, \quad \theta = 0, \quad D \omega = 0, \quad s = 0, \quad D h - ah = 0.$$

While on $x_3 = 1$ the boundary conditions are

$$P_r D \theta + Nu(P_r \theta - \xi_T f) = 0,$$

$$S_c D s + Nu_s(S_c s - \xi_s f) = 0,$$

$$\omega - P_r^{-1} \frac{\partial f}{\partial t} = 0,$$

$$D h + ah = 0,$$

$$\Delta_2 f - f B_0 - Cr P_r(-p + 2D \omega + Q P_r^{-1} h) = 0,$$

$$-Ma P_r^{-1} \Delta_2(P_r \theta - \xi_T f) - Ma_s S_c^{-1} \Delta_2(S_c s - \xi_s f) - (\Delta_2 \omega + D^2 \omega + Q D h) = 0,$$

Where $\Delta_2 = \nabla^2 - \frac{\partial^2}{\partial x_3^2}$ (two-dimensional Laplacian operator). Normal mode solution is sought for equations (4.5)-(4.8) in the form

$$\phi(t, x_i) = \phi(x_3) e^{i(px_1+qx_2)+\sigma t}, \quad \phi = \{\omega, h, p, \theta, s\}.$$

$$f(t, x_\alpha) = f_0 e^{i(px_1+qx_2)+\sigma t}, \quad f_0 \text{ is constant,}$$

where σ is an eigenvalue of the system.

$$(4.9) \quad \sigma P_r^{-1}(D^2 - a^2) \omega - \sigma Q P_m^{-1} P_r^{-1} Dh = (D^2 - a^2)^2 \omega - Q P_r^{-1} D^2 \omega,$$

$$(4.10) \quad \sigma \theta = \xi_T \omega + (D^2 - a^2) \theta,$$

$$(4.11) \quad \frac{\sigma}{Le} s = \xi_s \omega + (D^2 - a^2) s,$$

$$(4.12) \quad \sigma P_m^{-1} h = (D^2 - a^2) h + D\omega.$$

The boundary conditions on $x_3 = 0$ are

$$\omega = 0, \quad \theta = 0, \quad D\omega = 0, \quad s = 0, \quad Dh - ah = 0.$$

As a preamble to the formulation of the final boundary conditions on $x_3 = 1$, the pressure everywhere is given by the equation

$$p = \frac{1}{a^2} (D^3 \omega - a^2 D\omega + Q P_r^{-1} D^2 h - \sigma P_r^{-1} D\omega).$$

Hence the boundary conditions on $x_3 = 1$ are

$$P_r D\theta + Nu(P_r \theta - \xi_T f_0) = 0,$$

$$S_c Ds + Nu_s(S_c s - \xi_s f_0) = 0,$$

$$\omega = \sigma P_r^{-1} f_0,$$

$$Dh + ah = 0,$$

$$a^2(B_0 + a^2)f_0 - CrP_r(D^3 \omega - 3a^2 D\omega - Q P_r^{-1} D\omega) = \sigma Cr(Q P_m^{-1} h - D\omega),$$

$$(D^2 + a^2)\omega + Ma P_r^{-1}(P_r \theta - \xi_T f_0)a^2 + Ma_s S_c^{-1}(S_c s - \xi_s f_0)a^2 = 0.$$

5 Results and discussion

The Eigenvalue problem consists of a 10 ordinary differential equation of first order with 10 boundary conditions. This problem is solved using method based on series expansion of Chebyshev polynomials. In this paper we will discuss the numerical results in two ways, first, the Marangoni convection when free non-deformable surface is, second, the Marangoni convection when free surface is deformable.

