On Banach's fixed point theorem and formal balls

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Abstract. In this paper we present an alternative order-theoretic proof of the Banach fixed point theorem for selfmaps on complete metric spaces which is based on formal balls and, contrary to the case of Edalat and Heckmann [Theoret. Comput. Sci. 193 (1998), 53-73], does not derive from the well-known dcpo fixed point theorem.

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1 Introduction

In [2], Edalat and Heckmann established new connections between the theory of metric spaces and domain theory. In particular they introduced the ordered set of formal balls as a computational model for metric spaces. It was proved, among other results, that every metric space is complete if and only if the ordered set of formal balls is directed complete, and that it is separable if and only if the associated set of formal balls is a ω-continuous ordered set. Moreover, it was showed that every metric space is approximated by its set of formal balls, in the sense that every metric space is homeomorphic to the set of maximal formal balls endowed with the Scott topology. Furthermore, as an application of the theory presented in [2], Edalat and Heckmann gave a domain-theoretic proof of Banach's fixed point theorem based on the so-called dcpo fixed point theorem (Theorem 1 below).

Our main goal in this paper is to give a novel order-theoretic proof of the classical Banach's fixed point theorem which differs from that given by Edalat and Heckmann in that it is not based on the dcpo fixed point theorem. This is done employing only the dcpo completeness of the set of formal balls.

2 Preliminaries

From now on the letters $\mathbb{R}^+$ and ω will denote the set of nonnegative real numbers and the set of nonnegative integers numbers, respectively.

We briefly introduce very basics of domain theory. For a more extensive treatment of the subject we refer the reder to [1].

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An *ordered set* is a pair $(X, \leq)$ where X is a nonempty set and $\leq$ is a binary relation on X satisfying for all $x, y, z \in X$:

(i) $x \leq x$	(reflexivity)
(ii) $x \leq y$ and $y \leq x \Rightarrow x = y$	(antisymmetry)
(iii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$	(transitivity)

In the sequel, given $x, y \in X$, we will write $x < y$ if and only if $x \leq y$ and $x \neq y$.

An element $x \in X$ is *maximal* if $x \leq y$ implies $y = x$. Let $Y \subseteq X$. A *least* element of Y is an element $z \in Y$ such that $z \leq y$ for all $y \in Y$. A *bottom* element is a least element of X. An *upper bound* of Y is an element $x \in X$ such that $y \leq x$ for all $y \in Y$. The *supremum* of Y is the least of the set of all its upper bounds provided it exists. We will write $\sqcup Y$ to denote the supremum of Y when it exists. We will say that Y is *directed* if for every pair $x, y \in Y$ there exists $z \in Y$ such that $x \leq z$ and $y \leq z$.

An ordered set $(X, \leq)$ in which every directed subset has a supremum will be called a *directed-complete partial ordered set* (dcpo for short).

As usual, by an ascending sequence in an ordered set $(X, \leq)$ we mean a sequence $(x_n)_{n\in\omega}$ in X such that $x_n \leq x_{n+1}$for all $n \in \omega$.

Let $(X, \leq)$ and $(Y, \preceq)$ be two ordered sets. A mapping $\varphi : X \to Y$ is said to be *monotone* if $\varphi(x) \preceq \varphi(y)$ whenever $x \leq y$. As usual, by a continuous mapping φ between dcpo's $(X, \leq)$ and $(Y, \preceq)$ we mean a monotone mapping such that, $\varphi(\sqcup A) = \sqcup\varphi(A)$ for each directed subset A of X. In case of the mapping φ is a selfmap on X, then we will denote by $fix(\varphi)$ the set of fixed points of φ, i.e. $fix(\varphi) = \{x \in X : \varphi(x) = x\}$.

The next result, known as the *dcpo fixed point theorem*, plays a central role in domain theory.

Theorem 1. *Let $(X, \leq)$ be a dcpo with bottom element. Then every continuous mapping $f : X \to X$ has a least fixed point. Furthermore, it is given by $\sqcup_{n\in\omega} f^n(\perp)$, where $\perp$ is the bottom of X.*

3 Banach's fixed point theorem: A new order-theoretic proof

Let (X, d) be a metric space. A formal ball in the metric space (X, d) is a pair (x, r) with $x \in X$ and $r \in \mathbb{R}^+$. The set of formal balls of X is denoted by BX. An order can be defined on BX in the following way:

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s.$$

Note that $(x, r) \sqsubseteq (y, s)$ clearly forces $s \leq r$.

The maximal elements of BX are the formal balls $(x, 0)$.

Edalat and Heckmann proved the following useful equivalences which play a central role in our below discussion.

Theorem 2. *A metric space (X, d) is complete if and only if BX is a dcpo.*

Theorem 3. *For an ascending sequence $(x_n, r_n)_{n\in\omega}$ in BX and and element $(y, s) \in BX$, the following are equivalent:*

(i) (y, s) *is the least upper bound of* $(x_n, r_n)_{n\in\omega}$.

(ii) (y, s) *is an upper bound of* $(x_n, r_n)_{n\in\omega}$ *and* $lim_{n\to\infty} r_n = s$.

(iii) $lim_{n\to\infty} x_n = y$ *and* $lim_{n\to\infty} r_n = s$.

Next we give an alternative order-theoretic proof of the classical Banach's fixed point theorem which uses the dcpo completeness of the set of formal balls instead of the dcpo fixed point theorem (Theorem 1 above).

Theorem 4. *Let* f *be a mapping from a complete metric spaces* (X, d) *into itself such that there exists a real number* c *with* $0 \leq c < 1$, *satisfying*

$$d(f(x), f(y)) \leq cd(x, y) \tag{3.1}$$

for all $x, y \in X$. *Then* f *has a unique fixed point which is the limit of the orbit of any point in the space.*

Proof. Let $B_{f,c} : BX \to BX$ given by $B_{f,c}(x, r) = (f(x), cr)$. Then the monotonicity of $B_{f,c}$ follows easily from (3.1). Moreover, we note that $(z, t) \in BX$ is a fixed point of $B_{f,c}$ if and only if $z \in X$ is a fixed point of f and $t = 0$. Furthermore, it is obvious that if $B_{f,c}$ has a fixed point then, by (3.1), it is unique. Consequently, if f has a fixed point then this is also unique.

Let $x \in X$. Put $R_x = \frac{1}{1-c} d(x, f(x))$ and take $r \geq R_x$. Then $(x, r) \sqsubseteq B_{f,c}((x, r))$ and $B^n_{f,c}(x, r) \sqsubseteq B^{n+1}_{f,c}(x, r)$ for all $n \in \omega$. Applying Theorem 2 we obtain that the ascending sequence $(B^n_{f,c}(x, r))_{n\in\omega}$ has a least upper bound (y, s). Hence $B^n_{f,c}(x, r) \sqsubseteq (y, s) \sqsubseteq B_{f,c}((y, s))$ for all $n \in \omega$. It follows, by Theorem 3, that $\lim_{n\to\infty} c^n r = s$ and $\lim_{n\to\infty} f^n(x) = y$. So $s = 0$. Since $(y, s) \sqsubseteq B_{f,c}((y, s))$ we have that $(y, 0) \sqsubseteq (f(y), 0)$. Thus we conclude that $f(y) = y$ because $(y, 0)$ is a maximal element of BX. Therefore y is the unique fixed point of f. The proof is complete. ■

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References

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