

## Resonant Forcing of Barotropic Coastally Trapped Waves

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### ABSTRACT

The interaction of a longshore current with a longshore topographic feature is investigated in the barotropic case. It is shown that near resonance, when a long-wave speed is close to zero (in a fixed reference frame), there is enhanced generation of upstream and downstream coastally trapped waves. An evolution equation of the KdV-type is derived to describe the resonant behavior, and numerical solutions are discussed for a range of parameters describing the forcing terms, the detuning term and dissipation. The analogous situation of resonant generation due to wind stress is developed in an appendix.

### 1. Introduction

It is now widely recognized that coastally trapped waves are an important feature of the low-frequency motion on the continental shelf and slope, and there have been numerous theoretical and observational studies of the properties of these waves, their generation mechanisms, the scattering by longshore topographic features, and their decay due to frictional processes. However, relatively little attention has been given to the enhanced generation that would occur when the forcing mechanisms are resonant or near-resonant. Examples of such resonant forcing occur when the longshore wind stress has a component whose phase speed is close to that of a free wave speed, or when a longshore current encounters a longshore topographic feature with the flow speed close to critical (i.e., the speed of a free wave is close to zero in the fixed reference frame).

The purpose of this paper is to develop a theory for the resonant generation of barotropic coastally trapped waves for the case when the resonance occurs at the long-wave end of the spectrum. The main body of the theory is developed for the case when the forcing is due to the interaction of a longshore current with a longshore topographic feature; the case when the forcing is due to wind stress is similar and is developed in an appendix. We shall show that near resonance a forcing term of  $O(\alpha)$  will produce a response of  $O(\alpha^{1/2})$ , and that the dominant part of the response is a free long wave, whose amplitude is described by an evolution equation of the KdV-type (i.e., it contains a balance between time evolution, nonlinearities, wave dispersion, resonance detuning, dissipation and forcing). Some numerical solutions will be presented which will show the range of phenomena that can occur depending on the various parameters of the system. Analogous evolution equations of the KdV-type have been ob-

tained by Akylas (1984) and Cole (1985) for the resonant generation of water waves, by Grimshaw and Smyth (1986) for the resonant generation of internal waves, and by Patoine and Warn (1982) and Malanotte-Rizzoli (1984) for the resonant generation of Rossby waves.

When considering resonance due to the interaction of a longshore current with topography it is first necessary to determine the properties of the free waves which can exist in the coastal wave-guide in the presence of current shear. This topic, for the nondivergent barotropic case, has recently been reviewed by Collings and Grimshaw (1984), who demonstrated that both subcritical and supercritical wave modes may occur, thus indicating the possibility that there are near-critical, or near-resonant, wave modes (see also Collings and Grimshaw, 1980). Recently Hughes (1985a,b; 1986a,b,c) has developed a comprehensive theory of coastal hydraulics which has directly addressed the question of when is a current near a critical, or resonant, condition with respect to long wave modes. For the barotropic case Hughes (1985a) has demonstrated the existence of a critical current, adjoined in parameter space by two conjugate currents; the wider current was found to be subcritical, and the narrower current supercritical. Hughes (1985b, 1986a,c) has speculated that a number of current systems such as the California, Florida, Agulhas, East Australian, East Auckland currents and the East Cape Current off New Zealand may be subject to hydraulic control, or, in the terminology of this paper, candidates for resonant forcing. In a different context Whitehead et al. (1974), Gill (1977) and Pratt (1983, 1984) have considered the hydraulic theory of rotating-channel flows; this situation bears some resemblance to the hydraulics of coastal currents with Kelvin waves playing the role of the coastally trapped nondivergent waves that we will be concerned with.

In the remainder of this section we shall present the

barotropic equations of motion. We shall use nondimensional coordinates based on a horizontal length scale  $L$  (typical of the shelf width), a time scale  $|f|^{-1}$ , a velocity scale  $|f|L$ , and a vertical scale  $h_0$  (a typical depth). The sea-surface elevation is scaled by  $\mu^2 h_0$  where  $\mu^2 = f^2 L^2 (g h_0)^{-1}$  is the divergence parameter and is assumed to be small in the sequel. The nondimensional equations of motion are then

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k} \times \mathbf{u} + \nabla\zeta = \frac{\boldsymbol{\tau}}{H} \quad (1.1)$$

$$\mu^2 \frac{\partial \zeta}{\partial t} + \nabla \cdot (H\mathbf{u}) = 0, \quad (1.2)$$

where

$$H = h + \mu^2 \zeta. \quad (1.3)$$

Here  $\mathbf{u}$  is the fluid velocity,  $\zeta$  is the sea-surface elevation,  $h$  is the undisturbed depth,  $\boldsymbol{\tau}$  is a combination of wind stress and bottom stress,  $\mathbf{k}$  is a unit vector in the vertical direction,  $f$  is  $\pm 1$  according to whether the system is set in the Northern or Southern hemispheres, and  $d/dt$  is the conventional convective derivative. In the next section these equations will be expressed in curvilinear coordinates for which  $x = 0$  defines the coastline, and  $x \rightarrow \infty$  defines the deep ocean (see Fig. 1). The boundary condition at the coast is that

$$H\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } x = 0, \quad (1.4)$$

where  $\mathbf{n}$  is a unit vector orthogonal to the contours on which  $x$  is a constant. In the deep ocean we require that the waves be trapped and the appropriate condition is

$$H(\mathbf{u} - \mathbf{u}_0) \cdot \mathbf{n} = 0, \quad \text{as } x \rightarrow \infty. \quad (1.5)$$

Here  $\mathbf{u}_0$  is the basic longshore current and is not necessarily required to vanish in the deep ocean. The stress  $\boldsymbol{\tau}$  is given by

$$\boldsymbol{\tau} = \boldsymbol{\tau}_W - \boldsymbol{\tau}_B, \quad (1.6)$$

where

$$\boldsymbol{\tau}_B = r(\mathbf{u} - \mathbf{u}_0). \quad (1.7)$$

Here  $\boldsymbol{\tau}_W$  is the applied wind stress and  $\boldsymbol{\tau}_B$  is the bottom stress. In (1.7) we have assumed that the friction operates only on the waves, and that the basic longshore current  $\mathbf{u}_0$  is maintained by an appropriate body force. The friction coefficient is a function of depth, so that  $r = r(h)$ . An appropriate functional form could be  $r \propto h^{-1/2}$ , but we shall not need to specify the form of  $r(h)$  in the sequel.

In section 2 the equations of motion are formulated in the curvilinear coordinates, and the nonresonant interaction of a longshore current with a longshore topographic feature is discussed. Then in section 3 we consider the resonant case and derive the evolution equation. In section 4 numerical solutions of the evolution equation are described and discussed. The concluding section 5 summarizes our results. In appendix A we describe the general formulation of the equations

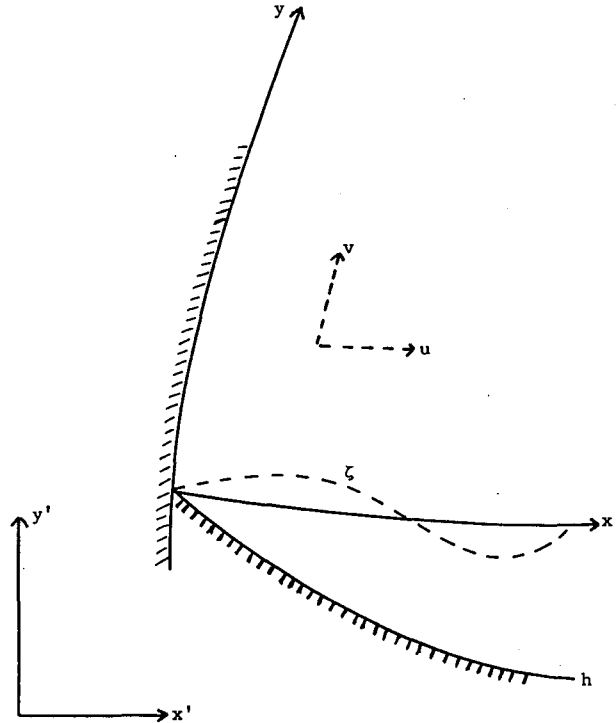


FIG. 1. The coordinate system.

in curvilinear coordinates, and in appendix B we develop the theory for resonant wind-stress forcing.

## 2. Nonresonant case

We shall consider here the generation of long coastally trapped waves by the interaction of a longshore current with longshore topographic variations. Thus, in the equation of motion (1.1) we set the wind stress  $\boldsymbol{\tau}_W = 0$  [see (1.6)]. Next let  $(x, y)$  be curvilinear coordinates such that the coastline is given by  $x = 0$ , and the deep ocean by  $x \rightarrow \infty$  (see Fig. 1). The equations of motion (1.1) and (1.2) in the  $(x, y)$ -coordinate system are described in appendix A. In the absence of any longshore topographic variations the depth  $h = h_0(x)$ , and  $u = 0$ ,  $v = v_0(x)$ ; here  $u, v$  are the velocity components orthogonal to the coordinate curves  $x, y = \text{constant}$  respectively. Here we shall assume that the depth  $h_0(x)$  is a monotonically increasing function of  $x$ , such that  $h_0(x) \rightarrow 1$  exponentially fast as  $x \rightarrow \infty$ ; at the coastline we shall assume that either  $h_0(0) = 0$ ,  $h_{0x}(0) \neq 0$  or that  $h_0(0) \neq 0$ . For the longshore current we shall assume that  $v_0(x) \rightarrow V_0$  exponentially fast as  $x \rightarrow \infty$ , where  $V_0$  is a constant. The case when  $V_0$  is nonzero is a simple model of a longshore current that is not coastally trapped (i.e., the offshore scale of the current may be larger than the width of the continental shelf).