1- Non-deformable free surface $Cr = 0$ The numerical results of this state are found by two stages: the first stage to find the thermal Marangoni number Ma versus the wave number a by considering chosen values of the non-dimensions constants P_r , P_m , Q , Nu , Nu_s , Le , Ma_s and B_0 . And the second stage is founding the critical values of thermal Marangoni number for deferent values of Q , Nu , Nu_s , Ma_s , we found the following results will be held:

1. When the fluid faces uniform vertical magnetic field, it helps to reduce the Marangoni convections as shown in figure (2).
2. If the free surface conducts, the heat is great, this cause great reducing the currents of Marangoni convections and this means that the stability will increase in the fluid as shown in figure (3).
3. If the free surface is permeable for the solute, this helps weakly to reduces the currents of Marangoni convections as shown in figure (4).
4. The increasing of the solute Marangoni number Ma_s means the increasing of Marangoni convection that leads the increase instability of the fluid as shown in figure (5).
5. The increasing of free surface deformation leads to generate stationary instability that controlled by forces of surface tension either the surface is conducting the heat source as shown in figure (6) or not as shown in figure (7).
6. There is a direct relation between the critical thermal Marangoni number Ma_c , and Nusselt number of heat Nu as shown in table (1) and this assure what we found in paragraph 2.
7. There is a direct relation between the critical thermal Marangoni number Ma_c and Nusselt number of solute as shown in table (2) and this assure what we found in paragraph 4.
8. There is a reversal relation between the critical thermal Marangoni number Ma_c and the solute Marangoni number Ma_s as shown in table (3) and this assure what we found in paragraph 4.
9. There is a direct relation between the critical thermal Marangoni number and Chandrasekhar number Q . If the free surface is the following cases: case 1, it doesn't conduct the heat, case 2, it isn't permeable for the solute, case 3, it doesn't conduct and is not permeable, and case 4, it conducts and it is permeable, it case 5, it is not permeable and it conducts as shown in table (4).

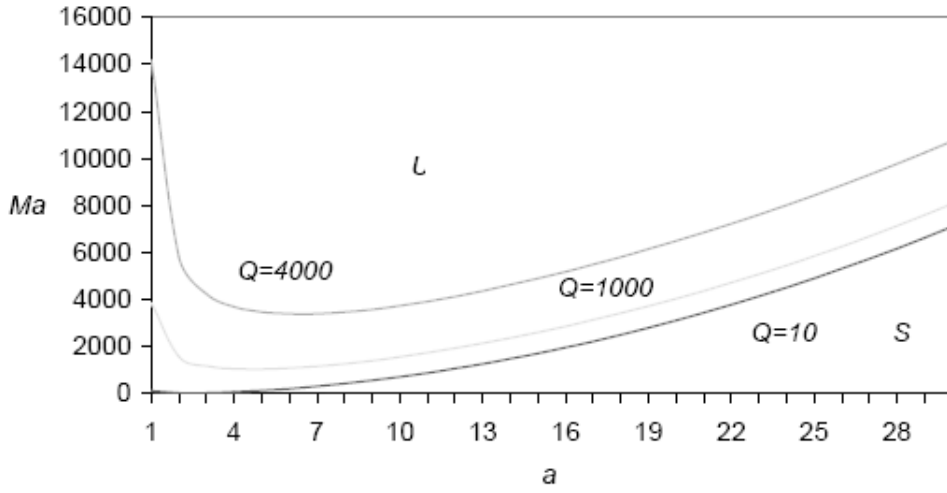


Fig.2. The relation between thermal Marangoni number Ma and wave number a for various of Chandrasekhar number Q when $Ma_s = 100$, $Nu = Nu_s = 0$, $B_0 = 0$, $P_r = P_m = 20$, $Le = 1$.

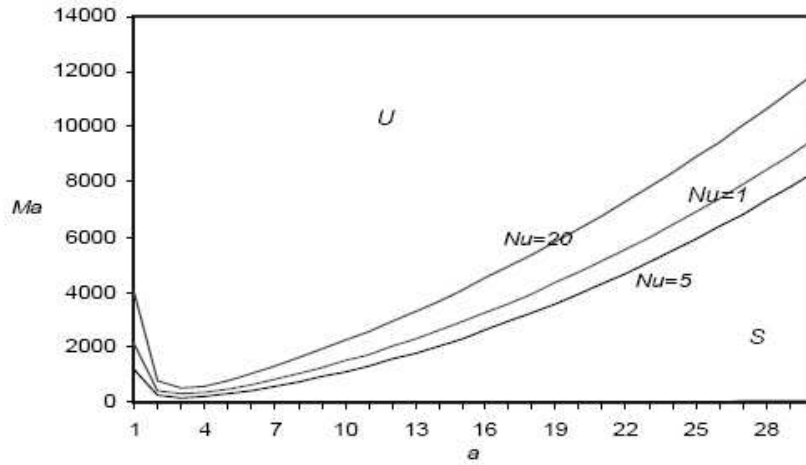


Fig.3. The relation between thermal Marangoni number Ma and wave number a for various of Nu when $Ma_s = 100$, $Nu_s = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$.

