First integrals for problems of calculus of variations on locally convex spaces

Eugénio A.M. Rocha and Delfim F.M. Torres

Abstract. The fundamental problem of calculus of variations is considered when solutions are differentiable curves on locally convex spaces. Such problems admit an extension of the Euler-Lagrange equations (Orlov, 2002) for continuously normally differentiable Lagrangians. Here, we formulate a Legendre condition and an extension of the classical theorem of Emmy Noether, thus obtaining first integrals for problems of the calculus of variations on locally convex spaces.

M.S.C. 2000: 49K27, 47J30, 46T20.
Key words: calculus of variations, locally convex spaces, Noether's theorem.

1 Introduction

The fundamental problem of the calculus of variations (CV) is studied in the setting of infinite dimensional differential geometry [10], i.e. where solutions are differentiable curves on locally convex spaces. The usual problem of CV concerns to find, among all functions with prescribed boundary conditions, those which minimize a given functional, i.e.

$$\begin{gathered}\int_\Omega f(x, u(x), \nabla u(x))\, dx \longrightarrow \min \\ \text{s.t.} \quad u \in X \quad \text{and} \quad u|_{\delta\Omega} = u_0,\end{gathered}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\nabla u \in \mathbb{R}^{nm}$, $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ is a continuous function, u_0 is a given function, and X is a Banach space. It is well known that problems of CV have very wide applications in several fields of mathematics, and in many areas of physics, economics, and biology. In recent years, part of the renewal of interest in variational methods finds its origins in nonlinear elasticity [4].

The present work deals with an extension of the setting of the previous problem by replacing $\mathbb{R}^m$ by a locally convex space E. Although, for technical reasons, we will only deal with the case $n = 1$. The problem of the CV on a locally convex space E is

Applied Sciences, Vol.10, 2008, pp. 207-218.

then

$$J[x] = \int_a^b L(t, x(t), \dot{x}(t))\, dt \longrightarrow \min\,, \tag{1.1}$$

where $L : [a,b] \times E \times E \to \mathbb{R}$, $x : [a,b] \to E$. But it is not completely defined without introducing the precise notions of differentiability and mapping regularity. It is well known that such functionals (1.1) are not, in general, Fréchet differentiable (see e.g. [3, 18]). Comparing with the classical CV, the main difficulties arise from the substitution of $\mathbb{R}^m$ by a locally convex space E; moving from a finite-dimension vector space with an inner product to an infinite-dimension vector space with only a family of semi-norms. The motivation for such problem is the first author interest [17] on studying calculus of variations and control theory on the infinite dimensional differential geometry setting (differential calculus of smooth mappings between subsets of sequentially complete locally convex spaces) developed by Frölicher, Kriegl and Michor [10]; so called convenient spaces.

A central result of CV is the first order necessary optimality condition, asserting that optimal solutions satisfy the Euler-Lagrange equation. Solutions of the Euler-Lagrange equations are called *extremals*. Extremals include optimal solutions but, in general, may also include non-optimal solutions. A function $C(t, x, v)$ which is preserved along all the extremals (i.e. $C(t, x(t), \dot{x}(t))$ is constant $\forall t \in [a, b]$ and for any extremal x) is called a *first integral*. The equation $C(t, x(t), \dot{x}(t)) = \text{constant}$ is the corresponding *conservation law*. Conservation laws are a useful tool to simplify the problem of finding minimizers [2, 11]. Emmy Noether was the first to establish a general theory of conservation laws in the calculus of variations [12]. Noether's theorem comprises a universal principle, connecting the existence of a group of transformations under which the functional to be minimized is invariant (the existence of variational symmetries) with the existence of conservation laws. Noetherian conservation laws play an important role on a vast number of disciplines, ranging from classical mechanics, where they find important interpretations such as conservation of energy, conservation of momentum, or conservation of angular momentum, to engineering, economics, control theory and their applications [7].

A very general approach to first order optimality conditions has been initiated in the sixties of the XX century by H. Halkin [8] and R.V. Gamkrelidze and G.L. Kharatishvili [5, 6]. In this work we use the notion of compactly normally differential functionals (introduced in [16]) and inductive scales of locally convex spaces: (1) to formulate a Legendre condition; (2) to extend the classical Noether's theorem to the calculus of variations on locally convex spaces.

The use of inductive scales of locally convex spaces is not a merely generalization, it is a need. First, as shown in [13], Banach manifolds are not suitable for many questions of global analysis. Second, we require the evaluation map $E^* \times E \to \mathbb{R}$ to be jointly continuous in order to be able to use integration by parts on some working space. However, if E is a locally convex space, and E^* is its dual equipped with any locally convex topology, then the jointly continuity of the evaluation map imply that, in fact, E is a normable space. Since, then, there are neighborhoods $U \subset E$ and $V \subset E^*$ of zero such that the image of $V \times U$ by the evaluation map is contained on $[-1, 1]$. But then U is contained in the polar of V, so it is bounded in E. Therefore, E admits a bounded neighborhood.