We propose to study long waves on this basic flow. Hence we rescale the longshore coordinate  $y$  and the time  $t$  and introduce the new variables

$$Y = \epsilon y, \quad T = \epsilon t \quad (2.1)$$

where  $\epsilon$  is a small parameter. Consistently with this rescaling we replace  $u$  with  $\epsilon u$ , the divergence parameter  $\mu$  with  $\epsilon\mu$ , and the friction parameter  $r$  with  $\epsilon^2 r$ . The longshore topographic variations are described in two steps. First, we put

$$x' = x + \alpha g(x, Y), \quad y' = y \quad (2.2)$$

where  $(x', y')$  are conventional Cartesian coordinates (see Fig. 1). Here  $\alpha$  is a small parameter measuring the magnitude of the topographic variations;  $\epsilon^{-1}$  measures the longshore length scale of the variability. The coastline is given by  $x = 0$ , or  $x' = \alpha g(0, \epsilon y')$ . Next, we shall suppose that the depth  $h$  is given by

$$h = h_0(x) + \alpha k(x, Y). \quad (2.3)$$

We shall suppose that  $g(x, Y)$  and  $k(x, Y) \rightarrow 0$  exponentially fast as  $x \rightarrow \infty$ , and also that they tend to zero as  $|Y| \rightarrow \infty$ . The longshore topographic variations are thus confined locally in both  $x$  and  $Y$ ; essentially,  $g(x, Y)$  describes the coastline variability and  $k(x, Y)$  describes the depth variability. Note that a convenient choice for  $g(x, Y)$  is  $g(0, Y)\{1 - h_0(x)\}/\{1 - h_0(0)\}$ .

The equations of motion are (1.1) and (1.2), or, in terms of the  $(x, y)$ -coordinate system, (A6) and (A7). Using the definitions (2.1), (2.2) and (2.3) these become

$$\begin{aligned} -fv + \zeta_x - \alpha fg_x v &= O(\epsilon^2), \\ fu + \zeta_Y + v_T + uv_x + vv_Y + \alpha fg_x u + \frac{\epsilon r_0}{h_0}(v - v_0(x)) &= O(\epsilon^2), \\ \{(h_0 + \alpha k)(1 + \alpha g_x)u\}_x &+ \{(h_0 + \alpha k)(1 + \alpha g_x)v\}_Y = O(\epsilon^2). \end{aligned} \quad (2.4)$$

Here  $r_0 = r(h_0)$ . The omitted terms on the right-hand side have not been displayed as they are not needed in either the present nonresonant case or in the resonant case discussed in the next section. The boundary conditions at the coast ( $x = 0$ ), or in the deep ocean ( $x \rightarrow \infty$ ), are that  $h_0 u = O(\epsilon^2)$  [see (A9)]. Since the forcing terms are  $O(\alpha)$  we initially seek a solution in which the response is also  $O(\alpha)$ . Thus we put

$$\begin{aligned} u &= \alpha u_1(x, Y, T) + O(\alpha^2, \alpha\epsilon, \epsilon^2), \\ v - v_0(x) &= \alpha v_1(x, Y, T) + O(\alpha^2, \alpha\epsilon, \epsilon^2), \\ \zeta - \zeta_0(x) &= \alpha \zeta_1(x, Y, T) + O(\alpha^2, \alpha\epsilon, \epsilon^2), \end{aligned} \quad (2.5)$$

where  $\zeta_{0x} = fv_0$ . It follows that

$$\begin{aligned} -fv_1 + \zeta_{1x} &= fg_x v_0, \\ (f + v_{0x})u_1 + \zeta_{1Y} + v_{1T} + v_0 v_{1Y} &= 0, \\ (h_0 u_1)_x + (h_0 v_1)_Y &= -(k + h_{0g_x})_Y v_0. \end{aligned} \quad (2.6)$$

From the third equation in (2.6) we can introduce a streamfunction  $\psi_1$  such that

$$h_0 u_1 = \psi_{1Y}, \quad h_0 v_1 + (k + h_{0g_x})v_0 = -\psi_{1x}. \quad (2.7)$$

Eliminating  $\zeta_1$  from the first two equations in (2.6) we find that

$$-\left(\frac{\partial}{\partial T} + v_0 \frac{\partial}{\partial Y}\right)\left(\frac{\psi_{1x}}{h_0}\right)_x + P_{0x} \psi_{1Y} = F_{1Y}, \quad (2.8)$$

where

$$P_0 = (f + v_{0x})/h_0.$$

The forcing term in (2.8) is independent of  $T$  and given by

$$F_1 = \frac{fv_0 k}{h_0} + \left(\frac{v_0^2 k}{h_0}\right)_x + (v_0^2 g_x)_x. \quad (2.9)$$

Equation (2.8) can be recognized as the  $O(\alpha)$  vorticity equation [see (A8)]. The boundary condition at the coast is

$$\psi_1 = 0, \quad \text{at } x = 0. \quad (2.10)$$

In the deep ocean an outer expansion is needed in which the independent variables are  $X = \epsilon x$ ,  $Y$  and  $T$ , and the dependent variables are  $V = \epsilon^{-1}(v - V_0)$ ,  $u$  and  $Z = (\zeta - \zeta_0)$ . We shall not give details here (but see §3) as it follows immediately that the outer boundary condition for  $\psi_1$  is

$$\psi_{1x} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (2.11)$$

We shall seek a solution of (2.8) in terms of the long-wave modes. These are obtained by considering the homogeneous form of (2.8) and seeking solutions with the separable form  $A(Y - cT)\psi(x)$ . It is readily shown that

$$(c - v_0)\left(\frac{\psi_x}{h_0}\right)_x + P_{0x}\psi = 0, \quad \begin{cases} \psi = 0 & \text{at } x = 0 \\ \psi_x \rightarrow 0 & \text{as } x \rightarrow \infty. \end{cases} \quad (2.12)$$

In general this equation has solutions for both real values of the speed  $c$  representing stable waves, and complex values of  $c$  representing unstable waves (see Collins and Grimshaw, 1980, 1984). Here, we shall suppose for simplicity that there are no unstable waves; a sufficient condition for this to occur is that  $P_{0x}$  is one-signed. For the stable waves  $c < v_m$  if  $f > 0$ , and  $c > v_M$  if  $f < 0$  where  $v_m = \min v_0(x)$ ,  $v_M = \max v_0(x)$ ; in order for stable waves to exist it is necessary for  $fP_{0x} < 0$  somewhere. Note, in particular, that we are excluding the possibility of critical levels where  $c = v_0$ . It will be useful in the sequel to have available an orthogonality condition between modes with different speeds. For this purpose we put

$$\psi = (c - v_0)\phi \quad (2.13)$$

and then (2.12) becomes

$$\left( (c-v_0)^2 \frac{\phi_x}{h_0} \right)_x + f(c-v_0) \left( \frac{1}{h_0} \right)_x \phi = 0, \quad (2.14)$$

$$\begin{cases} \phi = 0 & \text{at } x = 0 \\ \phi_x \rightarrow 0 & \text{as } x \rightarrow \infty. \end{cases} \quad (2.14)$$

Then, if  $[\phi_r(x), c_r]$  and  $[\phi_s(x), c_s]$  are two distinct modes (i.e.,  $c_r \neq c_s$ ), it may be shown that

$$\int_0^\infty P_{0x} \phi_r \phi_s dx = 0, \quad r \neq s. \quad (2.15)$$

It can be shown that this condition is related to the conservation law for pseudomomentum (compare the general analysis by Held (1985) for shear flows). Next, if  $[\phi_r(x), c_r]$ ,  $r = 1 \cdots R$  denotes the set of all long-wave modes, we put

$$\psi_1 = \sum_r A_r(Y, T) \psi_r(x). \quad (2.16)$$

Here  $R$  may be infinite. It is not known, in general, if the set of long-wave modes is complete, but we note that if  $v_0$  is constant (i.e.,  $v_0 \equiv V_0$ ) then  $R$  is infinite and the modes are complete (e.g., see Gill and Schumann, 1974). Further, if  $fP_{0x} < 0$  everywhere it may also be shown that the modes are again complete. Even when the modes are not complete it may be shown by solving (2.8) with a Fourier-Laplace transform in  $Y, T$  respectively, that (2.16), together with Eq. (2.17) (see below) for  $A_r$ , provides the asymptotic solution for  $\psi_1$  as  $T \rightarrow \infty$ .

