S-Linear Algebra in Economics and Physics

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Abstract. In this paper we develop some generalizations of classic algebraic concepts systematically used in the applications to Physics and Economics. This generalizations are located in the context of $\mathcal{S}$-Linear Algebra and this theory is assumed to be known. The paper begins with the proof that the convolution of two distributions can be viewed as a particular case of superposition, this with a glimpse to Physical applications. Then, the paper present several new concepts and results in $\mathcal{S}$-Linear Algebra. Section 7 is a wide reformulation, in the infinite-dimensional case, of the classic finite-dimensional state preference model of Financial Analysis.

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§1. The convolution as superposition

As it is already remarked in other papers about $\mathcal{S}$-Linear Algebra, the expansion

$$\int_{\mathbb{R}^n} u\delta = u,$$

justifies completely the following formal expression of the physicists (see [6, p. 78])

$$\int_{\mathbb{R}^n} \delta(x-p)\delta(y-x)dx = \delta(y-p).$$

In fact, for $u = \delta_p$, we infer

$$\int_{\mathbb{R}^n} \delta_p\delta = \delta_p. \tag{1.1}$$

A correct mathematical interpretation of the formal equality (1.1) is the convolution of δ_p with δ_0, but this interpretation cannot enjoy thoroughly the physicists. In fact, the operation of convolution involves only two distributions and not a family of distributions. According to the physicists the expression (1.1) is one of the case belonging to the large class of the *continuous expansions of infinite-dimensional vectors*.

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Well, now we shall see that the convolution is a particular case of superposition, restoring a common vision on the expression (0). Obviously, not all the superpositions can be viewed as convolutions, but only a particular class of them.

Note that, in the language of superpositions, if $a \in \mathcal{S}'_m$ and $b \in \mathcal{S}'_n$ are two distributions, the tensor product $a \otimes b$ is defined, for every function f in $\mathcal{S}_{m+n}$, by the following numerical superposition

$$(a \otimes b)(f) := \int_{\mathbb{R}^m} a\ (b(f(p, \cdot)))_{p \in \mathbb{R}^m}.$$

Recall now that the convolution $a * b$, with $a \in \mathcal{E}'_n$ and $b \in \mathcal{S}'_n$, is defined, for every ϕ in $\mathcal{D}_n$, by

$$(a * b)(\phi) = (a \otimes b)(\phi \circ +),$$

where $+$ is the standard addition in $\mathbb{R}^n$.

Theorem 1.1. *Let a and b be two tempered distributions, a with compact support. Then, the family $(\tau_p b)_{p \in \mathbb{R}^n}$ is a smooth family, and moreover,*

$$a * b = \int_{\mathbb{R}^n} a(\tau_p b)_{p \in \mathbb{R}^n}.$$

Proof. Put $v_p = \tau_p(b)$, for every n-tuple p, we see that, for every test function ϕ,

$$v(\phi)(p) = v_p(\phi) = \tau_p(b)(\phi) = b(\tau_{-p}(\phi)).$$

Hence, setting $f := \phi \circ +$, i.e.,

$$f(p, x) := \tau_{-p}(\phi)(x) = \phi(x + p),$$

for every pair (p, x) in $\mathbb{R}^n \times \mathbb{R}^n$, we read

$$v(\phi)(p) = b(f(p, \cdot)).$$

With standard techniques, it can be proved that $v(\phi)$ is a smooth function (in general not of class $\mathcal{S}$), then v is an $\mathcal{E}$-family. Being a a compact support distribution, we can consider the superposition of v with respect to a, obtaining

$$\left(\int_{\mathbb{R}^n} av \right)(\phi) = a(v(\phi)) = \int_{\mathbb{R}^n} a(b(f(p, \cdot)))_{p \in \mathbb{R}^n} = (a \otimes b)(f) = (a * b)(\phi). \blacksquare$$

So the correct interpretation of the above formal equality is the following one: *the vector δ_p is the linear superposition of the infinite continuous family of vectors $(\delta_y)_{y \in \mathbb{R}^n}$ with respect to the system of coefficients δ_p.*

Hence, for instance, we can (rigorously !) affirm that the most general state of a quantum-particle in one dimension (i.e. a complex tempered distribution on $\mathbb{R}$) is a linear superposition of "eigenstates" of the position operator

$$Q : \mathcal{S}'(\mathbb{R}, \mathbb{C}) \to \mathcal{S}'(\mathbb{R}, \mathbb{C}) : u \mapsto \mathbb{I}_{\mathbb{R}} u.$$

We examine further an expansion which is not a convolution. The Fourier expansion theorem justifies completely another formal expression used by physicists (see [6, p. 38, formula (10)])

$$\delta(x-p) = \int_{\mathbb{R}} \frac{1}{2\pi} e^{piy} e^{-iyx} dy. \tag{1.2}$$

In fact, a classic result gives

$$(a,1)\mathcal{F}^{-}(\delta_p) = \frac{a}{2\pi}\left[e^{pi(\cdot)}\right],$$

and thus, from the Fourier expansion theorem, set $a = 1$, one has

$$\delta_p = \int_{\mathbb{R}} \frac{1}{2\pi}\left[e^{pi(\cdot)}\right]\left(\left[e^{-i(x|\cdot)}\right]\right)_{x\in\mathbb{R}}.$$