2 Inductive scales of locally convex spaces

In what follows, F, E are locally convex spaces (LCSs) with the corresponding determining systems of semi-norms $\{\|\cdot\|_p\}_{p\in S_F}, \{\|\cdot\|^q\}_{q\in S_E}$, that are inductively ordered according to the increase of semi-norms. The set of linear continuous maps from F to E will be denoted by $\mathcal{L}(F,E)$. For a gentle introduction to locally convex topological vector spaces we refer the reader to [3, 9].

Let $A \in \mathcal{L}(F,E)$. For any $q \in S_E$, the normal index of A is the increasing multivalued mapping $n_A : S_E \to 2^{S_F}$ defined by

$$n_A(q) = \left\{ p \in S_F : \sup_{\|y\|_p \le 1} \|Ay\|^q < +\infty \right\},$$

and $N_{\mathcal{L}(F,E)} = \{n_A : A \in \mathcal{L}(F,E)\}$ is the set of all normal indices. We will consider the following inductive scale of LCSs [21]

$$\overrightarrow{(F,E)} = \{(X_n, \tau_n)\}_{n\in N_{\mathcal{L}(F,E)}} \quad \text{with} \quad X_n = \{A \in \mathcal{L}(F,E) : n_A \le n\},$$

i.e. a system of LCSs inductively ordered according to the continuous embedding $m \le n \Rightarrow X_m \subseteq X_n$; where each space X_n has the projective topology τ_n with respect to the determining system of semi-norms

$$\|A\|_p^q = \left\{ \sup_{\|y\|_p \le 1} \|Ay\|^q : p \in n(q), q \in S_E \right\}.$$

This inductive scale of LCSs generalize classical interpolation scales [20]. Properties of a scale are related with properties of the spaces of the scale and vice-versa [14]. Convergence in the scale $\overrightarrow{(F,E)}$ is the convergence in any space X_n of the scale. For Z a LCS, an operator $A \in \mathcal{L}(Z, \overrightarrow{(F,E)})$ if $A \in \mathcal{L}(Z, X_n)$ for some X_n and $n \in N_{\mathcal{L}(F,E)}$.

Definition 2.1. A mapping $\phi : F \to \overrightarrow{(F,E)}$ is

1. *continuous* at a point $y_0 \in F$ if $y \to y_0 \Rightarrow \phi(y) \to \phi(y_0)$ for some X_n.
2. *uniformly continuous* on a set $D \subset F$ if, for some X_n, the map $\bar{\phi} : D \to X_n$ is uniformly continuous.

Let F_1, F_2 be LCSs. The canonical isomorphism [15] correspondence $B(y_1, y_2) = (Ay_1)y_2$ between linear operators $A : F_1 \to \mathcal{L}(F_2, E)$ and bilinear operators $B : F_1 \times F_2 \to E$ justifies the following isometrically identification

$$\overrightarrow{(F_1, \overrightarrow{(F_2,E)})} \cong \overrightarrow{(F_1 \times F_2, E)}. \tag{2.1}$$

If $E \equiv \mathbb{R}$, the conjugate space has a normal decomposition into the following inductive scale of Banach spaces

$$\overrightarrow{(F,\mathbb{R})} \equiv \overrightarrow{F^*} = \{f \in \mathcal{L}(F,\mathbb{R}) : \|f\|^p \equiv \sup_{\|x\|_p \le 1} |f(x)| < +\infty \text{ and } p \in S_F\}.$$

In this case, we have the following canonical isometrical isomorphism of linear and bilinear operators in LCSs

$$\overrightarrow{(F_1, \overrightarrow{F_2^*})} \cong \overrightarrow{(F_1 \times F_2)}^*.$$

3 Compactly normal differentiability

Let F, E be LCSs, $y \in F$, and C a convex compact subset of F having y as limiting point.

Definition 3.1. A map $g : F \to E$ is

1. *normally differentiable* at the point y if $\Delta g(y,h) = g'(y)h + \phi_y(h)$ where $g'(y) \in \mathcal{L}(F,E)$ and
$$\forall q \in S_E \; \exists p \in S_F \; : \; \frac{\|\phi_y(h)\|^q}{\|h\|_p} \xrightarrow{h \to 0} 0.$$

2. *continuously normally differentiable* at y if g is normally differentiable in a neighborhood of y and the derived mapping $g' : F \to \overrightarrow{(F,E)}$ is continuous at y. This last condition means that $\tilde{g}' : [\alpha,\beta] \subset F \to X_n$ is continuous for some X_n, and that the compactness of $[\alpha,\beta]$ implies $n_{\tilde{g}'(y)} \leq n$ for $n \in N_{\mathcal{L}(F,E)}$ and all $y \in [\alpha,\beta]$.

3. *twice continuously normally differentiable* at y if g is continuously normally differentiable in a neighborhood of y and $g' : F \to X_n$ is normally differentiable at a vicinity of y for some X_n. In this case, $g''(y)$ will denote $(g')'(y)$ and, by the identification (2.1), $f'' : F \to \overrightarrow{(F \times F, E)}$.

4. *K-differentiable* (compactly normally differentiable) at a point y if for each C the restriction $f = g|_C$ is normally differentiable at the point y. The value $(f|_C)'(y)$ does not depend on the choice of subset C, and it is denoted by $g'_K(y)$.