Substituting (2.16) into (2.8) and using the orthogonality condition (2.15) it may be shown that

$$\frac{\partial A_r}{\partial T} + c_r \frac{\partial A_r}{\partial Y} = \frac{\partial F_r}{\partial Y}(Y), \quad (2.17)$$

where

$$I_r F_r = \int_0^\infty F_1 \phi_r dx,$$

$$I_r = \int_0^\infty P_{0x} \phi_r^2 dx.$$

The initial condition for (2.17) is that  $A_r(Y, 0) = 0$ , and hence the solution is given by

$$c_r A_r = F_r(Y) - F_r(Y - c_r T). \quad (2.18)$$

Equation (2.17) is a forced first-order wave equation of a kind now familiar in the theory of coastally trapped waves since the work of Gill and Schumann (1974). The solution (2.18) describes a steady disturbance in the region of the forcing, together with a freely propagating wave with speed  $c_r$ . Here the forcing function  $F_r$  is given in terms of the functions  $g(x, Y)$  and  $k(x, Y)$ , representing coastline and depth variability respectively, by (2.9) and the second equation of (2.17).

When  $c_r = 0$ , (2.18) is replaced by

$$A_r = T \frac{\partial F_r}{\partial Y}(Y). \quad (2.19)$$

Thus the expansion (2.5) is secular for large times, and a different scaling is required. This is the resonant case and is discussed in the next section. We note that  $c_r = 0$  implies that  $f v_0(x) > 0$  for all  $x$  since we are assuming that there are no critical levels.

### 3. Resonant case, $c_n \approx 0$

In this section we shall suppose that the  $n$ th mode is resonant. It is well known that for resonant phenomena an  $O(\alpha)$  forcing produces an  $O(\alpha^{1/2})$  response, and that the appropriate time scale is  $O(\alpha^{-1/2})$ . Hence we introduce the long-time variable

$$\tau = \alpha^{1/2} T. \quad (3.1)$$

At leading order the solution is described by the free long-wave mode  $\alpha^{1/2} A(Y, \tau) \psi_n(x)$ , whose speed  $c_n$  is  $O(\alpha^{1/2})$ , and whose amplitude  $A$  is undetermined at this stage. Assuming the balance  $\alpha = \epsilon^2$  we put

$$\left. \begin{aligned} u &= \alpha^{1/2} A_Y \frac{\psi_n}{h_0} + \alpha u_1 + O(\alpha^{3/2}) \\ v - v_0(x) &= -\alpha^{1/2} A \frac{\psi_{nx}}{h_0} + \alpha v_1 + O(\alpha^{3/2}) \\ \zeta - \zeta_0(x) &= \alpha^{1/2} A \left\{ \frac{v_0 \psi_{nx}}{h_0} - P_0 \psi_n \right\} \\ &\quad + \alpha \zeta_1 + O(\alpha^{3/2}) \\ c_n &= \alpha^{1/2} \Delta \end{aligned} \right\} \quad (3.2)$$

Here  $\Delta$  is a detuning parameter. It is useful to note here that the offshore fluid particle displacement is given by  $-\alpha^{1/2} A \phi_n / h_0$  to leading order.

On substituting the expressions (3.2) into the equations of motion (2.4) we find that

$$-f v_1 + \zeta_{1x} = f g_x v_0 - \Delta A \left( \frac{\psi_{nx}}{h_0} \right)_x,$$

$$(f + v_{0x}) u_1 + \zeta_{1Y} + v_{1T} + v_0 v_{1Y}$$

$$= \left( A_r + \frac{r_0}{h_0} A \right) \frac{\psi_{nx}}{h_0} + A A_Y \left( \frac{\psi_n}{h_0} \left( \frac{\psi_{nx}}{h_0} \right)_x - \frac{\psi_{nx}^2}{h_0^2} \right)_x,$$

$$(h_0 u_1)_x + (h_0 v_1)_Y = -(k + h_0 g_x)_Y v_0. \quad (3.3)$$

The third equation of (3.3) allows us to introduce a streamfunction  $\psi_1$  [see (2.7)], and then elimination of  $\zeta_1$  from the first two equations gives the following equation:

$$-\left( \frac{\partial}{\partial T} + v_0 \frac{\partial}{\partial Y} \right) \left( \frac{\psi_{1x}}{h_0} \right)_x + P_{0x} \psi_{1Y} = F_{1Y} + N_{1x}, \quad (3.4)$$

where

$$N_1 = \left( A_\tau + \Delta A_Y + \frac{r_0}{h_0} A \right) \frac{\psi_{nx}}{h_0} + AA_Y \left( \frac{\psi_n}{h_0} \left( \frac{\psi_{nx}}{h_0} \right)_x - \frac{\psi_{nx}^2}{h_0^2} \right).$$

Here we recall that  $F_1$  is the forcing term defined by (2.9). The boundary condition at the coast is again (2.10). Equation (3.4) has a similar structure to (2.8). However, it is not now convenient to seek a solution on terms of long wave modes similar to (2.16) since  $\psi_1$  does not now satisfy the outer boundary condition (2.11). Instead  $\psi_{1x}(x \rightarrow \infty)$  must be determined from an outer expansion which is described below.

First, however, we seek the solution of (3.4) subject to the boundary condition (2.10). Since the right-hand side is independent of  $T$ , we construct a solution for which  $\psi_1$  is also independent of  $T$ . The left-hand side of (3.4) is then an ordinary differential equation in  $x$  for  $\psi_{1Y}$ . The solution which satisfies the boundary condition (2.10) is

$$\psi_{1Y} = A_{1Y} \psi_n + \psi_n \int_0^x \frac{B_1 \chi}{v_0} dx - \chi \int_0^x \frac{B_1 \psi_n}{v_0} dx, \quad (3.5)$$

where  $B_1 = F_{1Y} + N_{1x}$ . Here  $A_1$  is an undetermined function of  $Y$  and  $\tau$  and can be regarded as an  $O(\alpha^{1/2})$  correction to the amplitude  $A$ ;  $\psi_n(x)$ ,  $\chi(x)$  are two independent solutions of (2.12) with  $c = 0$  and are normalized so that their Wronskian is given by

$$\frac{1}{h_0} (\psi_n \chi_x - \psi_{nx} \chi) = 1. \quad (3.6)$$

Note that  $\psi_n$  satisfies the boundary conditions (2.10) and (2.11), but  $\chi$  does not by virtue of (3.6). If we let  $x \rightarrow \infty$  in (3.5) it may be shown that

$$\int_0^\infty \frac{B_1 \psi_n}{v_0} dx + \left[ \frac{\psi_n \psi_{1xY}}{h_0} \right]_{x \rightarrow \infty} = 0. \quad (3.7)$$

Equation (3.7) is the compatibility condition which must be satisfied in order that the solution for  $\psi_1$  contains no secular terms. It will eventually provide an evolution equation for the amplitude  $A$ . Substituting for  $B_1$  from (3.5), using (2.9) for  $F_1$  and (3.4) for  $N_1$  we find that (3.7) becomes

$$A_\tau + \Delta A_Y + \nu A + \sigma AA_Y + \left[ \frac{\psi_n \psi_{1xY}}{I_n h_0} \right]_{x \rightarrow \infty} = F_{nY},$$

where

$$I_n \sigma = - \int_0^\infty v_0^2 \phi_n^3 \left( \frac{P_{0x}}{h_0 v_0} \right)_x dx, \\ I_n \nu = \int_0^\infty \frac{r_0}{h_0} P_{0x} \phi_n^2 dx. \quad (3.8)$$

Here  $\psi_n = -v_0 \phi_n$  [see (2.13)] and  $I_n$  and the forcing function  $F_n$  are defined in Eq. (2.17).

To complete the analysis we must construct the outer expansion in the deep ocean where  $h_0 \rightarrow 1$  and  $v_0 \rightarrow$

$V_0$ . The independent variables in the outer expansion are  $X = \epsilon x$ ,  $Y$  and  $T$ , and the dependent variables are  $V = \epsilon^{-1}(v - v_0)$ ,  $u$  and  $Z = (\zeta - \zeta_0)$ . Since  $g$  and  $k \rightarrow 0$  in the deep ocean, the equations of motion (1.1) and (1.2) become

$$\left. \begin{aligned} \epsilon(u_T + V_0 u_Y) - fV + Z_X &= O(\epsilon^2) \\ \epsilon(V_T + V_0 V_Y) + fu + Z_Y &= O(\epsilon^2) \\ \mu^2 \epsilon(Z_T + V_0 Z_Y + fV_0 u) + u_X + V_Y &= O(\epsilon^2) \end{aligned} \right\}. \quad (3.9)$$

Here we recall that the divergence parameter  $\mu$  in (1.2) has been replaced with  $\mu\epsilon$ . We seek a solution of these equations of the form

$$\left. \begin{aligned} u &= \alpha^{1/2} U_1 + O(\alpha) \\ V &= \alpha^{1/2} V_1 + O(\alpha) \\ Z &= \alpha^{1/2} Z_1 + O(\alpha) \end{aligned} \right\}. \quad (3.10)$$