So we can read the expression (1.2) as follows: *the vector δ_p is the linear superposition of the infinite continuous family of vectors $\left(\left[e^{-i(p|\cdot)}\right]\right)_{p\in\mathbb{R}}$ with respect to the system of coefficients* $(1/2\pi)[e^{pi(\cdot)}]$. Once more, we can affirm *rigorously* that *the most general state of a quantum-particle in one dimension (i.e. a complex tempered distribution on $\mathbb{R}$) is a linear superposition of "eigenstates" of the momentum operator*

$$P : \mathcal{S}'(\mathbb{R},\mathbb{C}) \to \mathcal{S}'(\mathbb{R},\mathbb{C}) : u \mapsto -i\hbar u'.$$

§2. Systems of coordinates in an S-linearly independent family

It is simple to prove that, if v is an $\mathcal{S}$-linearly independent family and if $u \in {}^{\mathcal{S}}\mathrm{span}\,(\mathrm{v})$, then there exists a unique $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$. So, we can give the following

Definition 2.1 (system of coordinates). Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family and $u \in {}^{\mathcal{S}}\mathrm{span}(\mathrm{v})$. The only tempered distribution $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$ is denoted by $[u|v]$ and is called the system of coordinates of u in v.

Definition 2.2 (definition of coordinate operator in an $\mathcal{S}$-linearly independent family). Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family. The coordinate operator in w is the following operator

$$[\,\cdot\ \mid w] : {}^{\mathcal{S}}\mathrm{span}(\mathrm{w}) \to \mathcal{S}'_{\mathrm{m}} : \mathrm{u} \mapsto [\mathrm{u} \mid \mathrm{w}]\,. \ \square$$

*Example (on the Dirac family and the (a,b)-***Fourier family).** Let δ be the Dirac family in $\mathcal{S}'_n$. For all $u \in \mathcal{S}'_n$, we have $[u \mid \delta] = u$, and hence $[\cdot \mid \delta] = (\cdot)_{\mathcal{S}'_n}$. Let f be the (a,b)-Fourier family in $\mathcal{S}'_n$. For each $u \in \mathcal{S}'_n$ we have $[u \mid f] = \mathcal{F}^{-}_{(h,\omega)}(u)$, and hence $[\cdot \mid f] = \mathcal{F}^{-}_{(h,\omega)}$. $\triangle$

Theorem 2.1. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family. Then,*

$$[\,\cdot\ \mid w] \in \mathrm{Hom}({}^{\mathcal{S}}\mathrm{span}(\mathrm{w}), \mathcal{S}'_{\mathrm{m}}).$$

Proof. Let $\lambda \in \mathbb{C}$ and $u, v \in {}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$, then we have

$$u + \lambda v = \int_{\mathbb{R}^m} [u \mid w]\, w + \lambda \int_{\mathbb{R}^m} [v \mid w] w = \int_{\mathbb{R}^m} ([u \mid w] + \lambda [v \mid w])\, w,$$

and thus, we infer

$$[u + \lambda v \mid w] = [u \mid w] + \lambda\, [v \mid w]\,. \quad \blacksquare$$

Theorem 2.2. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $A : \mathcal{S}'_n \to \mathcal{S}'_n$ be an invertible $\mathcal{S}$-linear operator. Then, the following assertions hold true*

1) w is $\mathcal{S}$-linearly independent if and only if the family Aw is $\mathcal{S}$-linearly independent;

2) ${}^{\mathcal{S}}\mathrm{span}(\mathrm{Aw}) = \mathrm{A}\left({}^{\mathcal{S}}\mathrm{span}(\mathrm{w})\right)$*;*

3) if w is $\mathcal{S}$-linearly independent, then for each $u \in A\left({}^{\mathcal{S}}\mathrm{span}(\mathrm{w})\right)$, we have

$$[u \mid Aw] = [A^{-1}u \mid w].$$

Proof. 1) Let w be $\mathcal{S}$-linearly independent and let a belong to $\mathcal{S}'_m$ such that

$$\int_{\mathbb{R}^m} aA(w) = 0_{\mathcal{S}'_n}.$$

Applying A^{-1}, we obtain

$$0_{\mathcal{S}'_n} = A^{-1} 0_{\mathcal{S}'_n} = A^{-1} \int_{\mathbb{R}^m} aA(w) = \int_{\mathbb{R}^m} aA^{-1}A(w) = \int_{\mathbb{R}^m} aw.$$

Since w is $\mathcal{S}$-linearly independent we deduce $a = 0_{\mathcal{S}'_n}$, and then Aw is $\mathcal{S}$-linearly independent too.

2) Let $u \in A\left({}^{\mathcal{S}}\mathrm{span}(\mathrm{w})\right)$. Then, there exists an $a \in \mathcal{S}'_m$ such that $u = A\int_{\mathbb{R}^m} aw$. Thus, we have

$$u = \int_{\mathbb{R}^m} aAw,$$

so $u \in {}^{\mathcal{S}}\mathrm{span}(\mathrm{Aw})$, and hence $A\left({}^{\mathcal{S}}\mathrm{span}(\mathrm{w})\right) \subseteq {}^{\mathcal{S}}\mathrm{span}(\mathrm{Aw})$.