Lemma 3.1. *A mapping $g : F \to E$ is continuously normally differentiable at a convex compact set $C \subset F$ if and only if*
$$\frac{\|g(x+h) - g(x) - g'(x)h\|^s}{\|h\|_m} \to 0 \text{ as } h \to 0, x \in C,$$
for some normal index $n_A \in N_{\mathcal{L}(F,E)}$ and any $s \in S_E$ and $m \in n_A(s)$.

Remark 3.1. If F is a Banach space then $\phi_y(h) = o(\|h\|)$ and g' is the Fréchet derivative of g. In such case, we denote the derivative by a dot, $\dot{g}$.

We denote by $C^1([a,b];E)$ the space of continuous differentiable mappings $x : [a,b] \to E$ with a determining system of semi-norms $\{\|x\|_1^p\}_{p \in P}$ where
$$\|x\|_1^p = \sup_{a \leq t \leq b} \|x(t)\|_p + \sup_{a \leq t \leq b} \|\dot{x}(t)\|_p.$$

4 Euler-Lagrange and Legendre conditions

Let $F \equiv [a,b] \times E \times E$. The following theorem proves that optimal solutions of the problem of the Calculus of Variations, for K-differential functionals, verify an Euler-Lagrange equation.

Theorem 4.1 (*[16]*). *Let a function $L(t,x,v)$ be continuously normally differentiable on $[a,b]\times E\times E$. If the functional $\mathrm{J}[\cdot]$ (1.1) has an extremum at a point $x\in C^1([a,b];E)$, then $\mathrm{J}[\cdot]$ is K-differentiable at x and we have*

$$J'_K[x]h=\int_a^b\left[\frac{\partial L}{\partial x}(t,x(t),\dot{x}(t))h(t)+\frac{\partial L}{\partial v}(t,x(t),\dot{x}(t))\dot{h}(t)\right]dt=0. \tag{4.1}$$

Observe that $\dot{x}$ is the Fréchet derivative of x, where $[a,b]\to\overrightarrow{([a,b],E)}$ is identified with $[a,b]\to E$. For a given x, let $\mu_x:[a,b]\to F$ be defined by $\mu_x(t)=(t,x(t),\dot{x}(t))$. For any $(\bar{t},\bar{x},\bar{v})\in F$ define $L^t_{(\bar{x},\bar{v})}:\mathbb{R}\to\mathbb{R}$ as $L^t_{(\bar{x},\bar{v})}(t)=L(t,\bar{x},\bar{v})$, $L^x_{(\bar{t},\bar{v})}:E\to\mathbb{R}$ as $L^x_{(\bar{t},\bar{v})}(x)=L(\bar{t},x,\bar{v})$, and $L^v_{(\bar{t},\bar{x})}:E\to\mathbb{R}$ as $L^v_{(\bar{t},\bar{x})}(v)=L(\bar{t},\bar{x},v)$. Partial derivatives are defined as usual

$$\frac{\partial L}{\partial t}(\bar{t},\bar{x},\bar{v})=(L^t_{(\bar{x},\bar{v})})'(\bar{t})\,,\ \frac{\partial L}{\partial x}(\bar{t},\bar{x},\bar{v})=(L^x_{(\bar{t},\bar{v})})'(\bar{x})\,,\ \frac{\partial L}{\partial v}(\bar{t},\bar{x},\bar{v})=(L^v_{(\bar{t},\bar{x})})'(\bar{v})\,, \tag{4.2}$$

hence, for a given extremal $\hat{x}$, we have $\frac{\partial L}{\partial x}\circ\mu_{\hat{x}}:[a,b]\to\overrightarrow{E^*}$ and $\frac{\partial L}{\partial v}\circ\mu_{\hat{x}}:[a,b]\to\overrightarrow{E^*}$. Now, by virtue of the jointly continuous of the evaluation map on $\overrightarrow{E^*}\times E$, the space $\mathcal{C}\equiv\{A\otimes B\in\mathcal{L}([a,b],\overrightarrow{E^*})\times\mathcal{L}([a,b],E): A \text{ and } B \text{ are differentiable mappings}\}$ is a derivation algebra, i.e. it admits the Leibniz product rule and the usual integration by parts.

The Euler-Lagrange equation is obtained as a corollary of Theorem 4.1 in the usual way, considering the integration by parts of the second addend of (4.1)

$$\int_a^b\left(\frac{\partial L}{\partial v}\circ\mu_{\hat{x}}(t)\right)\dot{h}(t)\,dt=\left(\frac{\partial L}{\partial v}\circ\mu_{\hat{x}}(t)\right)h(t)\Big|_a^b-\int_a^b\frac{d}{dt}\left[\frac{\partial L}{\partial v}\circ\mu_{\hat{x}}(t)\right]h(t)\,dt\,,$$

and using an extension of the fundamental lemma of the calculus of variations.