Since we are maintaining the balance  $\alpha = \epsilon^2$ , the expansion (3.10) describes a quasi-geostrophic solution in the deep ocean. This is given by

$$fU_1 = -Z_{1Y}, \quad fV_1 = Z_{1X},$$

where

$$\left( \frac{\partial}{\partial T} + V_0 \frac{\partial}{\partial Y} \right) (Z_{1XX} + Z_{1YY}) - \mu^2 f^2 Z_{1T} = 0. \quad (3.11)$$

The solution (3.10) as  $X \rightarrow 0$  must be matched with the solution (3.2) as  $x \rightarrow \infty$ . Hence it may be shown that

$$\left. \begin{aligned} Z_1(X \rightarrow 0) &= -fA\psi_n(x \rightarrow \infty) \\ Z_{1X}(X \rightarrow 0) &= -f\psi_{1x}(x \rightarrow \infty) \end{aligned} \right\}. \quad (3.12)$$

The matching conditions also confirm the boundary condition (2.11) for  $\psi_n$ . Since the first boundary condition in (3.12) for  $Z_1$  is independent of  $T$ , we seek a solution for  $Z_1$  which is likewise independent of  $T$ . This can be constructed using a Fourier transform in  $Y$ , and the deep ocean outer boundary condition that  $U_1 \rightarrow 0$  as  $X \rightarrow \infty$ . The result is

$$Z_1/f\psi_n(\infty) = -\frac{1}{2\pi} \int_{-\infty}^\infty \exp(i\eta Y - |\eta|X) \mathcal{F}(A) d\eta, \quad (3.13)$$

where  $\mathcal{F}(A) = \int_{-\infty}^\infty \exp(-i\eta Y) A(Y, \tau) dY$ . Here  $\mathcal{F}(A)$  is the Fourier transform of  $A$ . The second boundary condition in (3.12) now shows that

$$\psi_{1x}(x \rightarrow \infty) = \psi_n(\infty) \mathcal{B}(A), \quad (3.14)$$

where  $\mathcal{B}(A) = -(1/2\pi) \int_{-\infty}^\infty |\eta| \exp(i\eta Y) \mathcal{F}(A) d\eta$ . Here  $\mathcal{B}(A)$  is a pseudodifferential operator and is equivalent to the Hilbert transform which appears in the evolution equation describing internal solitary waves in a deep fluid (Benjamin, 1967, or Davis and Acrivos, 1967). Note the curious fact that the divergence parameter  $\mu$  plays no role in this solution.

We are now in a position to complete the derivation of the evolution equation by substituting (3.14) into (3.8), with the result that

$$A_\tau + \Delta A_Y + \nu A + \sigma A A_Y + \lambda \mathcal{B}(A_Y) = F_n Y, \quad (3.15)$$

where  $I_n \lambda = \psi_n^2(\infty)$ . The evolution equation (3.15) contains a balance between time evolution, resonance detuning, dissipation, nonlinearity, wave dispersion, and forcing. Analogous equations of the KdV-type in which the dispersive term  $\mathcal{B}(A_Y)$  in (3.15) is replaced by  $A_{YYY}$  have been obtained by Akylas (1984) and Cole (1985) for the resonant generation of water waves, by Grimshaw and Smyth (1986) for the resonant generation of internal waves, and by Patoine and Warn (1982) and Malanotte-Rizzoli (1984) for the resonant generation of Rossby waves. Equation (3.15) can be recognized as a forced version of the equation which in the absence of dissipation ( $\nu = 0$ ) describes solitary coastally trapped waves (for instance, see Smith, 1975, or Grimshaw, 1977). Indeed, the unforced, nondissipative equation is the deep-fluid internal solitary-wave equation derived by Benjamin (1967) and Davis and Acrivos (1967). Discussion of Eq. (3.15) is taken up in the next section.

It remains to determine an initial condition for (3.15). This is obtained by matching the expansion (3.2) as  $\tau \rightarrow 0$  with the expansion (2.5) as  $T \rightarrow \infty$ . The result is

$$\left. \begin{aligned} A(Y, 0) &= 0 \\ A_\tau(Y, 0) &= \frac{\partial F_n}{\partial Y} \end{aligned} \right\}. \quad (3.16)$$

The first condition in (3.16) is the required initial condition. The second condition is then automatically satisfied by solutions of (3.15) and agrees with the resonant solution (2.19) found in section 2.

#### 4. Numerical solutions for resonant case

In this section we shall discuss numerical solutions of the evolution equation (3.15). First let us rescale Eq. (3.15) by putting

$$\begin{aligned} \tau^* &= |\lambda| \tau, & Y^* &= (\text{sign } \lambda) Y, & A^* &= \frac{\sigma}{6\lambda} A, \\ F^* &= -\frac{\sigma}{6\lambda^2} F_n, & \Delta^* &= \frac{\Delta}{\lambda}, & \nu^* &= \frac{\nu}{|\lambda|}. \end{aligned} \quad (4.1)$$

The result is

$$\begin{aligned} A^* + \Delta^* A_{Y^*} + \nu^* A^* + 6A^* A_{Y^*} + \mathcal{B}(A_{Y^*}) + F^* Y^* &= 0, \\ A^*(Y^*, 0) &= 0. \end{aligned} \quad (4.2)$$

Henceforth we shall omit the asterisks in our discussion of (4.2). We consider only localized forcing functions  $F(Y)$ , such that  $F \rightarrow 0$  as  $Y \rightarrow \pm\infty$ . The forcing function will be characterized by two parameters  $F_0$  and  $\xi$  where we put

$$F(Y) = F_0 \hat{F}(\xi Y). \quad (4.3)$$

Here  $\hat{F} \geq 0$  for all  $Y$  and has a maximum value of 1 at  $Y = 0$  and  $\rightarrow 0$  as  $Y \rightarrow \pm\infty$ . Thus  $F_0$  is the maximum, or minimum, of  $F$  according as  $F_0$  is positive, or negative. The positive parameter  $\xi$  measures the length-scale of the forcing, i.e.,  $\xi^{-1}$  is the half-width of the forcing. Equation (4.2) was integrated numerically using a pseudospectral method analogous to that developed by Fornberg and Whitham (1978) for the Korteweg-de Vries equation. The results are not sensitive to the shape of  $\hat{F}$ , and all the results shown are for the forcing function

$$\hat{F}(\xi Y) = \text{sech}^2(\xi Y). \quad (4.4)$$

The results are shown in Figs. 2–7 and are discussed in detail below. For computational convenience, in the numerical solutions the forcing function was centered at  $Y = 85$ . In general the numerical solutions are similar to those obtained by Grimshaw and Smyth (1986) and Smyth (1986) for the forced Korteweg-de Vries equation [i.e., the dispersive term  $\mathcal{B}(A_Y)$  in (4.2) is replaced by  $A_{YYY}$ ]. We refer the reader to these papers for a comprehensive analytical description of the numerical solutions; a similar analytical description could also be given here but for brevity will be omitted. Note that to obtain the results of these papers, the transformation  $\Delta \rightarrow -\Delta$ ,  $Y \rightarrow -Y$  must be made, as well as replacing the dispersive term  $\mathcal{B}(A_Y)$  with  $A_{YYY}$  in (4.2).

One of the important distinctions which emerges from our numerical solutions is that between positive forcing ( $F_0 > 0$ ) and negative forcing ( $F_0 < 0$ ). Positive (negative) forcing corresponds to the situation when the forcing has the same (opposite) polarity to the solitary wave solutions of the unforced nondissipative equation [i.e., (4.2) with  $F \equiv 0$  and  $\nu = 0$ ]. Through the scaling (4.1) the criterion for positive or negative forcing can be correlated with the physical features of the coastal waveguide. Thus the forcing is positive or negative according as  $(\sigma I_n)(F_n I_n)$  is negative or positive. Here  $\sigma I_n$  is given by (3.8) and  $F_n I_n$  by (2.17). To estimate the sign of these two expressions we first observe that  $-A\phi_n(x)$  is proportional to the offshore fluid particle displacement. Assuming that  $\phi_n(x)$  is normalized so that  $-\phi_n > 0$  on average, it follows that when  $A$  is positive (negative) fluid particles are displaced offshore (onshore) on average. Note that this is likely to be a valid interpretation for the first mode, but becomes more difficult to interpret for the higher modes, and the reader is cautioned that the discussion which follows may not have any general validity, and in any particular situation it is best to compute (2.17) and (3.8) directly. Proceeding nevertheless, we shall assume that  $P_0(x)$  is dominated by the term  $f/h_0$  and that gradients of  $h_0$  dominate those of  $v_0$ . It then follows from (3.8) that  $\sigma I_n$  has the sign of  $(h_0^{-2})_{xx}$ ; thus we expect  $\sigma I_n$  to be positive (negative) when the bottom topography is predominantly convex (concave). In obtaining this result

we have used the fact that  $fv_0 > 0$ . With the same hypotheses  $F_1$  (2.9) has the same sign as  $k$  for topographic forcing (i.e.,  $k \neq 0, g = 0$ ), and the same sign as  $g_{xx}$  for coastline forcing (i.e.,  $k = 0, g \neq 0$ ). Also, we note that  $g_{xx}$  will be proportional to  $-g(0, Y)h_{0xx}$  if we make the canonical choice for  $g(x, Y)$ . In general, we expect the bottom curvature to be convex on average, and hence we expect  $\sigma I_n$  to be positive; it follows from (3.15) and (4.1) that  $A^*$  and  $A$  then have the same polarity. Further, for topographic forcing we expect  $F_0$  to have the same sign as  $k$ ; for instance, a submarine canyon produces  $F_0 > 0$ . For coastline forcing we expect  $F_0$  to have the same sign as  $-g(0, Y)$ ; for instance, a headland produces  $F_0 < 0$ .