To prove the converse, let $u \in {}^{\mathcal{S}}\mathrm{span}(\mathrm{Aw})$. Then, there exists an $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} aAw$, and hence,

$$u = A\int_{\mathbb{R}^m} aw,$$

which yields $u \in A\left({}^{\mathcal{S}}\mathrm{span}(\mathrm{w})\right)$, hence ${}^{\mathcal{S}}\mathrm{span}(\mathrm{Aw}) \subseteq \mathrm{A}\left({}^{\mathcal{S}}\mathrm{span}(\mathrm{w})\right)$. Concluding, we get

$${}^{\mathcal{S}}\mathrm{span}(\mathrm{Aw}) = \mathrm{A}\left({}^{\mathcal{S}}\mathrm{span}(\lambda\mathrm{w})\right).$$

3) For every $u \in {}^{\mathcal{S}}\mathrm{span}(\mathrm{A}^{-1}\mathrm{w})$ we have

$$u = \int_{\mathbb{R}^m} [u \mid A^{-1}w] A^{-1}w,$$

and applying A,

$$Au = \int_{\mathbb{R}^m} [u \mid A^{-1}w] AA^{-1}w = \int_{\mathbb{R}^m} [u \mid A^{-1}w] w,$$

so, Au belongs to ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$ and $[Au \mid w] = [u \mid A^{-1}w]$. ∎

By the Dieudonné-Schwartz theorem we immediately deduce the following characterization.

Theorem 2.3. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then the following assertions are equivalent:*

1) v is $\mathcal{S}$-linearly independent and ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v})$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed;

2) $\int_{\mathbb{R}^m}(\cdot, v)$ is an injective topological homomorphism for $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$ and $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$;

3) $[\cdot \mid v]$ is a topological isomorphism for the two weak topologies $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ and $\sigma(\mathcal{S}'_m, \mathcal{S}_m)$).

Proof. We have to prove only the equivalence between 2 and 3, but the operator $\int_{\mathbb{R}^m}(\cdot, v)$ is an injective weakly *-topological homomorphism if and only if (by definition of topological homomorphisms) the inverse of its restriction to the pair $(\mathcal{S}'_m, {}^{\mathcal{S}}\mathrm{span}(\mathrm{v}))$, i.e., the application $[\cdot \mid v]$, is a topological isomorphism, with respect to the topology induced by $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ on ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v})$ and to $\sigma(\mathcal{S}_m, \mathcal{S}'_m)$, in and only if ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v})$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closed. ∎

§3. The S-linearity of the coordinate operator

Theorem 3.1. Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family. Then the following assertions are equivalent:

1) $[\,\cdot \mid w]$ is an $\mathcal{S}$-operator;
2) $[\,\cdot \mid w]$ is an $\mathcal{S}$-homomorphism;
3) $\int_{\mathbb{R}^m}(\,\cdot\,, w)$ is an $\mathcal{S}$-homomorphism.

Proof. 1) implies 2). In fact, let v be a family in ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$ such that $[v \mid w]$ is of class $\mathcal{S}$, we have $v = \int_{\mathbb{R}^m}[v \mid w]w$ and being $\int_{\mathbb{R}^m}(\,\cdot\,, w)$ an $\mathcal{S}$-operator, v is of class $\mathcal{S}$.

2) implies 3). In fact, let a be a family in $\mathcal{S}'_m$ such that $v := \int_{\mathbb{R}^m} aw$ is a family of class $\mathcal{S}$, we have by $\mathcal{S}$-linear independence that $a = [v \mid w]$ and so, being $[\cdot \mid w]$ an $\mathcal{S}$-operator, a is necessarily of class $\mathcal{S}$.

3) implies 1). In fact, let v be an $\mathcal{S}$-family in ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$, we have to prove that $[v \mid w]$ is of class $\mathcal{S}$. But, $v = \int_{\mathbb{R}^m}[v \mid w]w$ and so, being $\int_{\mathbb{R}^m}(\,\cdot\,, w)$ an $\mathcal{S}$-homomorphism, $[v \mid w]$ is of class $\mathcal{S}$. ∎

Theorem 3.2 ($\mathcal{S}$-closedness of the ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$). *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family such that $[\,\cdot \mid w]$ is an $\mathcal{S}$-operator. Then, ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$ is $\mathcal{S}$-closed in $\mathcal{S}'_n$.*

Proof. To prove that ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$ is $\mathcal{S}$-closed, let k be a natural number, $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ be a family in ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$ and $a \in \mathcal{S}'_k$. Then, $v = \int_{\mathbb{R}^m}[v \mid w]w$, in fact

$$v_p = \int_{\mathbb{R}^m} [v_p \mid w]\, w = \int_{\mathbb{R}^m} [v \mid w]_p\, w = \left(\int_{\mathbb{R}^m} [v \mid w] w \right)_p ,$$

for any $p \in \mathbb{R}^k$. Thank to the $\mathcal{S}$-linearity of the $\mathcal{S}$-linear combinations, we have

$$\int_{\mathbb{R}^k} av = \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} [v \mid w] w \right) = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} a\,[v \mid w] \right) w,$$

and thus $\int_{\mathbb{R}^k} av \in {}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$. ■

Open problem. We do not know if $\mathcal{S}$-closedness of ${}^{\mathcal{S}}\mathrm{span}\,(\mathrm{w})$ implies that the operator $[\,\cdot\, \mid w]$ is an $\mathcal{S}$-operator. However, note that, if ${}^{\mathcal{S}}\mathrm{span}\,(\mathrm{w})$ is σ_n^*-closed, then $\widehat{w}$ is surjective; then, for every $\mathcal{S}$-family v in ${}^{\mathcal{S}}\mathrm{span}\,(\mathrm{w})$, holding $\widehat{v}(g) = [v \mid w](\widehat{w}(g))$, we have that $[v \mid w]$ is an $\mathcal{S}$-family, and so, $[\,\cdot\, \mid w]$ is an $\mathcal{S}$-operator.