Corollary 4.2 ([16]). *Let a function $L(t,x,v)$ be twice continuously normally differentiable on $[a,b]\times E\times E$. If functional $\mathrm{J}[\cdot]$ (1.1) has an extremum at a point $\hat{x}\in C^1([a,b];E)$, then $\hat{x}$ satisfies the Euler-Lagrange equation (in $\overrightarrow{E^*}$)*

$$\frac{\partial L}{\partial x}\circ\mu_{\hat{x}}(t)-\frac{d}{dt}\frac{\partial L}{\partial v}\circ\mu_{\hat{x}}(t)=0. \tag{4.3}$$

Following the classical terminology [2, 11], we call *extremals* to the solutions of (4.3); *first integrals* to functions which are kept constant along all the extremals.

Other necessary conditions exist apart from the Euler-Lagrange equation. In what follows we obtain, from the second variation, the so called Legendre condition. Let us observe that the Euler-Lagrange equation is just a consequence of the Fermat lemma *(if a function $f:E\to\mathbb{R}$ has a local extremum at a point x and is normally differentiable at this point, then we have $f'(x)=0$)*, and the Legendre condition is a consequence of the necessary condition of second order *(if a function $f:E\to\mathbb{R}$ has a local minimum at x and is twice normally differentiable at this point, then not only $f'(x)=0$ but also $f''(x)\geq 0$)*. Such conditions are proved by the standard methods on [1].

Consider the natural extensions of the partial derivatives defined on (4.2) to higher orders.

Theorem 4.3. *Let a function $L(t,x,v)$ be twice continuously normally differentiable on $[a,b]\times E\times E$. If functional* J$[\cdot]$ *(1.1) has an extremum at a point $x\in C^1([a,b];E)$, then x satisfies the Legendre condition*

$$\frac{\partial^2 L}{\partial v\partial v}\circ\mu_x(t)\geq 0\quad \forall t\in[a,b]. \tag{4.4}$$

Proof. As shortcut define $X\equiv C^1([a,b];E)$, which is a locally convex space. Recall that $J:X\to\mathbb{R}$, $J'_K[\cdot]:X\to\overrightarrow{X^*}$ and $J''_K[\cdot]:X\to\overrightarrow{X\times X^*}$. Theorem 4.1 ensures that $J'_K[x]$ is a K-differentiable mapping at x, so it is (locally) normally differentiable at x. We will show that $J''_K[x]$ is also a K-differentiable mapping at x.
Consider an arbitrary convex compact set $C\subset X$ where x is a limiting point. The sets $A=\{y\in E: y\in x([a,b]), x\in C\}$ and $B=\{z\in E: z\in\dot{x}([a,b]),\dot{x}\in C\}$ are convex compacts in E. By the definition of normal differentiability, we have

$$\begin{aligned}
J'_K[x+h_2]h_1-J'_K[x]h_1 &= \int_a^b\left[\frac{\partial L}{\partial x}(t,x+h_2,\dot{x}+h_2)h_1\right.\\
&\left.+\frac{\partial L}{\partial v}(t,x+h_2,\dot{x}+h_2)\dot{h}_1\right]-\left[\frac{\partial L}{\partial x}(t,x,\dot{x})h_1+\frac{\partial L}{\partial v}(t,x,\dot{x})\dot{h}_1\right]dt\\
&=\int_a^b\left[\left(\frac{\partial^2 L}{\partial v\partial v}\circ\mu_x\right)\dot{h}_1\dot{h}_2+\left(\frac{\partial^2 L}{\partial v\partial x}\circ\mu_x\right)\dot{h}_2h_1\right.\\
&\left.+\left(\frac{\partial^2 L}{\partial x\partial v}\circ\mu_x\right)\dot{h}_1h_2+\left(\frac{\partial^2 L}{\partial x\partial x}\circ\mu_x\right)h_1h_2\right]dt+\int_a^b r_t(h_2,\dot{h}_2)\,dt,
\end{aligned}$$

where r_t is the residual term of the increments at the point $(t,x(t),\dot{x}(t))$. Since L is twice continuously normally differentiable on $C\equiv[a,b]\times A\times B$, we can apply lemma 3.1 to r_t

$$\frac{|r_t(h_2(t),\dot{h}_2(t))|}{\|h_2(t)\|_m+\|\dot{h}_2(t)\|_m}\to 0 \text{ as } (h_2(t),\dot{h}_2(t))\to 0.$$

Let $R_x(h_2)=\int_a^b r_t(h_2(t),\dot{h}_2(t))\,dt$. The uniform convergence and the above condition imply $\frac{R_x(h_2)}{\|h_2\|_1^m}\to 0$ as $h_2\to 0$, which implies the K-differentiability of $J''_K[x]$ at x.