The two other important parameters are  $\Delta$  and  $\nu$ . The  $\Delta$  measures the amount of resonance detuning [see (3.2)]. If  $\Delta > 0 (< 0)$  we shall say that the flow is subcritical (supercritical). Here the terminology derives from classical hydraulic theory. To demonstrate this we first recall the definitions of  $I_n$  (2.17) and  $\lambda$  (3.15), and use the fact that  $fP_{0x}$  is predominantly negative (i.e., only stable waves occur) to infer that  $\lambda$  has the opposite sign to  $f$ . Then, since  $fv_0 > 0$  (i.e., there are no critical layers), it follows that subcritical (supercritical) flow corresponds to long waves of phase speed  $c_n$  propagating against (with) the longshore current  $v_0$ . Thus the classification agrees with that expected from hydraulic theory (see also Hughes, 1985a, for a hydraulic theory of coastal currents). Consistent with this classification we shall refer to  $Y > 0 (< 0)$  as the upstream (downstream) region. The friction parameter  $\nu$  is defined in (3.8) and is always non-negative. In discussing our numerical results we shall consider both positive and negative forcing for a range of values of  $\Delta$ , i.e., for a subcritical case ( $\Delta > 0$ ), a resonant case ( $\Delta = 0$ ) and a supercritical case ( $\Delta < 0$ ). Each set of results will be for a fixed value of  $\nu$ , and three such sets will be considered; these will be for  $\nu = 0$  (nondissipative), for a small value of  $\nu$  (weakly dissipative) and for a moderately large value of  $\nu$  (strongly dissipative). The numerical solutions are not sensitive to the value

of the remaining parameter  $\xi$  and all our results are shown for a fixed value of  $\xi$ . Generally, increasing  $\xi$  increases the effects of dispersion vis-à-vis the effects of nonlinearity. Also the numerical results are not very sensitive to the value of  $|F_0|$ , and hence we shall only show results for a single representative value.

a. Positive forcing, nondissipative ( $F_0 > 0, \nu = 0$ )

The numerical results are shown in Figs. 2, 3 and 4. In Fig. 2 we show the resonant case  $\Delta = 0$ . In the forcing region the solution becomes locally steady as  $\tau \rightarrow \infty$ , with a downstream depression and a rise in level over the obstacle. The downstream depression is terminated by a modulated wavetrain, whose crests scale with  $Y/\tau$  as  $\tau \rightarrow \infty$ , thus suggesting they can be modeled by a similarity solution of the unforced, non-dissipative equation (4.2). Note that the unforced equation can be used upstream and downstream since the forcing is effectively confined to a distance  $\xi^{-1}$  on each side of the maximum point. On the upstream side of the forcing another modulated wavetrain is being generated, with each wave being closely approximated by the solitary-wave solutions of the unforced, non-dissipative equation (4.2). These are given by (see Benjamin 1967, and Davis and Acrivos, 1967)

$$A = a(1 + l^2(Y - v\tau)^2)^{-1} \tag{4.5}$$

where

$$v - \Delta = \frac{3a}{2} = l.$$

In general, the numerical solution is qualitatively similar to that obtained by Grimshaw and Smyth (1986) for the positive forcing of the Korteweg-de Vries equations, and much of the analysis developed by these authors has its counterpart here. Thus the amplitude and spacing of the solitary-like upstream waves can be approximately predicted by regarding these waves as a train of  $N$  solitary waves, with spacing  $h$ , where  $N \sim v\tau h^{-1}$ , and then applying the laws for conservation of mass and energy. The result is

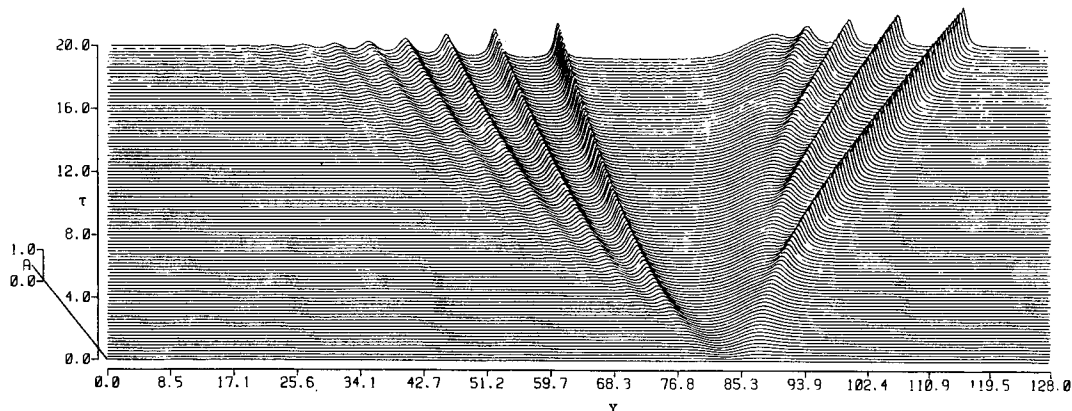


FIG. 2. The numerical solution for  $F_0 = 0.5, \nu = 0, \Delta = 0$  and  $\xi = 0.3$ .

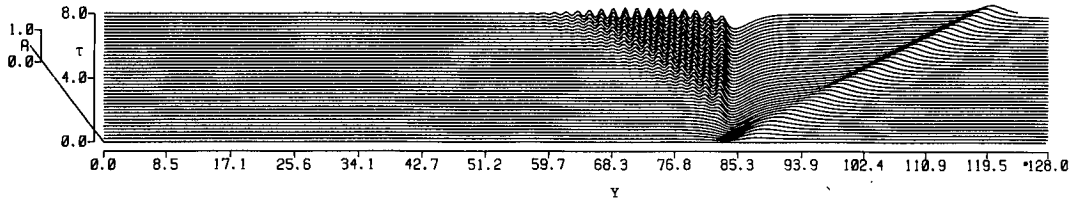


FIG. 3. The numerical solution for  $F_0 = 1$ ,  $\nu = 0$ ,  $\Delta = 4$  and  $\xi = 0.3$ .

$$\left. \begin{aligned} a &= \frac{2A_+(\Delta + 4A_+)}{(\Delta + 3A_+)} \\ h &= \frac{2\pi}{3} \left\{ \frac{(\Delta + 3A_+)^2 + 3A_+^2}{A_+(\Delta + 3A_+)^2} \right\} \end{aligned} \right\}, \quad (4.6)$$

where  $A_+$  is the mean level just upstream of the forcing region. Further if  $A_-$  is the level of the depression downstream of the forcing, then it is readily shown from the steady, nondissipative form of (4.2) that

$$\Delta + 3(A_+ + A_-) = 0. \quad (4.7)$$

It remains to find a suitable estimate for  $A_{\pm}$ . Grimshaw and Smyth (1986) show that for wide forcing regions (i.e.,  $\xi < 1$ ) hydraulic theory may be used to estimate  $A_{\pm}$ . This is obtained by omitting the dispersive term [i.e.,  $\mathcal{B}(A_Y)$  in (4.2)] and solving the resulting equation in the vicinity of the forcing region. The result is

$$6A_{\pm} = \pm(12F_0)^{1/2} - \Delta. \quad (4.8)$$

Substitution of (4.8) into (4.6) then gives the required approximate expressions for  $a$  and  $h$ , which are found to be in good agreement with the numerical results. Similar solutions to that shown in Fig. 2 are found when  $\Delta$  lies in the resonant band  $\Delta_- < \Delta < \Delta_+$ . Here  $\Delta_{\pm}$  are  $\geq 0$  and are determined in part by the requirement that  $A_{\pm} \geq 0$ . Using the approximate expression (4.8) for  $A_{\pm}$  valid for wide forcing regions we estimate that

$$\Delta_{\pm} = \pm(12F_0)^{1/2}. \quad (4.9)$$

For the forced Korteweg-de Vries equation Grimshaw and Smyth (1986) identified a parameter range  $\frac{1}{2}\Delta_+ < \Delta < \Delta_+$  in which there was a transition from the resonant behavior shown in Fig. 2 to the completely subcritical behavior shown in Fig. 3. The transition

zone features a weakening of the upstream waves and a corresponding intensification of the downstream waves. A similar transition zone occurs here but we shall not give any further details.