Theorem 3.3 (the $\mathcal{S}$-linearity of the coordinate operator). *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family such that $[\,\cdot\, \mid w]$ is an $\mathcal{S}$-operator. Then, one has*

$$[\,\cdot\, \mid w] \in {}^{\mathcal{S}}\mathrm{Hom}({}^{\mathcal{S}}\mathrm{span}(\mathrm{w}), \mathcal{S}'_{\mathrm{n}}).$$

Proof. The operator $[\,\cdot\, \mid w]$ is of class $\mathcal{S}$ for assumption. For each natural k, for any $a \in \mathcal{S}'_k$ and for every family $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ in ${}^{\mathcal{S}}\mathrm{span}(\mathrm{w})$,

$$\int_{\mathbb{R}^k} av = \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} [v \mid w]\, w \right) = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} a\,[v \mid w] \right) w,$$

and thus, by the definition of system of contravariant components one has

$$[\,\cdot\, \mid w] \left(\int_{\mathbb{R}^k} av \right) = \left[\int_{\mathbb{R}^k} av \mid w \right] = \int_{\mathbb{R}^k} a\,[v \mid w]\,. \ ■$$

Theorem 3.4. *Let v be an $\mathcal{S}$-linearly independent family in $\mathcal{S}'_n$, let $[\cdot \mid v]$ be an $\mathcal{S}$-operator and let F be the collection of all the subset of $\mathcal{S}'_n$ containing v and $\mathcal{S}$-closed. Then ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v}) = \bigcap \mathrm{F}$.*

Proof. Since F_i is $\mathcal{S}$-closed and contains v, we have ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v}) \subseteq \mathrm{F_i}$, for every $i \in I$, and consequently ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v}) \subseteq \bigcap \mathrm{F}$. Since $[\cdot \mid v]$ is an $\mathcal{S}$-operator, ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v})$ is $\mathcal{S}$-closed, moreover it contains v, thus ${}^{\mathcal{S}}\mathrm{span}(\mathrm{v}) \in \mathrm{F}$, and hence $\bigcap F \subseteq {}^{\mathcal{S}}\mathrm{span}(\mathrm{v})$. ■

§4. Change of basis

Notation (the set of the $\mathcal{S}$-bases of a subspace). Let $X \subseteq \mathcal{S}'_n$ be a subspace. In the following we shall use the notation

$${}^{\mathcal{S}}\mathcal{B}(\mathbb{R}^m, X)$$

for the set of the families $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ such that $\mathrm{Im}(v) \subseteq X$ and that are $\mathcal{S}$-basis for X.

Definition 4.1 (the family of change for two $\mathcal{S}$-bases). Let $v \in {}^{\mathcal{S}}\mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ and $w \in {}^{\mathcal{S}}\mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$. We call family of change from v to w, the following family

$$[w \mid v] := ([w_p \mid v])_{p \in \mathbb{R}^m}\,.$$

Theorem 4.1 (on the change of basis). *Let* $v \in {}^{\mathcal{S}}\mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ *and* $w \in {}^{\mathcal{S}}\mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$*. Then*

$$[v \mid w] \in {}^{\mathcal{S}}\mathcal{B}(\mathbb{R}^n, \mathcal{S}'_m).$$

Moreover, for every $u \in \mathcal{S}'_n$*, we have*

$$[u \mid w] = \int_{\mathbb{R}^n} [u \mid v]\,[v \mid w].$$

Proof. Since

$$v = \int_{\mathbb{R}^m} [v \mid w] w,$$

we have

$$\widehat{v}(\phi) = [v \mid w]\left(\widehat{w}(\phi)\right),$$

for every test function $\phi \in \mathcal{S}'_n$; so, being $\widehat{w}$ surjective (w is an $\mathcal{S}$-basis and thus it is invertible), $[v \mid w]$ is an $\mathcal{S}$-family. Moreover, the same equality shows that

$$\widehat{v} \circ (\widehat{w})^{-1} = [v \mid w]^{\wedge},$$

and then $[v \mid w]$ is invertible, that is an $\mathcal{S}$-basis.

Now, applying the $\mathcal{S}$-linearity of the $\mathcal{S}$-linear combinations, we have

$$u = \int_{\mathbb{R}^n} [u \mid v]\, v = \int_{\mathbb{R}^n} [u \mid v] \left(\int_{\mathbb{R}^m} [v \mid w]\, w \right) = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} [u \mid v]\,[v \mid w] \right) w,$$

and thus by definition of system of coordinates in an $\mathcal{S}$-basis

$$[u \mid w] = \int_{\mathbb{R}^n} [u \mid v]\,[v \mid w]. \quad \blacksquare$$

§5. $\mathcal{S}$-connected and $\mathcal{D}$-connected sets

It is interesting to note that every tempered distribution u in $\mathcal{S}'_n$ belongs to many $\mathcal{S}$-families and to many $\mathcal{D}$-families.