Hence, the second variation for $J[x]$ reads as follows, for all $x,h\in C^1([a,b];E)$,

$$\begin{aligned}
J''_K[x](h,h) &= \int_a^b\left[\left(\frac{\partial^2 L}{\partial v\partial v}\circ\mu_x(t)\right)\dot{h}(t)^2+2\left(\frac{\partial^2 L}{\partial v\partial x}\circ\mu_x(t)\right)\dot{h}(t)h(t)\right]dt\\
&+\int_a^b\left(\frac{\partial^2 L}{\partial x\partial x}\circ\mu_x(t)\right)h(t)^2\,dt,
\end{aligned}$$

with the corresponding Jacobi eigenvalue equation

$$-\frac{d}{dt}\left(R(t)\dot{h}(t)\right)+P(t)h(t)=\lambda h(t),\quad h\in C^1([a,b];E),$$

where we have defined

$$R(t)=\frac{\partial^2 L}{\partial v\partial v}\circ\mu_x(t)\quad\text{and}\quad P(t)=\frac{\partial^2 L}{\partial x\partial x}\circ\mu_x(t)-\frac{d}{dt}\frac{\partial^2 L}{\partial v\partial x}\circ\mu_x(t).$$

Now, if we observe that $\frac{d}{dt}h(t)^2 = 2\dot{h}(t)h(t)$, then integration by parts of the second variation yields

$$J''_K[x]\,h = \int_a^b R(t)\dot{h}(t)^2 + P(t)h(t)^2\,dt \qquad \forall x, h \in C^1([a,b];E). \tag{4.5}$$

The necessary condition of second order implies that $J''_K[x]\,(h,h) \geq 0$ for all $h \in C^1([a,b];E)$. Therefore, (4.4) follows from (4.5). Namely, if $R(t_0) < 0$ for a $t_0 \in [a,b]$, then one can always choose an $h \in C^1([a,b];E)$ having very large $\dot{h}(t_0)$ and small $h(t_0)$, so that $J''_K[x]\,(h,h) < 0$ holds; however, this is not possible. □

5 Invariance and conservation laws

To obtain a Noether theorem on locally convex spaces, we will need further regularity of the solution curves x on the LCS E. Similarly to $C^1([a,b];E)$, we denote by $C^2([a,b];E)$ the space of twice continuous differentiable mappings $x : [a,b] \to E$ with a determining system of semi-norms $\{\|x\|_2^p\}_{p\in P}$ where

$$\|x\|_2^p = \sup_{a\leq t\leq b} \|x(t)\|_p + \sup_{a\leq t\leq b} \|\dot{x}(t)\|_p + \sup_{a\leq t\leq b} \|\ddot{x}(t)\|_p.$$

Therefore, the problem of calculus of variations is

$$J[x] = \int_a^b L\left(t, x(t), \dot{x}(t)\right) dt \longrightarrow \min\,, \tag{5.1}$$

where $L : [a,b] \times E \times E \to \mathbb{R}$ is twice continuously normally differentiable and $x \in C^2([a,b];E)$.

Let us introduce a local Lie group h^s with generators T e X. A transformation h in the space $\mathbb{R} \times E$ is a twice continuously normally differentiable mapping $h : \mathbb{R} \times E \to \mathbb{R} \times E$ with $h(t,x) = (\bar{t}, \bar{x})$ defined by the equations

$$\bar{t} = h_t\left(t, x\right)\,, \quad \bar{x} = h_x\left(t, x\right)\,,$$

for h_t and h_x given functions. The symmetry transformations, which define the invariance of problem (5.1), are transformations which depend on a real parameter $s \in \mathbb{R}$. Let s vary continuously in an open interval $|s| < \varepsilon$, for small ε, and h^s be a family of transformations defined by

$$\bar{t} = h_t^s(t,x) = h_t\left(t, x, s\right)\,, \quad \bar{x} = h_x^s(t,x) = h_x\left(t, x, s\right)\,, \tag{5.2}$$

where h_t and h_x are analytical functions in $[a,b] \times E \times (-\varepsilon, \varepsilon)$. The one-parameter family of transformations h^s is a local Lie group if and only if it satisfies the *local closure property*; contains the *identity* (without loss of generality, we assume that the identity transformation is obtained for parameter $s = 0$); and *inverse* exist for each s sufficiently small. Since normally differentiable mappings admit a Taylor formula in the asymptotic form [16], if h^s defined by (5.2) is a local Lie group, then one can expand $h_t = (t,x,s)$ and $h_x = (t,x,s)$ in Taylor series about $s = 0$:

$$
\begin{aligned}
\bar{t} &= h_t\,(t,x,0) + \frac{\partial h_t}{\partial s}\,(t,x,0)\,s + \frac{1}{2}\frac{\partial^2 h_t}{\partial s^2}\,(t,x,0)\,s^2 + \cdots \\
&= t + T\,(t,x)\,s + o\,(s)\ , \\
\bar{x} &= h_x\,(t,x,0) + \frac{\partial h_x}{\partial s}\,(t,x,0)\,s + \frac{1}{2}\frac{\partial^2 h_x}{\partial s^2}\,(t,x,0)\,s^2 + \cdots \\
&= x + X\,(t,x)\,s + o\,(s)\ ,
\end{aligned}
\tag{5.3}
$$

where the quantities $T(t,x) = \frac{\partial h_t}{\partial s}\,(t,x,0)$, and $X(t,x) = \frac{\partial h_x}{\partial s}\,(t,x,0)$ are called the *generators* of h^s. A local Lie group h^s induces, in a natural way, a local Lie group $\tilde{h}^s$ in the space $\mathbb{R}\{t\} \times E\{x\} \times E\{\dot{x}\}$:

$$
\tilde{h}^s : \begin{cases} \bar{t} & = h_t\,(t,x,s)\ , \\ \bar{x} & = h_x\,(t,x,s)\ , \\ \dot{\bar{x}} & = \frac{d\bar{x}}{dt} = \frac{\frac{\partial h_x}{\partial t} + \frac{\partial h_x}{\partial x}\dot{x}}{\frac{\partial h_t}{\partial t} + \frac{\partial h_t}{\partial x}\dot{x}}\,. \end{cases}
$$