In Fig. 3 we show a subcritical case,  $\Delta > \Delta_+$ . The most prominent feature here is the downstream stationary lee-waves, the local depression in the forcing region, and the generation of a finite number of upstream solitary waves (just one appears in Fig. 3) to compensate for the depression. As  $\Delta \rightarrow \infty$ , with  $F_0$  fixed, the solution reduces to a classical lee-wave configuration (see, for instance, Patoine and Warn, 1982). In Fig. 4 we show a supercritical case,  $\Delta < \Delta_-$ . The most prominent features are the locally stationary elevation over the forcing region and the downstream modulated wavetrain. Note that whereas the compensation for a depression is a positive displacement resulting in upstream solitary waves (the subcritical case, Fig. 3), the compensation for an elevation is a negative displacement resulting in downstream modulated waves (the supercritical case, Fig. 4).

#### b. Positive forcing, weakly dissipative ( $F_0 > 0$ , $\nu = 0.1$ )

In Fig. 5 we show the resonant case  $\Delta = 0$ . It is generally similar to the corresponding nondissipative case shown in Fig. 2. The most noticeable effect of the weak dissipation is the reduced amplitude of the waves, both upstream and downstream. In particular, whereas in the nondissipative case the leading wave is the largest, the opposite is now the case. This is presumably because it is the first to be generated, and hence it has had a longer time to be affected by dissipation. The subcritical and supercritical cases are also generally similar to the corresponding nondissipative cases shown in Figs. 3 and 4, respectively, with the upstream and downstream waves substantially reduced in amplitude.

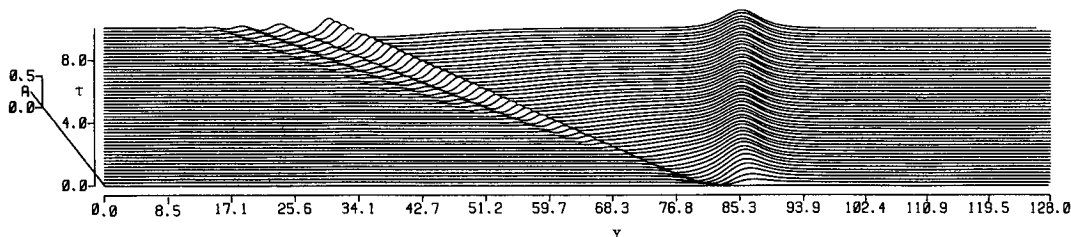


FIG. 4. The numerical solution for  $F_0 = 1$ ,  $\nu = 0$ ,  $\Delta = -4$  and  $\xi = 0.3$ .



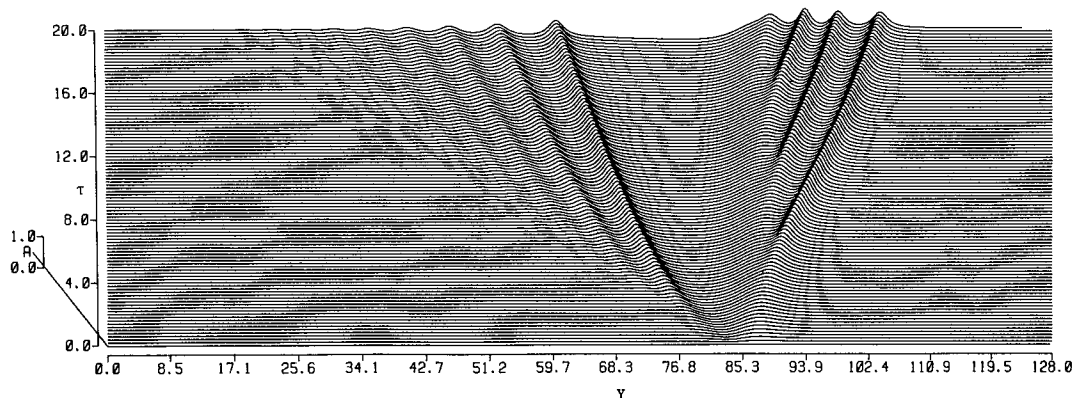


FIG. 5. The numerical solution for  $F_0 = 0.5$ ,  $\nu = 0.1$ ,  $\Delta = 0$  and  $\xi = 0.3$ .

*c. Positive forcing, strongly dissipative ( $F_0 > 0$ ,  $\nu = 1.0$ )*

In Fig. 6 we show the resonant case  $\Delta = 0$ . The dissipation is now so strong that no waves are generated, and the solution is a stationary state in the forcing region, for which the primary balance is between the dissipative term and the forcing. The subcritical and supercritical cases are also dominated by the stationary state in the forcing region, since the dissipation is sufficiently strong to obliterate the upstream and downstream waves. For the subcritical (supercritical) case the stationary state is one of depression (elevation) over the forcing region.

*d. Negative forcing, nondissipative ( $F_0 < 0$ ,  $\nu = 0$ )*

The numerical results for the resonant case,  $\Delta = 0$ , are shown in Fig. 7. The major differences between this case, and the corresponding case for positive forcing shown in Fig. 2, are the nonstationary character of the solution in the forcing region, and the absence of any pronounced upstream or downstream waves. If the forcing is sufficiently strong, eventually the disturbance in the forcing region builds up to a sufficient level for

one or more solitary waves to be emitted upstream, with the simultaneous production of some radiation downstream. Some very small-scale downstream radiation can be seen in Fig. 7 associated with a rise in amplitude of the transient disturbance in the forcing region. Unfortunately it is not possible to show this more clearly with our present numerical scheme as the large amplitudes needed for the transient solution in the forcing region are associated with high curvature leading to a breakdown of the numerical scheme. However, the general nature of the response can be inferred from the corresponding solutions obtained by Grimshaw and Smyth (1986) for the forced Korteweg-de Vries equation, for which these numerical problems are not so acute. The subcritical case ( $\Delta < 0$ ) will not be displayed as it can be described in terms of the corresponding case for positive forcing shown in Fig. 3, in that the dominant feature is again the downstream lee-wave train. However, in contrast to Fig. 3, the local depression over the forcing region is replaced by a local, transient elevation, and the compensating upstream solitary waves are replaced by a compensating upstream modulated wavetrain. The supercritical case ( $\Delta < 0$ ) will also not be displayed as it can be described in terms

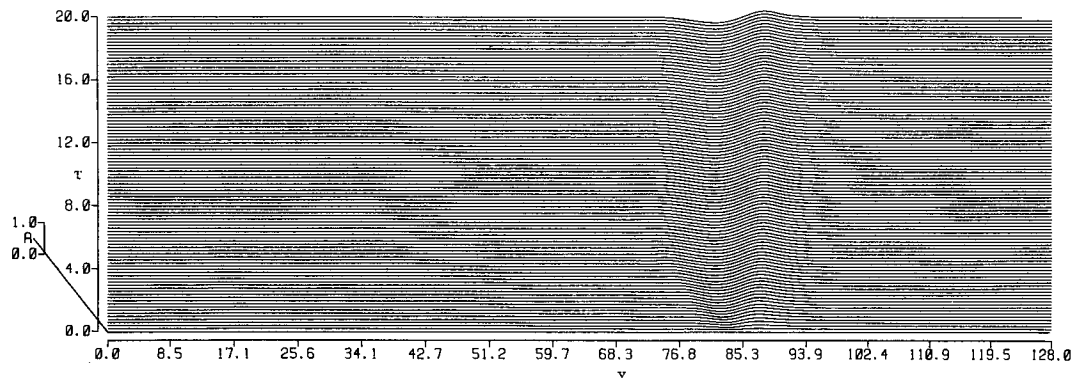


FIG. 6. The numerical solution for  $F_0 = 0.5$ ,  $\nu = 1.0$ ,  $\Delta = 0$  and  $\xi = 0.3$ .

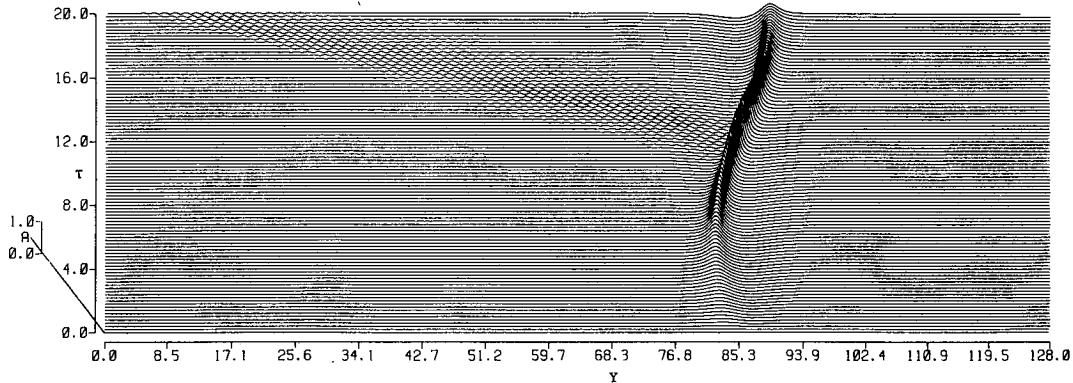


FIG. 7. The numerical solution for  $F_0 = -0.2$ ,  $\nu = 0$ ,  $\Delta = 0$  and  $\xi = 0.3$ .

of the corresponding case for positive forcing shown in Fig. 4. Thus, in contrast to Fig. 4, a local depression is formed over the forcing region and to compensate a finite number of upstream solitary waves are formed.