Theorem 5.1. *For every* u *in* $\mathcal{S}'_n$ *and for every not-identically zero test function* f *in* $\mathcal{S}_m$ *(resp.* $\mathcal{D}_m$*), there exists an* $\mathcal{S}$*-family (resp.* $\mathcal{D}$*-family) in* $\mathcal{S}'_n$ *containing* u *and such that the associated operator* $\widehat{v}$ *is proportional to the operator* $\langle u, \cdot \rangle f$*, where* $\langle \cdot, \cdot \rangle$ *is the canonical bilinear form on* $\mathcal{S}'_n \times \mathcal{S}_n$*.*

Proof. Let f be an $\mathcal{S}$-function ($\mathcal{D}$-function) in $\mathcal{S}_m$ (resp. $\mathcal{D}_m$) not identically 0, and let p_0 be an m-vector such that $f(p_0) \neq 0$. The family defined by

$$v_p = \frac{f(p)}{f(p_0)} u,$$

for every m-vector p, is an $\mathcal{S}$-family ($\mathcal{D}$-family) containing u. Indeed, we have $v_{p_0} = u$, and

$$v(\phi)(p) = v_p(\phi) = \left(\frac{f(p)}{f(p_0)}u\right)(\phi) = \frac{f(p)}{f(p_0)}u(\phi) = \left(\frac{u(\phi)}{f(p_0)}f\right)(p),$$

thus $v(\phi) = (u(\phi)/f(p_0))\, f$, and so $v(\phi)$ is an $\mathcal{S}$-function ($\mathcal{D}$-function) in $\mathcal{S}_m$ (resp. $\mathcal{D}_m$). ■

By the preceding result we immediately deduce a natural sufficient condition in order that a set contains at least one $\mathcal{D}$-family, or $\mathcal{S}$-family, through every its point.

Recall that a subset S of a vector space V is said to be *star-shaped in the origin* if it contains, for every s in S, the closed segment joining s with the origin of V. On the other hand, if S contains, for every s, the segment joining s with the origin but not the origin, S is said to be a *blunt star-shaped set in the origin.*

Theorem 5.2. *Let S be a (blunt) star-shaped set in the origin of the space $\mathcal{S}'_n$. Then, for every u in S, there is a $\mathcal{D}$-family ($\mathcal{S}$-family) contained in S and passing through u.*

Proof. It is sufficient to choose a smooth function f defined on $\mathbb{R}^m$, with compact support, real, non-negative, with values lower or equal than 1, and such that $f(0_m) = 1$. Then, for every u in S, the family of distributions indexed by $\mathbb{R}^m$, defined by $v_p = f(p)u$, is a $\mathcal{D}$-family contained in S (v describes the segment joining u with the origin of $\mathcal{S}'_n$) and containing u.

In the blunt case it is necessary to consider a function f of class $\mathcal{S}$, real, non-negative, with values lower or equal than 1, everywhere different from 0 and such that $f(0_m) = 1$. ■

Remark. We can see more, let u_0 and u_1 two tempered distributions in $\mathcal{S}'_n$ and f_0, f_1 two $\mathcal{S}$-functions in $\mathcal{S}_1$, such that $f_i(j) = \delta_{ij}$, for every $i, j = 0, 1$. Define a family v in $\mathcal{S}'_n$ as follows

$$v_p = f_0(p)u_0 + f_1(p)u_1,$$

for every real number p. We have

$$v_0 = f_0(0)u_0 + f_1(0)u_1 = u_0,$$

and, analogously, $v_1 = u_1$, hence v contains both u_0 and u_1. Moreover,

$$\begin{aligned} v(\phi)(p) &= v_p(\phi) = \\ &= (f_0(p)u_0 + f_1(p)u_1)(\phi) = \\ &= f_0(p)u_0(\phi) + f_1(p)u_1(\phi) = \\ &= (u_0(\phi)f_0 + u_1(\phi)f_1)(p), \end{aligned}$$

so the function $v(\phi)$ is a linear combination of f_0 and f_1, and so $v(\phi)$ is in $\mathcal{S}_1$.

In general, every finite sequence $u = (u_i)_{i=1}^k$ of tempered distributions is a subfamily of many $\mathcal{S}$-families in $\mathcal{S}'_n$. It is enough to consider a system $(f_i)_{i=1}^k$ of functions in $\mathcal{S}_1$ such that $f_i(j) = \delta_{ij}$ for every $i, j = 1, ..., k$, and define a family v in $\mathcal{S}'_n$ as follows

$$v_p = \sum_{i=1}^{k} f_i(p)u_i,$$

for every real number p. The above discussion shows us that every finite linear combinations of distributions can be always viewed as superposition of an $\mathcal{S}$-family.

We can go beyond. But first we give the following

Definition 5.1 (of $\mathcal{S}$-connected pair in a subset of $\mathcal{S}'_n$ and $\mathcal{S}$-connected subset of $\mathcal{S}'_n$). Let X be a subset of $\mathcal{S}'_n$ and let $x, y \in X$. The pair (x, y) is said to be an $\mathcal{S}$-connected ($\mathcal{D}$-connected) pair in X if and only if there is an $\mathcal{S}$-family ($\mathcal{D}$-family) v indexed by $\mathbb{R}^m$ containing x and y and contained in X. X is said $\mathcal{S}$-connected ($\mathcal{D}$-connected) if, for every $x, y \in X$, the pair (x, y) is an $\mathcal{S}$-connected ($\mathcal{D}$-connected) pair in X.

Theorem 5.3. *Let S be a star-shaped set in the origin of the space $\mathcal{S}'_n$. Then, S is $\mathcal{D}$-connected and, consequently, $\mathcal{S}$-connected.*

Proof. Every finite sequence of tempered distributions $(u_i)_{i=1}^n$ is a subfamily of a particular kind of $\mathcal{D}$-family. Consider a system $(f_i)_{i=1}^k$ of functions in $\mathcal{D}_1$ such that $f_i = \tau_i(f_0)$, for every i, with f_0 a smooth function fulfilling the following properties: 1) $f_0(0) = 1$; 2) $f_0(x) \in [0, 1]$, for every real x; 3) $\mathrm{suppf_0} = \overline{\mathrm{B}}(0, 1/2)$. Consequently the functions f_i fullfil the following: 1) $f_i(i) = 1$, for every i; 2) $f_i(x) \in [0, 1]$, for every real x; 3) $\mathrm{suppf_i} = \overline{\mathrm{B}}(\mathrm{i}, 1/2)$. Define the family v in $\mathcal{S}'_n$ as

$$v_p = \sum_{i=1}^{k} f_i(p) u_i,$$

for every real number p. It is simple to see that v is a $\mathcal{D}$-family contained in S and passing through every u_i. ■