This group is called in the $\mathbb{R}^n$-setting the *extended group*. Noticing that

$$
\begin{aligned}
\frac{\partial h_t}{\partial t} &= 1 + s\frac{\partial T}{\partial t} + o(s)\,, \quad & \frac{\partial h_t}{\partial x} &= s\frac{\partial T}{\partial x} + o(s)\,, \\
\frac{\partial h_x}{\partial t} &= s\frac{\partial X}{\partial t} + o(s)\,, \quad & \frac{\partial h_x}{\partial x} &= 1 + s\frac{\partial X}{\partial x} + o(s)\,,
\end{aligned}
$$

then

$$
\begin{aligned}
\dot{\bar{x}} &= \frac{s\frac{\partial X}{\partial t} + \left(1 + s\frac{\partial X}{\partial x}\right)\dot{x} + o(s)}{1 + s\frac{\partial T}{\partial t} + s\frac{\partial T}{\partial x}\dot{x} + o(s)} = \frac{\dot{x} + sX' + o(s)}{1 + sT' + o(s)} \\
&= \dot{x} + (X' - \dot{x}T')\,s + o(s) \\
&= \dot{x} + Vs + o(s)\,,
\end{aligned}
\tag{5.4}
$$

with $V(t,x,\dot{x}) = X'(t,x) - \dot{x}T'(t,x)$ the generator associated with the derivative.

The integral functional $J[\cdot]$ of the fundamental problem of the calculus of variations (5.1),

$$
J\,[x] = \int_a^b L\,(t, x(t), \dot{x}(t))\,dt\,,
$$

is said to be *invariant* under a local Lie group h^s if, and only if,

$$
\begin{aligned}
\int_{t_1}^{t_2} L\left(t, x(t), \frac{dx}{dt}(t)\right) dt + o(s) &= \int_{\bar{t}_1}^{\bar{t}_2} L\left(\bar{t}, \bar{x}\,(\bar{t})\,, \frac{d\bar{x}}{d\bar{t}}\,(\bar{t})\right) d\bar{t} \\
&= \int_{t_1}^{t_2} L\left(h_t^s, h_x^s, \frac{\frac{\partial h_x^s}{\partial t} + \dot{x}\frac{\partial h_x^s}{\partial x}}{\frac{\partial h_t^s}{\partial t} + \dot{x}\frac{\partial h_t^s}{\partial x}}\right) \frac{dh_t^s}{dt} dt\,,
\end{aligned}
\tag{5.5}
$$

where $\bar{t}_1 = h_t(t_1, x(t_1), s)$, $\bar{t}_2 = h_t(t_2, x(t_2), s)$, $h_t^s = h_t(t, x(t), s)$, $h_x^s = h_x(t, x(t), s)$, and (5.5) is verified for all $|s| < \varepsilon$, for all $x \in C^2([a,b]; E)$, and for all $[t_1, t_2] \subseteq [a,b]$.

Condition (5.5) is equivalent to

$$\text{(5.6)} \quad \left.\frac{d}{ds}\int_{\bar{t}_1}^{\bar{t}_2} L\left(\bar{t},\bar{x}\left(\bar{t}\right),\frac{d\bar{x}}{d\bar{t}}\left(\bar{t}\right)\right)d\bar{t}\right|_{s=0} = 0$$

$$\Leftrightarrow \left.\frac{d}{ds}\int_{t_1}^{t_2} L\left(h_t^s,h_x^s,\frac{\frac{\partial h_x^s}{\partial t}+\dot{x}\frac{\partial h_x^s}{\partial x}}{\frac{\partial h_t^s}{\partial t}+\dot{x}\frac{\partial h_t^s}{\partial x}}\right)\frac{dh_t^s}{dt}dt\right|_{s=0} = 0\,.$$

The requirement that (5.6) hold for every subinterval $[t_1,t_2]$ of $[a,b]$ permits to remove the integral from consideration, and to put the focus on the Lagrangian $L(\cdot,\cdot,\cdot)$.

Definition 5.1 (Invariance/symmetry). The fundamental problem of the calculus of variations on locally convex spaces (5.1) is said to be *invariant* under the local Lie group h^s if, and only if,

$$\text{(5.7)} \quad \left.\frac{d}{ds}\left\{L\left(h_t^s,h_x^s,\frac{\frac{\partial h_x^s}{\partial t}+\dot{x}\frac{\partial h_x^s}{\partial x}}{\frac{\partial h_t^s}{\partial t}+\dot{x}\frac{\partial h_t^s}{\partial x}}\right)\frac{dh_t^s}{dt}\right\}\right|_{s=0} = 0\,.$$

We then say that h^s is a *symmetry* for the problem.