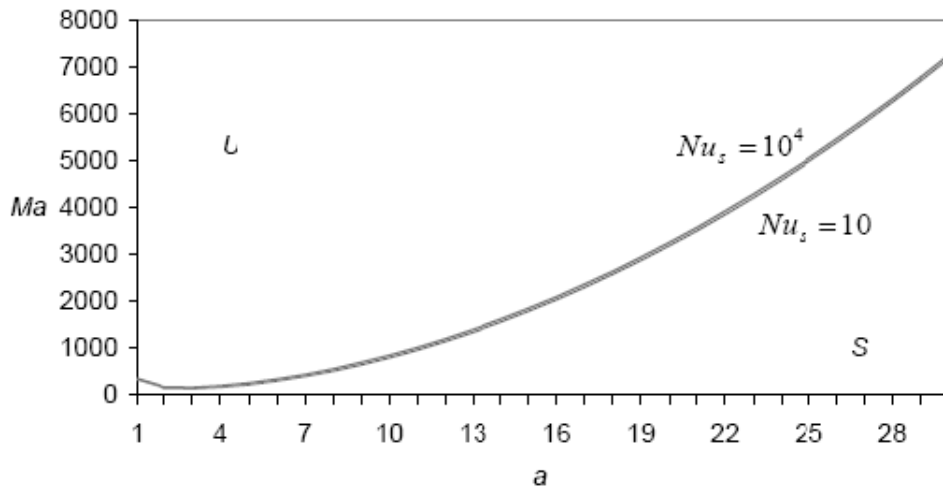


Fig.4. The relation between thermal Marangoni number Ma and wave number a for various of Nu_s when $Ma_s = 100$, $Nu = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$, $Q = 50$.

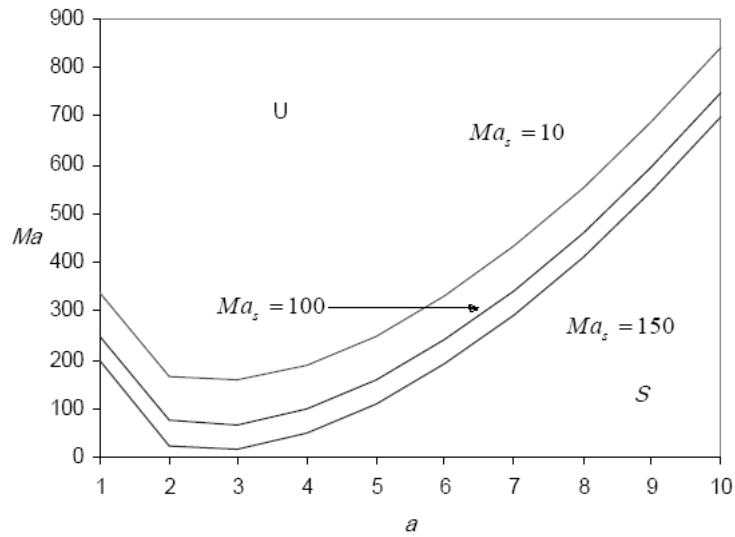


Fig.5. The relation between thermal Marangoni number Ma and wave number a for various of Ma_s when $Q = 50$, $Nu = Nu_s = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$.