As a consequence, if we say $\mathcal{S}$-closed a part F of $\mathcal{S}'_n$ such that every superposition of each $\mathcal{S}$-family of F lies in F, we conclude the following

Corollary. *Every $\mathcal{S}$-closed star-shaped subset of $\mathcal{S}'_n$ is a subspace of $\mathcal{S}'_n$.*

§6. $\mathcal{D}_{\mathcal{L}^1}$-closed sets and $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-closedness

Following Schwartz, if $p \in [1, +\infty]$, we shall denote by $\mathcal{D}_{\mathcal{L}^p}$ the vector space of the smooth complex functions defined on $\mathbb{R}^n$ whose derivatives belong to $\mathcal{L}^p(\mathbb{R}^n, \mathbb{C})$. The natural topology on this space is, by definition, the topology generated by the family of seminorms $(q_k)_{k \in \mathbb{N}_0^n}$, where, for multi-index k, q_k is defined on $\mathcal{D}_{\mathcal{L}^p}$ by

$$q_k(f) = \left\| f^{(k)} \right\|_{\mathcal{L}^p}.$$

When $\mathcal{D}_{\mathcal{L}^p}$ is endowed with its natural topology, the associated topological vector space is denoted simply by $(\mathcal{D}_{\mathcal{L}^p})$. It is a complete locally convex topological vector space with a denumerable fundamental system of neighborhood of the origin, it is then metrizable and so a Fréchet space. A sequence $f = (f_i)_{i \in \mathbb{N}}$ converges to the zero-function in $(\mathcal{D}_{\mathcal{L}^p})$ if and only if it converges to 0 in the topological vector space $(\mathcal{L}^p)$ with all its derivatives.

Lemma. *Let f be a C^1-function defined on $\mathbb{R}^{\geq}$ and of class $\mathcal{L}^1$ with its derivative. Then, the series $\sum (f(k))_{k=1}^{\infty}$ is absolutely convergent, and moreover*

$$\sum_{k=1}^{\infty} |f(k)| \leq \|f\|_{\mathcal{L}^1} + \|f'\|_{\mathcal{L}^1}.$$

Proof. Let k be a positive integer, and let m_k be the minimum point of $|f|$ on the interval $[k-1,k]$. For every k, denoted by l the Lebesgue measure on $\mathbb{R}$, we have

$$|f(m_k)|\, l([k-1,k]) \leq \int_{k-1}^{k} |f|\, dl.$$

Hence, for every $n \geq 1$,

$$\sum_{k=1}^{n} |f(m_k)| \leq \sum_{k=1}^{n} \int_{k-1}^{k} |f|\, dl = \int_{0}^{n} |f|\, dl.$$

This implies that the series

$$\sum (|f(m_k)|)_{k=1}^{\infty}$$

is convergent, and that

$$\sum_{k=1}^{\infty} |f(m_k)| \leq \lim_{n\to\infty} \int_{0}^{n} |f|\, dl = \int_{0}^{+\infty} |f|\, dl = \|f\|_{\mathcal{L}^1}.$$

On the other hand, by the Torricelli-Barrow theorem, for every k,

$$f(k) - f(m_k) = \int_{m_k}^{k} f' dl,$$

and so

$$|f(k)| = \left| f(m_k) + \int_{m_k}^{k} f' dl \right| \leq |f(m_k)| + \left| \int_{m_k}^{k} f' dl \right| \leq |f(m_k)| + \int_{k-1}^{k} |f'|\, dl,$$

by this inequality, the series

$$\sum (|f(k)|)_{k=1}^{\infty}$$

converges, and moreover

$$\sum_{k=1}^{\infty} |f(k)| \leq \sum_{k=1}^{\infty} |f(m_k)| + \sum_{k=1}^{\infty} \int_{k-1}^{k} |f'|\, dl \leq \|f\|_{\mathcal{L}^1} + \|f'\|_{\mathcal{L}^1},$$

that is the conclusion. ■

Lemma. *The distribution $\sum_{i=1}^{\infty} \delta_i$ belongs to the space $(\mathcal{D}_{\mathcal{L}^1})'$.*

Proof. We have to prove that the distribution $\sum_{i=1}^{\infty} \delta_i$ it is a continuous form on the space $(\mathcal{D}_{\mathcal{L}^1})$. Let $f = (f_j)_{j\in J}$ be a sequence convergent to the zero-function in $(\mathcal{D}_{\mathcal{L}^1})$, then f converges to the zero-function in the topological vector space $(\mathcal{L}^1)$ with all its derivatives. We have to prove that the sequence

$$\left(\left(\sum_{i=1}^{\infty} \delta_i \right) (f_j) \right)_{j\in J}$$

is convergent to 0. By the above lemma we have

$$\sum_{k=1}^{\infty} |f_j(k)| \leq \|f_j\|_{\mathcal{L}^1} + \left\|f_j'\right\|_{\mathcal{L}^1},$$

and, since f converges to the zero-function in $(\mathcal{D}_{\mathcal{L}^1})$, the right hand converges to 0, implying the claim. ■

Remark. Let us see an alternative proof of the second lemma. It is simple to see that every delta-distribution belongs to $(\mathcal{D}_{\mathcal{L}^1})$, and thus every finite linear combination of delta-distributions. So the distribution $\sum_{i=1}^{\infty} \delta_i$ is the punctual limit of a sequence of continuous linear forms on $(\mathcal{D}_{\mathcal{L}^1})$. Since $(\mathcal{D}_{\mathcal{L}^1})$ is barreled (it is a Fréchet space) the Banach-Steinhaus theorem holds true, and we conclude, once more, that $\sum_{i=1}^{\infty} \delta_i$ is a continuous linear form on $(\mathcal{D}_{\mathcal{L}^1})$.