Theorem 5.1 (Necessary and sufficient condition of invariance). *The fundamental problem of the calculus of variations (5.1) is invariant under a local Lie group (5.2) with generators T and X if, and only if,*

$$\text{(5.8)} \quad (L^t_{(x(t),\dot{x}(t))})'(t)T(t,x(t)) + (L^x_{(t,\dot{x}(t))})'(x(t))X(t,x(t)) \\ + (L^v_{(t,x(t))})'(\dot{x}(t))\big(X'(t,x(t)) - \dot{x}(t)T'(t,x(t))\big) + L\circ\mu_x(t)T'(t,x(t)) = 0\,.$$

Proof. Carrying out the differentiation of equation (5.7) we obtain:

$$\text{(5.9)} \quad L\left.\frac{d}{ds}\left(\frac{d\bar{t}}{dt}\right)\right|_{s=0} + \left.\frac{d}{ds}L\left(\bar{t},\bar{x},\frac{d\bar{x}}{d\bar{t}}\right)\right|_{s=0} = 0\,.$$

Recalling (4.2), and having in mind that by (5.3) and (5.4)

$$\left.\frac{d}{ds}\left(\frac{d\bar{t}}{dt}\right)\right|_{s=0} = \left.\frac{d}{ds}\left(\frac{d}{dt}\left(t+sT+o(s)\right)\right)\right|_{s=0} = T'\,,$$
$$\left.\frac{d}{ds}L\left(\bar{t},\bar{x},\frac{d\bar{x}}{d\bar{t}}\right)\right|_{s=0} = \frac{\partial L}{\partial t}T + \frac{\partial L}{\partial x}X + \frac{\partial L}{\partial v}\left(X' - \dot{x}T'\right)\,,$$

we obtain from (5.9) the intended conclusion. □

Theorem 5.2 (Noether's Theorem on Locally Convex Spaces). *If*

$$J[x] = \int_a^b L\left(t,x(t),\dot{x}(t)\right)dt$$

is invariant under a local Lie group h^s with generators T and X, then

$$\text{(5.10)} \quad \Big[L\circ\mu_x(t) - \dot{x}(t)(L^v_{(t,x(t))})'(\dot{x}(t))\Big]\,T(t,x(t)) \\ + (L^v_{(t,x(t))})'(\dot{x}(t))X(t,x(t)) = \mathit{constant}$$

$\forall t\in[a,b]$, and along all the solutions x of the Euler-Lagrange equation (4.3).

Remark 5.1. If we introduce the Hamiltonian function $H(\cdot,\cdot,\cdot)$ by

$$H(t,x,\dot{x}) = -L(t,x,\dot{x}) + \dot{x}\frac{\partial L}{\partial v}(t,x,\dot{x}) \,,$$

with $\frac{\partial L}{\partial v}$ as in (4.2), then the conservation law (5.10) can be written in the form

$$\left[\frac{\partial L}{\partial v}\circ\mu_x(t)\right] X(t,x(t)) - [H\circ\mu_x(t)]\, T(t,x(t)) = \text{constant}\,.$$

Proof. Direct calculations show that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}X\right) - X\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{d}{dt}\frac{\partial L}{\partial v}X + \frac{\partial L}{\partial v}X' - X\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial v}X'\,, \tag{5.11}$$

$$\begin{aligned}\frac{dL}{dt}T - \frac{\partial L}{\partial x}\dot{x}T - \frac{\partial L}{\partial v}\ddot{x}T &= \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial v}\ddot{x}\right)T - \frac{\partial L}{\partial x}\dot{x}T - \frac{\partial L}{\partial v}\ddot{x}T \\ &= \frac{\partial L}{\partial t}T\,,\end{aligned} \tag{5.12}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial v}\dot{x}T\right) - \dot{x}T\frac{d}{dt}\frac{\partial L}{\partial v} &= \frac{d}{dt}\frac{\partial L}{\partial v}\dot{x}T + \frac{\partial L}{\partial v}\frac{d}{dt}(\dot{x}T) - \dot{x}T\frac{d}{dt}\frac{\partial L}{\partial v} \\ &= \frac{\partial L}{\partial v}(\ddot{x}T + \dot{x}T')\,.\end{aligned} \tag{5.13}$$

Substituting (5.12) in the necessary and sufficient condition of invariance (5.8) we obtain that

$$\frac{dL}{dt}T - \frac{\partial L}{\partial x}\dot{x}T - \frac{\partial L}{\partial v}\ddot{x}T + \frac{\partial L}{\partial x}X + \frac{\partial L}{\partial v}X' - \frac{\partial L}{\partial v}(\dot{x}T') + LT' = 0\,; \tag{5.14}$$

substituting (5.11) in (5.14) we get

$$\begin{aligned}\frac{dL}{dt}T - \frac{\partial L}{\partial x}\dot{x}T - \frac{\partial L}{\partial v}\ddot{x}T + \frac{\partial L}{\partial x}X + \frac{d}{dt}\left(\frac{\partial L}{\partial v}X\right)& \\ - X\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial v}(\dot{x}T') + LT' = 0\,.&\end{aligned} \tag{5.15}$$