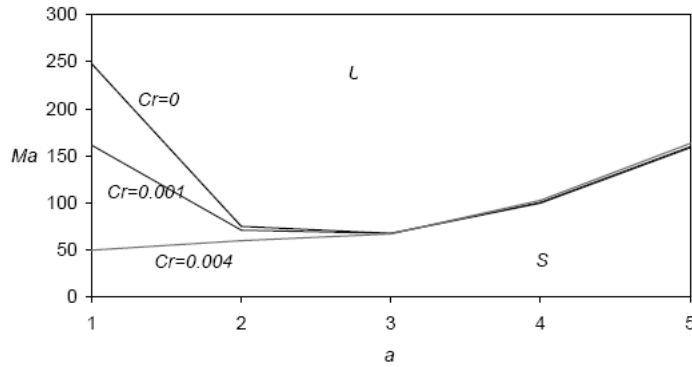


Fig.6. The relation between thermal Marangoni number Ma and wave number a for various of Cr when the free surface conducts the heat and $Ma_s = 100$, $Nu = Nu_s = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$, $Q = 50$.

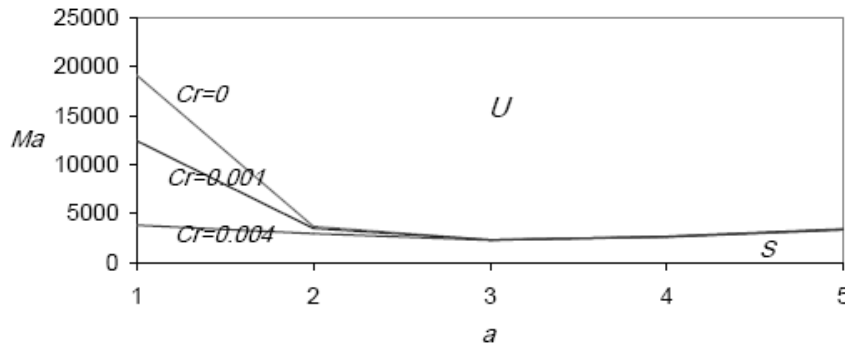


Fig.7. The relation between thermal Marangoni number Ma and wave number a for various of Cr when the free surface insulating the heat and $Ma_s = 100$, $Nu_s = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$, $Q = 50$.

| Nu | Ma_c | a_c |
|------|---------------|-------|
| 0 | 192.862266 | 2.189 |
| 5 | 568.113841 | 2.189 |
| 10 | 943.365415 | 2.189 |
| 15 | 1318.61698903 | 2.189 |
| 20 | 1693.86856327 | 2.189 |
| 25 | 2069.12013752 | 2.189 |
| 30 | 2444.37171177 | 2.189 |
| 35 | 2819.62328602 | 2.189 |
| 40 | 3194.87486026 | 2.189 |
| 45 | 4144.31829932 | 1.94 |
| 50 | 4582.29034882 | 1.94 |

Table (1):- Critical Marangoni number Ma_c and wave number a_c for various Nu when $Q = 50$, $Nu_s = 0$, $Ma_s = 0$, $P_r = P_m = Le = 10$, $B_0 = 0.1$.

| Nu_s | Ma_c | a_c | Ma_s | Ma_c | a_c | Ma_s |
|--------|------------|--------|--------|------------|--------|--------|
| 0 | 182.862266 | 2.1890 | 10 | 92.862266 | 2.1890 | 100 |
| 200 | 191.554073 | 2.1890 | 10 | 183.232839 | 2.1890 | 100 |
| 400 | 191.602867 | 2.1890 | 10 | 183.75288 | 2.1890 | 100 |
| 600 | 191.619253 | 2.1890 | 10 | 183.927483 | 2.1890 | 100 |
| 800 | 191.627470 | 2.1890 | 10 | 184.015058 | 2.1890 | 100 |
| 100 | 191.632407 | 2.1890 | 10 | 184.067683 | 2.1890 | 100 |
| 1200 | 191.635702 | 2.1890 | 10 | 184.102800 | 2.1890 | 100 |
| 1400 | 191.638057 | 2.1890 | 10 | 184.127899 | 2.1890 | 100 |
| 1600 | 191.639823 | 2.1890 | 10 | 184.146733 | 2.1890 | 100 |

Table (2):- Critical Marangoni number Ma_c and wave number a_c for various when Nu_s when $Ma_s = 0, P_r = P_m = Le = 10, B_0 = 0.1, Q = 50, Nu = 0$.

