A note on proper conformal vector fields in Bianchi type I space-times

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Abstract. Direct integration technique is used to study the proper conformal vector fields in non conformally flat Bianchi type-1 space-times. Using the above mentioned technique we have shown that a very special class of the above space-time admits proper conformal vector fields.

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1 Introduction

This paper investigates the existance of proper conformal vector fields in Bianchi type-1 space-times by using the direct integration technique. The conformal vector field which preserves the metric structure upto a conformal factor carries significant interest in Einstein's theory of general relativity. It is therefore important to study these symmetries.

Through out M is represents a four dimensional, connected, hausdorff space-time manifold with Lorentz metric g of signature $(-,+,+,+)$. The curvature tensor associated with g_{ab} through Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$ and the Ricci tensor components are $R_{ab} = R^c{}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively.

Any vector field X on M can be decomposed as

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab}, \tag{1.1}$$

where $h_{ab}(= h_{ba}) = L_X g_{ab}$ and $F_{ab} = -F_{ba}$ are symmetric and skew symmetric tensors on M, respectively. Such a vector field X is called conformal vector field if the local diffeomorphisms ψ_t (for appropriate t) associated with X preserve the metric structure up to a conformal factor i.e. $\psi_t^* g = \phi g$, where ϕ is a nowhere zero positive function on M and ψ_t^* is a pullback map on M [3]. This is equivalent to the condition that

$$h_{ab} = 2\phi g_{ab},$$

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equivalently

$$g_{ab,c}X^c + g_{ac}X^c_{,b} + g_{bc}X^c_{,a} = 2\phi g_{ab}, \tag{1.2}$$

where $\phi : M \to R$ is the smooth conformal function on M, then X is a called conformal vector field. If ϕ is constant on M, X is homothetic (proper homothetic if $\phi \neq 0$) while $\phi = 0$ it is Killing [3]. If the vector field X is not homothetic then it is called proper conformal. It follows from [3] that for a conformal vector field X, the bivector F and the function ϕ satisfy (putting $\phi_a = \phi_{,a}$)

$$F_{ab;c} = R_{abcd}X^d - 2\phi_{[a}g_{b]c}, \tag{1.3}$$

$$\phi_{a;b} = -\frac{1}{2}L_{ab;c}X^c - \phi L + R_{c(a}F^c_{b)}, \tag{1.4}$$

where $L_{ab} = R_{ab} - \frac{1}{6}Rg_{ab}$.

2 Main results

Consider a Bianchi type-1 space-time in the usual coordinate system (t, x, y, z) with line element [1]

$$ds^2 = -dt^2 + h(t)dx^2 + k(t)dy^2 + f(t)dz^2, \tag{2.1}$$

where f, k and h are some nowhere zero functions of t only. The possible Segre type of the above space-time is $\{1, 111\}$ or one of its degeneracies. It follows from [2, 4] the above space-time admits three linearly independent Killing vector fields which are

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}. \tag{2.2}$$

A vector field X is said to be a conformal vector field if it satisfy equation (1.2). One can write (1.2) explicitly using (2.1) we have

$$X^0_{,0} = \phi, \tag{2.3}$$

$$-X^0_{,1} + hX^1_{,0} = 0, \tag{2.4}$$

$$-X^0_{,2} + kX^2_{,0} = 0, \tag{2.5}$$

$$-X^0_{,3} + fX^3_{,0} = 0, \tag{2.6}$$

$$\dot{h}X^0 + 2hX^1_{,1} = 2h\phi, \tag{2.7}$$

$$hX^1_{,2} + kX^2_{,1} = 0, \tag{2.8}$$

$$hX^1_{,3} + fX^3_{,1} = 0, \tag{2.9}$$

$$\dot{k}X^0 + 2kX^2_{,2} = 2k\phi, \tag{2.10}$$

$$kX^2_{,3} + fX^3_{,2} = 0, \tag{2.11}$$

$$\dot{f}X^0 + 2fX^2_{,2} = 2f\phi. \tag{2.12}$$

Equations (2.3), (2.4), (2.5) and (2.6) give

$$\begin{aligned} X^0 &= \int \phi(t)dt + A^1(x,y,z), \\ X^1 &= A^1_x(x,y,z)\int \frac{dt}{h} + A^2(x,y,z), \\ X^2 &= A^1_y(x,y,z)\int \frac{dt}{k} + A^3(x,y,z), \\ X^3 &= A^1_z(x,y,z)\int \frac{dt}{f} + A^4(x,y,z), \end{aligned} \tag{2.13}$$

where $A^1(x,y,z)$, $A^2(x,y,z)$, $A^3(x,y,z)$ and $A^4(x,y,z)$ are functions of integration. In order to determine $A^1(x,y,z)$, $A^2(x,y,z)$, $A^3(x,y,z)$ and $A^4(x,y,z)$ we need to integrate the remaining six equations. To avoid details, here we will present only the result when the above space-time (2.1) admits proper conformal vector field. It follows from the above calculations; there exist only one possibility when the above space-time (2.1) admits proper conformal vector field which is:
Case 1: Four conformal vector fields:

In this case the space-time (2.1) becomes

$$ds^2 = -dt^2 + V^2(t)(e^{-2d_1N(t)}dx^2 + e^{-2d_{11}N(t)}dy^2 + e^{-2d_{13}N(t)}dz^2) \tag{2.14}$$

and conformal vector field is

$$X^0 = V(t), X^1 = d_1x + d_2, X^2 = d_{11}y + d_{12}, X^3 = d_{13}z + d_{14}, \tag{2.15}$$

where $V(t) = \int \phi(t)dt + d_8$, $N(t) = \int \frac{dt}{V(t)}$, $d_1, d_2, d_8, d_{11}, d_{12}, d_{13}, d_{14} \in R(d_1 \neq d_{11}, d_1 \neq d_{13}, d_{13} \neq d_{11}, d_1 \neq 0, d_{11} \neq 0, d_{13} \neq 0)$ and ϕ is no where zero function of t only. The above space-time (2.14)admits four independent conformal vector fields in which three are Killing vector fields which are given in (2.2) and one is proper conformal vector field which is

$$Z = (V(t), d_1x, d_{11}y, d_{13}z). \tag{2.16}$$

One can easily check that the above vector field (2.16) is not a homothetic vector field by substituting it into the homothetic equations.

Now consider the case when $d_{11} = d_{13}, d_{11} \neq d_1$ and the above space-time (2.14) becomes

$$ds^2 = -dt^2 + V^2(t)(e^{-2d_1N(t)}dx^2 + e^{-2d_{11}N(t)}(dy^2 + dz^2)). \tag{2.17}$$

The above space-time admits five independent conformal vector fields in which four independent Killing vector fields which are: $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ and $z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ and one proper conformal vector field which is given in (2.16). The cases when $d_{11} = d_1, d_{11} \neq d_{13}$ and $d_1 = d_{13}, d_{11} \neq d_1$ are exactly the same.

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