*e. Negative forcing, weakly dissipative ( $F_0 < 0$ ,  $\nu = 0.1$ )*

No numerical results will be shown as they generally show the same features as the nondissipative case (d) except that where waves are formed they are reduced in amplitude.

*f. Negative forcing, strongly dissipative ( $F_0 < 0$ ,  $\nu = 1.0$ )*

No numerical results will be shown as they can be described in terms of the corresponding case for positive forcing shown in Fig. 6. The dissipation is so strong that no waves are generated, and the solution is a stationary state in the forcing region. In the resonant case  $\Delta = 0$  the primary balance is between the dissipative term and the forcing, and the solution is analogous to that shown in Fig. 6 but with the opposite polarity. For the subcritical (supercritical) case the stationary state over the forcing region is one of elevation (depression).

## 5. Summary

In this paper we have described the resonant generation of coastally trapped waves when a longshore current interacts with a longshore topographic feature. The analogous theory when wind stress provides the forcing is developed in an appendix. The most significant predictions of the theory are the generation of disturbances trapped in the forcing region defined by the topographic feature, and the simultaneous generation of upstream and downstream nonlinear waves. The precise nature of the response depends primarily on the polarity of the forcing and the strength of the dissipation. Although the conditions for resonance have been precisely defined we have not directly addressed

the question as to which current systems are the more likely candidates for applications of the theory developed here. However, we note that Hughes (1985, 1986a,b) has speculated that a number of coastal currents are potential candidates for critical, or resonant, flow, and has commented that the stationary and transient eddies often observed in these currents may be forced by longshore topographic features. A notable example here is the California Current whose eddy field is extremely complicated and strongly affected by the horizontal shear of the current and the bottom topography (e.g., see Hickey, 1979). Clearly, further work is needed to establish the viability of the resonant mechanism proposed in this paper. However, it is pertinent to note that resonant forcing needs only a topographic feature of  $O(\alpha)$  to produce a response of  $O(\alpha^{1/2})$ , and that exact resonance is not required. Indeed, the theory described in section 4(a) can be used to deduce that the bandwidth for resonant forcing is given by

$$|c_n| < |2\sigma\alpha(\max|F_n|)|^{1/2} \quad (5.1)$$

where  $c_n$  is the long-wave phase speed of the resonant mode,  $F_n$  is related to the forcing by (2.17), and  $\sigma$  is defined by (3.8). Finally, we note that although this theory has been restricted to barotropic waves, we expect a similar theory could be developed which included the effects of stratification, although this would be considerably more complicated.

*Acknowledgments.* I am grateful for the assistance of Dr. Noel Smyth who provided the numerical routines for solving the evolution equation (4.2).

## APPENDIX A

### Curvilinear Coordinates

Let  $(x, y)$  be curvilinear coordinates such that the coastline is given by  $x = 0$ , and the deep ocean by  $x \rightarrow \infty$ . We shall suppose that they are related to conventional Cartesian coordinates  $(x', y')$  (see Fig. 1) by

$$x' = x'(x, y), \quad y' = y'(x, y). \quad (A1)$$

In these coordinates the element of distance  $ds$  is given by

$$ds^2 = Edx^2 + 2Fdx dy + Gdy^2 \quad (A2)$$

where  $E = \mathbf{a} \cdot \mathbf{a}$ ,  $F = \mathbf{a} \cdot \mathbf{b}$ ,  $G = \mathbf{b} \cdot \mathbf{b}$ . Here  $\mathbf{a}$ ,  $\mathbf{b}$  are vectors in the  $x$ ,  $y$  directions respectively given by

$$\mathbf{a} = (x'_x, y'_x), \quad \mathbf{b} = (x'_y, y'_y). \quad (A3)$$

Also we let

$$J = \mathbf{k} \cdot \mathbf{a} \times \mathbf{b} = (EG - F^2)^{1/2}, \quad (A4)$$

which is the Jacobian of the transformation (A1).

Next we define velocity components  $u$ ,  $v$  by

$$\mathbf{u} = u\mathbf{a} + v\mathbf{b}, \quad (A5)$$

where  $u = dx/dt$ ,  $v = dy/dt$ . Here  $u$ ,  $v$  are components of  $\mathbf{u}$  in directions orthogonal to the coordinate curves  $x = \text{constant}$ ,  $y = \text{constant}$ , respectively (i.e.,  $u = \mathbf{u} \cdot \nabla x$ ,  $v = \mathbf{u} \cdot \nabla y$ ). Then it may be shown that the momentum equation (1.1) becomes

$$E \frac{du}{dt} + F \frac{dv}{dt} + \frac{1}{2} E_x u^2 + E_y uv + (F_y - \frac{1}{2} G_x) v^2 + \zeta_x - Jfv = \tau \cdot \mathbf{a}/H,$$

$$F \frac{du}{dt} + G \frac{dv}{dt} + (F_x - \frac{1}{2} E_y) u^2 + G_x uv + \frac{1}{2} G_y v^2 + \zeta_y + Jfu = \tau \cdot \mathbf{b}/H, \quad (A6)$$

where  $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$ . The equation for conservation of mass (1.2) becomes

$$u^2 J \zeta_t + (HJu)_x + (HJv)_y = 0. \quad (A7)$$

The vorticity equation may be obtained by eliminating  $\zeta$  from (A6).

$$\frac{d}{dt} \left( \frac{f + \eta}{H} \right) = \frac{1}{JH} \left\{ \frac{\partial}{\partial x} \left( \frac{\tau \cdot \mathbf{b}}{H} \right) - \frac{\partial}{\partial y} \left( \frac{\tau \cdot \mathbf{a}}{H} \right) \right\}, \quad (A8)$$

where  $J\eta = (Fu + Gv)_x - (Eu + Fv)_y$ . The boundary conditions (1.4) at the coast and (1.5) in the deep ocean, become, respectively

$$Hu = 0, \quad \text{at } x = 0, \quad \text{and as } x \rightarrow \infty. \quad (A9)$$

The coordinate system used in the main body of this paper is (2.2), or

$$x' = x + \alpha g(x, Y), \quad y' = y, \quad (A10)$$

where  $Y = \epsilon y$ . For this transformation it is readily shown that

$$\mathbf{a} = (1 + \alpha g_x, 0), \quad \mathbf{b} = (\epsilon \alpha g_Y, 1), \quad (A11)$$

and that

$$\left. \begin{aligned} E &= 1 + 2\alpha g_x + \alpha^2 g_x^2 \\ F &= \epsilon \alpha g_Y (1 + \alpha g_x) \\ G &= 1 + \epsilon^2 \alpha^2 g_Y^2 \\ J &= 1 + \alpha g_x \end{aligned} \right\}. \quad (A12)$$

APPENDIX B

Resonant Forcing by Wind Stress

In this appendix we complement the analysis of sections 2 and 3 by considering the generation of long coastally trapped waves by wind stress. Thus in the equations of motion (1.1) we retain the term  $\tau_w$ , but, in order to be consistent with the long-wave scaling (2.1) we replace  $\tau_w$  with  $\epsilon \alpha \tau_w$ . Here the factor  $\alpha$  is introduced as it is the parameter measuring the magnitude of the forcing terms. Further, we shall assume that  $\tau_w$  has a large scale relative to the offshore topographic scale and hence  $\tau_w$  is a function of  $X = \epsilon x$ ,  $Y$  and  $T$ . The equations of motion (2.4) are then modified by the inclusion of the term  $\alpha \tau_w^{(y)}(0, Y, T)h^{-1}$  on the right-hand side of the second equation in (2.4). Here  $\tau_w^{(y)}$  denotes the component of wind stress in the  $y'$ -direction (see Fig. 1).