Finally, using (5.13) in the last equation (5.15) one obtains

$$\begin{aligned}\frac{dL}{dt}T - \frac{\partial L}{\partial x}\dot{x}T - \frac{\partial L}{\partial v}\ddot{x}T + \frac{\partial L}{\partial x}X + \frac{d}{dt}\left(\frac{\partial L}{\partial v}X\right)& \\ - X\frac{d}{dt}\frac{\partial L}{\partial v} + \frac{\partial L}{\partial v}\ddot{x}T - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\dot{x}T\right) + \dot{x}T\frac{d}{dt}\frac{\partial L}{\partial v} + LT' = 0\,,&\end{aligned}$$

which, by simplification, takes form

$$\begin{aligned}\frac{dL}{dt}T + \frac{\partial L}{\partial x}(X - \dot{x}T) + \frac{d}{dt}\left(\frac{\partial L}{\partial v}X\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\dot{x}T\right) - \frac{d}{dt}\frac{\partial L}{\partial v}(X - \dot{x}T) + LT' = 0& \\ \Leftrightarrow \frac{d}{dt}\left(LT + \frac{\partial L}{\partial v}X - \frac{\partial L}{\partial v}\dot{x}T\right) + (X - \dot{x}T)\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v}\right) = 0\,.&\end{aligned}$$

By definition, along the solutions of the Euler-Lagrange equations (4.3) $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v} = 0$, and one gets the desired conclusion:

$$\frac{d}{dt}\left(LT + \frac{\partial L}{\partial v}X - \frac{\partial L}{\partial v}\dot{x}T\right) = 0\,.$$

□

The theory of the calculus of variations on locally convex spaces is under development. Much remains to be done, in particular, in the vectorial setting. Even for the scalar case, it would be interesting, for example, to have a version of the DuBois-Reymond necessary condition on locally convex spaces. With such condition one can try to prove Theorem 5.2 for more general classes of admissible functions, following the scheme in [19].

Acknowledgments

The authors acknowledge the financial support of the Research Unit CEOC and the FCT/FEDER project POCI/MAT/55524/2004.

References

[1] N. Bourbaki, *Topological Vector Spaces*, Springer Verlag, 1981.

[2] B. Brunt, *The Calculus of Variations*, Springer Verlag, 2004.

[3] R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Academic Press, 1977.

[4] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer Verlag, 1989.

[5] R. V. Gamkrelidze and G. L. Kharatishvili, *Extremal problems in linear topological spaces I*, Math. Systems Theory 1 (1967), 229–256.

[6] R. V. Gamkrelidze and G. L. Kharatishvili, *Extremal problems in linear topological spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969), 781–839.

[7] A. Gugushvili, O. Khutsishvili, V. Sesadze, G. Dalakishvili, N. Mchedlishvili, T. Khutsishvili, V. Kekenadze and D. F. M. Torres, *Symmetries and Conservation Laws in Optimal Control Systems*, Georgian Technical University, 2003.

[8] H. Halkin, *Topological aspects of optimal control of dynamical polysystems*, Contributions to Differential Equations 3 (1964), 377–385.

[9] H. Jarchow, *Locally Convex Spaces*, Teubner, 1981.

[10] A. Kriegl and P. W. Michor, *The Convenient Setting of Global Analysis*, Amer. Math. Soc., 1997.

[11] J. D. Logan, *Applied Mathematics: a Contemporary Approach*, John Wiley & Sons, 1987.

[12] E. Noether, *Invariant variation problems*, Transport Th. Stat. Phys. 1 (1971), 186–207. [Translated from the German (Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1918), 235–257)].

[13] H. Omori, *On Banach Lie gorups acting on finite dimensional manifolds*, Tôhoku Math. J. 30 (1978), 223–250.

[14] I. V. Orlov, *Normal indices of linear and nonlinear mappings in locally convex spaces and space scales*, Spectral and Evolution Problems 11 (2001), 18–29.

[15] I. V. Orlov, *Canonical isomorphisms of linear and bilinear operators in locally convex spaces*, Dop. NAN Ukr. 17 (2001), 196–205.

[16] I. V. Orlov, *Normal differentiability and extrema of functionals in a locally convex space*, Cybernet. Systems Anal. 38 (2002), 493–502.

[17] E. M. Rocha, *On computation of the logarithm of the Chen-Fliess series for nonlinear systems.* In: A. Zinober et al. (ed.), Nonlinear and Adaptive Control, Lect. Notes Control Inf. Sci. 281 (2003), 317–326.

[18] I. V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Amer. Math. Soc., 1994.

[19] D. F. M. Torres, *Proper extensions of Noether's symmetry theorem for nonsmooth extremals of the calculus of variations*, Commun. Pure Appl. Anal. 3 (2004), 491–500.

[20] H. Triebel, *Theory of Function Spaces*, Birkhäuser Verlag, 1983.

[21] L. R. Volevich and S. G. Gindikin, *Generalized Functions and Equations in Convolution*, Nauka, 1994.

Authors' address:

Eugénio A. M. Rocha, Delfim F. M. Torres
Department of Mathematics, University of Aveiro,
3810-193 Aveiro, Portugal.
E-mail addresses: eugenio@ua.pt, delfim@ua.pt