| Ma_s | Ma_c | a_c |
|--------|----------|--------|
| 0 | 192.8622 | 2.1890 |
| 10 | 182.8622 | 2.1890 |
| 20 | 172.8622 | 2.1890 |
| 30 | 162.8622 | 2.1890 |
| 40 | 152.8622 | 2.1890 |
| 50 | 142.8622 | 2.1890 |
| 60 | 132.8622 | 2.1890 |
| 70 | 122.8622 | 2.1890 |
| 80 | 112.8622 | 2.1890 |
| 90 | 102.8622 | 2.1890 |
| 100 | 92.8622 | 2.1890 |

Table (3):- Critical Marangoni number Ma_c and wave number a_c for various Ma_s when $Q = 50, Nu = Nu_s = 0, P_r = P_m = Le = 10, B_0 = 0.1$.

| Q | $Nu = 0, Nu_s = 0$ | | $Nu = 0, Nu_s \rightarrow \infty$ | |
|-----|--------------------|-------|-----------------------------------|-------|
| | Ma_c | a_c | Ma_c | a_c |
| 0 | -20.393 | 1.99 | 79.586 | 1.99 |
| 10 | 3.690 | 2.17 | 101.230 | 2.17 |
| 20 | 26.505 | 2.18 | 122.077 | 2.18 |
| 30 | 48.929 | 2.18 | 142.871 | 2.18 |
| 40 | 71.031 | 2.18 | 163.600 | 2.18 |
| 50 | 92.862 | 2.18 | 189.257 | 2.18 |
| 60 | 114.460 | 2.18 | 204.839 | 2.18 |
| 70 | 135.856 | 2.18 | 225.347 | 2.18 |
| 80 | 157.076 | 2.18 | 245.782 | 2.18 |
| 90 | 178.141 | 2.18 | 266.147 | 2.18 |

Table (4):- Critical Marangoni number Ma_c and wave number a_c for various Q when $(Nu = 0, Nu_s = 0), (Nu = 0, Nu_s \rightarrow \infty), B_0 = 0.1, P_r = P_m = Le = 10, Ma_s = 0$.

| Q | $Nu \rightarrow \infty, Nu_s \rightarrow \infty$ | | $Nu \rightarrow \infty, Nu_s = 0$ | |
|----|--|-------|-----------------------------------|-------|
| | Ma_c | a_c | Ma_c | a_c |
| 0 | 358168.699 | 2.18 | -102585.21 | 1.81 |
| 10 | 440129.695 | 2.18 | -5870.598 | 2.15 |
| 20 | 520018.482 | 2.18 | 74571.912 | 2.18 |
| 30 | 599907.269 | 2.18 | 152751.780 | 2.18 |
| 40 | 675000.549 | 2.18 | 229533.490 | 2.18 |
| 50 | 750596.076 | 2.18 | 305126.713 | 2.18 |
| 60 | 825171.766 | 2.18 | 379700.127 | 2.18 |
| 70 | 898859.951 | 2.18 | 453388.482 | 2.18 |
| 80 | 971773.192 | 2.18 | 526300.645 | 2.18 |
| 90 | 1044005.875 | 2.18 | 598525.659 | 2.18 |

Table (4'):- Critical Marangoni number Ma_c and wave number a_c for various Q when $(Nu \rightarrow \infty, Nu_s \rightarrow \infty), (Nu \rightarrow \infty, Nu_s = 0), B_0 = 0.1, P_r = P_m = Le = 10, Ma_s = 0$.

2- deformable free surface $Cr \neq 0$. The numerical results of this state is find the thermal Marangoni number Ma versus the wave number a by considering chosen values of the non-dimensions constants $P_r, P_m, Q, Nu, Nu_s, Le, Ma_s, B_0$. When $Cr = 0.001$, so we found the following results:

1. When the fluid faces uniform vertical magnetic field, it helps to reduce the Marangoni convections as shown in figure (8).
2. If the free surface conducts, the heat great, this cause great reducing the currents of Marangoni convections as shown in figure (9).
3. If the free surface is permeable for the solute, this helps to reduce the currents of Marangoni convections as shown in figure (10).
4. The increasing of the solute Marangoni number increases the Marangoni convections as shown in figure (11).
5. When we put $Cr = 0$, we obtain the first state and this means that the state is a generalization for the pervious state. Moreover, we observe that the overstable Marangoni convection appears when we take Cr as the values not equal zeros. Also, the deformation and increasing in Marangoni number of solute lead to the overstable Marangoni convection quickly.