First let us follow the development of section 2 and seek a solution of the form (2.5). Equations (2.6) are then modified by the inclusion of the term  $\tau_w^{(y)}(0, Y, T)h^{-1}$  on the right-hand side of the second equation, and Eq. (2.8) contains an extra forcing term  $G$  on the right-hand side where

$$G_1 = \tau_w^{(y)}(0, Y, T) \left( \frac{1}{h} \right)_x \quad (B1)$$

The boundary condition at the coast is again (2.10). However, the outer boundary condition (2.11) must now be modified. To determine the replacement for (2.11) we consider the outer expansion for which the independent variables are  $X = \epsilon x$ ,  $Y$  and  $T$ , and the dependent are  $V = \epsilon^{-1}(v - v_0)$ ,  $u$  and  $Z = (\zeta - \zeta_0)$ . The equations of motion in the outer region are then (3.9) with the modification that terms  $\alpha \tau_w^{(x)}$  and  $\alpha \tau_w^{(y)}$  must be included in the first and second equations, respectively. To leading order the solution of these equations is given by

$$\left. \begin{aligned} \epsilon f u &= -\alpha Z_{1Y} + O(\alpha^2, \alpha \epsilon, \epsilon^2) \\ \epsilon f V &= \alpha Z_{1X} + O(\alpha^2, \alpha \epsilon, \epsilon^2) \end{aligned} \right\}, \quad (B2)$$

where

$$\left( \frac{\partial}{\partial T} + V_0 \frac{\partial}{\partial Y} \right) (Z_{1XX} + Z_{1YY}) - \mu^2 f^2 Z_{1T} = f \left( \frac{\partial \tau_w^{(y)}}{\partial X} - \frac{\partial \tau_w^{(x)}}{\partial Y} \right).$$

This, of course, is just the quasi-geostrophic solution being forced by wind-stress curl. The solution (B2) as  $X \rightarrow 0$  must be matched with the solution (2.5) as  $x \rightarrow \infty$ . Hence it may be shown that

$$\left. \begin{aligned} Z_1(X \rightarrow 0) &= 0 \\ Z_{1X}(X \rightarrow 0) &= f v_1(x \rightarrow \infty) \end{aligned} \right\}. \quad (B3)$$

Equations (B2) are then solved subject to the first condition in (B3) and the deep ocean outer boundary condition that  $Z_{1Y} \rightarrow 0$  as  $X \rightarrow \infty$ , where we are assuming that  $\tau_w$  also vanishes as  $X \rightarrow \infty$ . The result, combined with the second condition in (B3) and (2.7), determines the replacement for the outer boundary condition (2.11). This is

$$\psi_{1x} \rightarrow -V_w(Y, T). \quad (\text{B4})$$

Here,  $V_w = f^{-1}Z_{1X}(X \rightarrow 0)$  and is the longshore current generated at the outer boundary of the continental shelf by the action of wind-stress curl in the deep ocean. The necessity to include this forcing term has been shown by Allen (1976).

The solution of the modified form of equation (2.8) is now obtained by putting

$$\left. \begin{aligned} \psi_1 &= \hat{\psi}_1 - V_w \int_0^x h_0 dx' \\ \text{and} \quad \hat{\psi}_1 &= \sum_r A_r(Y, T) \psi_r(x) \end{aligned} \right\} \quad (\text{B5})$$

Here  $\psi_r(x)$  are the same long wave modes defined in section 2. We find that, in place of (2.17),

$$\frac{\partial A_r}{\partial T} + c_r \frac{\partial A_r}{\partial Y} = \frac{\partial F_r}{\partial Y} + \tau_w^{(y)}(0, Y, T) J_r + \frac{\partial V_w}{\partial Y} K_r, \quad (\text{B6})$$

where

$$\left. \begin{aligned} I_r J_r &= \int_0^\infty \phi_r \left( \frac{1}{h} \right)_x dx \\ I_r K_r &= \psi_r(\infty) \end{aligned} \right\},$$

while  $F_r$  and  $I_r$  are defined in (2.17). In the absence of the topographic term  $F_r$  and the mean current  $v_0(x)$ , Eq. (B6) is the forced first-order wave equation derived by Gill and Schumann (1974) for wind stress forcing and modified by Allen (1976) to include the effects of wind-stress curl forcing in the deep ocean. With the initial condition  $A_r(Y, 0) = 0$ , Eq. (B6) can be solved by standard methods. Resonance will occur whenever either  $\tau_w^{(y)}$  or  $V_w$  become functions of the single variable  $Y - c_n T$  as  $T \rightarrow \infty$ , for some long-wave speed  $c_n$ .

For the resonant case we now suppose that

$$\left. \begin{aligned} \tau_w^{(y)}(0, Y, T) &\sim G(Y - c_w T) \quad \text{as } T \rightarrow \infty \\ V_w(Y, T) &\sim H(Y - c_w T) \quad \text{as } T \rightarrow \infty \end{aligned} \right\}, \quad (\text{B7})$$

where  $c_n - c_w = \alpha^{1/2} \Delta$ . Here  $\Delta$  is a detuning parameter, the counterpart of the corresponding parameter in section 3 [see (3.2)]. The analysis now proceeds in a manner similar to that described in section 3. Thus we again introduce the long-time variable  $\tau$  (3.1), and the expansion (3.2) for  $u, v, \zeta$  where in the expression for  $\zeta$ ,  $v_0$  is replaced with  $v_0 - c_w$ . Also now the amplitude  $A = A(\hat{Y}, \tau)$  where

$$\hat{Y} = Y - c_w T. \quad (\text{B8})$$

We again obtain Eq. (3.4) where now an extra term  $G_1$  (B1) appears on the right-hand side. Unless  $c_w = 0$  the topographic forcing term  $F_1$  does not produce a resonance, and can be omitted in the subsequent discussion; the modifications necessary if  $c_w = 0$  are obvious and left to the reader. Hence we seek a solution for which  $\psi_1$  is a function of  $x, \hat{Y}$  and  $\tau$ , and then (3.4) reduces to an ordinary differential equation in  $x$  for  $\psi_{1\hat{Y}}$ . The solution is given by (3.5) where  $\hat{Y}$  replaces  $Y$ ,  $v_0 - c_w$  replaces  $v_0$ , and  $G(\hat{Y})(1/h)_x$  replaces  $F_{1Y}$  in  $B_1$ . Similarly the compatibility condition (3.7) again holds with  $v_0 - c_w$  replacing  $v_0$ . The outer expansion is described by (3.9) with terms  $\alpha \tau_w^{(x)}$  and  $\alpha \tau_w^{(y)}$  added to the first and second equations, respectively. With the balance  $\alpha = \epsilon^2$  we again seek a solution of the form (3.10) and find that  $Z_1$  now satisfies equation (B2) rather than (3.11). Matching with the interior solution produces the conditions (3.12). The solution for  $Z_1$  is now split into two parts. One part,  $Z_1^{(1)}$  satisfies the homogeneous equation (B2) [i.e., (3.11)] with the boundary condition  $Z_1^{(1)}(x \rightarrow 0) = -f A_n \psi_n(x \rightarrow \infty)$ . It is given by

$$Z_1^{(1)}/f \psi_n(\infty) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\eta \hat{Y} - \gamma X) \mathcal{F}(A) d\eta, \quad (\text{B9})$$

where

$$\gamma = \left( \eta^2 + \frac{\mu^2 f^2 c_w}{c_w - V_0} \right)^{1/2}.$$

The other part  $Z_1^{(2)}$  is the response to the wind-stress curl and produces the longshore current  $V_w$  (i.e.,  $V_w = f^{-1}Z_{1X}^{(2)}, X \rightarrow 0$ ). Finally, we obtain the evolution equation, which replaces (3.15). This is

$$\begin{aligned} A_\tau + \Delta A_{\hat{Y}} + \nu A + \sigma A A_{\hat{Y}} + \lambda \mathcal{B}^{(\mu)}(A_{\hat{Y}}) \\ = J_n G(\hat{Y}) + K_n H_{\hat{Y}}(\hat{Y}), \end{aligned} \quad (\text{B10})$$

where  $\mathcal{B}^{(\mu)}(A) = -(1/2\pi) \int_{-\infty}^{\infty} \gamma \exp(i\eta \hat{Y}) \mathcal{F}(A) d\eta$ . When the divergence parameter  $\mu = 0$ , the pseudodifferential operator  $\mathcal{B}^{(\mu)}(A)$  reduces to  $\mathcal{B}(A)$  [see (3.14)]. Since  $f(c_w - V_0) < 0$  for stable waves, we would normally expect  $c_w(c_w - V_0) > 0$  and  $\gamma$  is then always real-valued and positive; this is certainly the case when  $V_0 = 0$ . The operator  $\mathcal{B}^{(\mu)}(A)$  for  $\mu$  nonzero can then be expected to behave similarly to  $\mathcal{B}(A)$ . On the other hand,

$$\begin{aligned} \gamma \rightarrow (\mu^2 f^2 c_w / c_w - V_0)^{1/2} \{ 1 + (c_w - V_0) \eta^2 / 2 c_w \mu^2 f^2 \}, \\ \text{as } \mu \rightarrow \infty, \end{aligned} \quad (\text{B11})$$

and then  $\mathcal{B}^{(\mu)}(A)$  reduces to a term proportional to  $A$  plus a term proportional to  $A_{\hat{Y}}$ . Equation (B10) then becomes a forced, dissipative Korteweg-de Vries equation. The exceptional case when  $c_w(c_w - V_0) < 0$  leads to a range of values of  $\eta$  for which  $\gamma$  is pure imaginary, with  $V_0 \eta \text{Im} \gamma < 0$ ; these values of  $\eta$  correspond to radiating Poincaré waves, and their effect in the operator  $\mathcal{B}^{(\mu)}(A)$  is to induce a form of radiation damping.

We shall not describe any numerical solutions for

(B10) as, in general, we expect the solutions to be similar to those obtained for (3.15) [in scaled form, (4.2)]. Indeed, in the limit  $\mu \rightarrow 0$ , (B10) reduces to (3.15) with a different forcing term. On the other hand, in the limit  $\mu \rightarrow \infty$ , (B10) reduces to a forced Korteweg-de Vries equation [assuming that  $c_W(c_W - V_0) > 0$ ], whose numerical solutions have been comprehensively discussed by Grimshaw and Smyth (1986) and Smyth (1986).

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