When $p = +\infty$ the space $\mathcal{D}_{\mathcal{L}^p}$ is denoted also by $\mathcal{B}_n$. Since a continuous function belonging to $\mathcal{L}^\infty$ is bounded, $\mathcal{B}_n$ is the vector space of the smooth functions that are bounded with all their derivatives. Moreover, $\dot{\mathcal{B}}_n$ denotes the subspace of $\mathcal{B}_n$ containing the function vanishing at infinite with all their derivatives; $\left(\dot{\mathcal{B}}_n\right)$ shall be the associated topological vector space endowed with the topology induced by $(\mathcal{B}_n)$. It is clear that $\mathcal{S}_n$ is included in $\dot{\mathcal{B}}_n$, and it is also evident that the topological vector space $(\mathcal{S}_n)$ is continuously imbedded in the space $\left(\dot{\mathcal{B}}_n\right)$, consequently $\left(\dot{\mathcal{B}}_n\right)' \subset \mathcal{S}_n'$.

Let us see the sum of a convergent series of tempered distribution as a superposition.

Theorem 6.1. *Let $\sum (u_k)_{k=1}^{\infty}$ be a weakly* convergent series in $\mathcal{S}_n'$. Then, there is a $\dot{\mathcal{B}}_1$-family, more precisely, a $\mathcal{D}_{\mathcal{L}^1}$ -family, which contains the series as sub-family.*

Proof. Let $\sum (u_k)_{k=1}^{\infty}$ be such series and assume it is weakly* convergent to a tempered distribution u^*. Consider a sequence $f = (f_i)_{i=1}^{\infty}$ of functions in $\mathcal{D}_1$ such that $f_i = \tau_i(f_0)$, for every i, where f_0 a smooth function in $\mathcal{D}_1$ with the following properties: 1) $f_0(0) = 1$, for every positive integer i; 2) $f_0(x) \in [0,1]$, for every real x; 3) $\mathrm{suppf_0} = \overline{\mathrm{B}}(0, 1/2)$. Define the family v in $\mathcal{S}_n'$ as follows

$$v_p := {}^{\sigma(\mathcal{S}_n', \mathcal{S}_n)} \sum_{i=1}^{\infty} f_i(p) u_i,$$

for every real number p. Note that the sequence of partial sums of the series $\sum (f_i(p)u_i)_{i=1}^{\infty}$ is definitely constant, and then the series is $\sigma(\mathcal{S}_n', \ \mathcal{S}_n)$-convergent. Moreover, $v_j = u_j$, for every natural j. We have to prove that, if g is a test function of class $\mathcal{S}_n$, then $v(g)$ is of class $\dot{\mathcal{B}}$. We have

$$v(g)(p) = v_p(g) = \sum_{i=1}^{\infty} f_i(p) u_i(g) = \begin{cases} u_j(g) f_j(p) & if\ p \in \overline{B}(j, 1/2)\ and\ j \in \mathbb{N} \\ 0 & elsewhere \end{cases}.$$

So the function

$$v(g) = \sum_{i=1}^{\infty} u_i(g) f_i$$

is smooth and vanishing at infinity with all its derivatives. In fact, being the numerical series $\sum (u_i(g))_{i=1}^{\infty}$ convergent, for every test function g, we have

$$\lim_{i\to\infty} |u_i(g)| = 0,$$

hence

$$\lim_{p\to\infty} |v(g)(p)| = \lim_{j\to\infty} |u_j(g)f_j| \leq \max f_0 \cdot \lim_{i\to\infty} |u_i(g)| = 0.$$

Analogously, for every natural k, we have

$$\lim_{p\to\infty} \left| v(g)^{(k)}(p) \right| = \lim_{j\to\infty} \left| u_j(g)f_j^{(k)} \right| \leq \max f_0^{(k)} \cdot \lim_{i\to\infty} |u_i(g)| = 0.$$

Hence $v(g)$ belongs to $\dot{\mathcal{B}}_1$.

To prove that v is of class $\mathcal{D}_{\mathcal{L}^1}$, note that, for every integer $k \geq 0$,

$$\begin{aligned} \left| \int_{\mathbb{R}} v(g)^{(k)} dl \right| &= \left| \int_{\mathbb{R}} \sum_{i=1}^{\infty} u_i(g) f_i^{(k)} dl \right| = \left| \sum_{i=1}^{\infty} u_i(g) \int_{\mathbb{R}} f_i^{(k)} dl \right| = \\ &= \left| \sum_{i=1}^{\infty} u_i(g) \right| \left| \int_{\mathbb{R}} f_0^{(k)} dl \right|, \end{aligned}$$

thus $v(g)$ is smooth and of class $\mathcal{L}^1$ with all its derivatives, then, following Schwartz, $v(g)$ belongs to the space $\mathcal{D}_{\mathcal{L}^1}$. ■