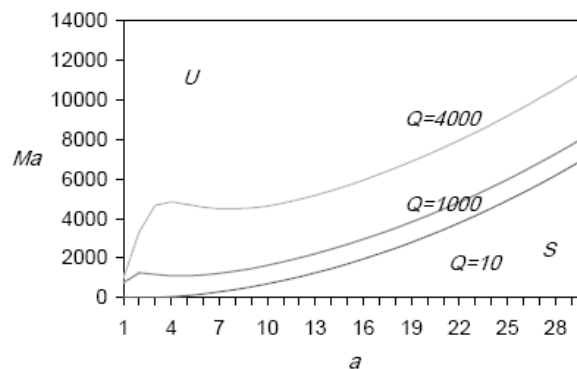


Fig.8. The relation between thermal Marangoni number Ma and wave number a for various of Chandrasekhar number when $Ma_s = 100$, $Nu = Nu_s = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$.

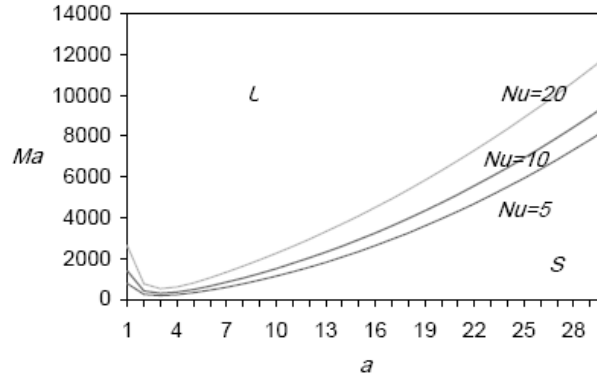


Fig.9. The relation between thermal Marangoni number Ma and wave number a for various of Nu when $Ma_s = 100$, $Nu_s = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$, $Q = 50$.

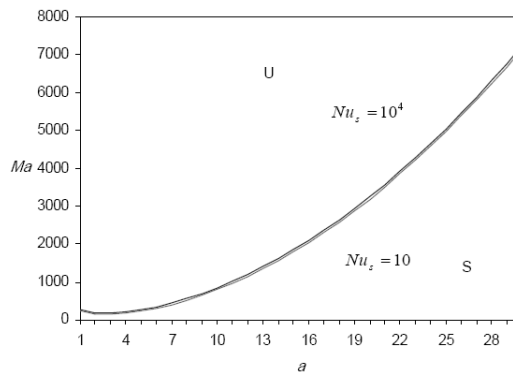


Fig.10. The relation between thermal Marangoni number Ma and wave number a for various of Nu_s when $Ma_s = 100$, $Nu = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$, $Q = 50$.

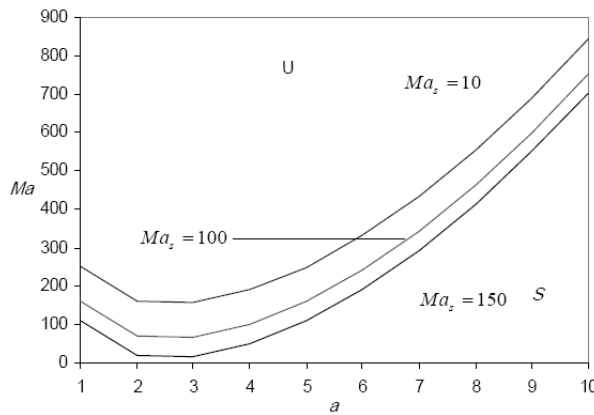


Fig.11. The relation between thermal Marangoni number Ma and wave number a for various of Ma_s when $Q = 50$, $Nu = Nu_s = 0$, $P_r = P_m = 20$, $B_0 = 0.1$, $Le = 1$.

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