Corollary. *Let $\sum (u_k)_{k=1}^{\infty}$ be a weakly* convergent series in $\mathcal{S}'_n$ to a tempered distribution u^*. Then u^* is a $\mathcal{D}_{\mathcal{L}^1}$-superposition of v. As a consequence, each $\mathcal{D}_{\mathcal{L}^1}$ -closed subset of $\mathcal{S}'_n$ is sequentially weakly* closed.*

Proof. Consider the series of distributions $\sum (\delta_i)_{i=1}^{\infty}$ in $(\mathcal{D}_{\mathcal{L}^1})'$; it is convergent in $(\mathcal{D}_{\mathcal{L}^1})'$. In fact, for every s in $\mathcal{D}_{\mathcal{L}^1}$, the series $\sum (s(i))_{i=1}^{n}$ is convergent, and we have

$$\sum_{i=1}^{\infty} \delta_i(s) = \sum_{i=1}^{\infty} s(i).$$

Let v be the family of class $\mathcal{D}_{\mathcal{L}^1}$ built in the proof of the above theorem, we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}} \sum_{i=1}^{\infty} \delta_i v \right)(g) &= \left(\sum_{i=1}^{\infty} \delta_i \right)(v(g)) = \sum_{i=1}^{\infty} v_i(g) = \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u_j(g) f_j(i) = \sum_{i=1}^{\infty} u_i(g) = u^*(g). \end{aligned}$$

Hence u^* is a $\mathcal{D}_{\mathcal{L}^1}$-superposition of v. Now, a set F is sequentially weakly* closed if and only if contains the sum of every series in F sequentially weakly* convergent, and this concludes the proof. ■

§7. The state preference model

We first recall the standard setting of the state preference model.

Economic model. We consider a market, and we observe it only two times, called the initial time and the final time. Assume that in the market there are n goods. For every $j \in \underline{n}$, each unit of the j-th good can assume m-possible values at the final time. These values depend on m states of the world. The value of a unit of the j-th good, in the i-th state of the world, is the real number a_{ij}.

Definition. A state preference model is a system (G, S, a), where G is an ordered set with n elements, called the set of the goods of the model, S is an ordered set with m elements, called the set of the states of the world, and a is an (m, n)-matrix, called the values-matrix of the model. For every positive integer $j \leq n$, the vector $C_j(a) = (a_{ij})_{i \in \underline{m}}$ (the j-th column of a) is called the values-vector of the j-th good.

Definition (of portfolio). We call portfolio of the considered market, every n-tuple $x \in \mathbb{R}^n$.

Economic interpretation. If $j \in \underline{n}$, the j-th component of x is the quantity of the j-th good, bought if it is positive, sold (at overdraft) if x_j is negative.

If we consider a portfolio x and we have to calculate the value of the portfolio in the i-th state of the world, we have simply to calculate the following number

$$v_i(x) = \sum_{j=1}^{n} a_{ij} x_j = R_i(a) \cdot x.$$

Definition (of values-representation). The vector of $\mathbb{R}^m$ defined by

$$ax = (R_i(a) \cdot x)_{i=1}^{m},$$

is called the a-representation of the portfolio x.

Economic interpretation. A portfolio x can be represented by the m-vector ax, its a-representation, whose components are the values that the portfolio x takes on the m states of the world. In these conditions, x is a vector of quantities, ax is a vector of values.

The matrix a generates in a natural way a preference relation.

Definition (the preference relation generated by a**).** We say that a portfolio x is preferred or indifferent to x' with respect to a, and we write $x \succeq_a x'$, if $ax \geq ax'$ (i.e., $a(x - x')$ is a vector with non negative components). In other words, for every state of the world s, the value of x in s is greater or equal to the value of x' in s.

Definition (the price of a portfolio). Let p be an n-vector, the price of a portfolio x relative to p is the product $(p \mid x)_n = p \cdot x$.

Definition (no-arbitrage price vectors) . An n-vector p such that, for every n-portfolios x and x', one has

$$x \succeq_a x' \Rightarrow px \geq px',$$

is said compatible with $\succeq_a$ or a no-arbitrage price vector.

Definition (the a-representation of a system of prices). Let A be the linear operator canonically associated to the matrix a. Let p be an n-vector. An m-tuple $q \in \mathbb{R}^m$, such that, for every portfolio x, we have

$$(p \mid x)_{\mathbb{R}^n} = (q \mid Ax)_{\mathbb{R}^m},$$

is called an a-representation of p in $\mathbb{R}^m$.

Theorem (characterization of the representations of a price-vector). *Let (G, S, a) be a state preference model, and let A be the operator canonically associated to a. Then, an m -vector q is a representation of p if and only if $p = A^*u$, where A^* is the Euclidean-adjoint of A.*

Proof. It is well known that, there exists an operator $A^* : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$(u \mid Ax)_{\mathbb{R}^m} = (A^*u \mid x)_{\mathbb{R}^n},$$

for every m-tuple u and every n-tuple x, A^* is called the Euclidean-adjoint of A (A^* is the operator canonically associated to ${}^t a$). By definition q is a representation of p if and only if

$$(p \mid x)_{\mathbb{R}^n} = (q \mid Ax)_{\mathbb{R}^m},$$

for every portfolio x, or equivalently,

$$(p \mid x)_{\mathbb{R}^n} = (A^*q \mid x)_{\mathbb{R}^n}.$$

The last equality holds if and only if $p = A^*u$, as desired. ■

Economic interpretation. In the state preference model, every price vector $p \in \mathrm{imA}^*$, on the space of quantities $\mathbb{R}^n$, can be represented by the m-vectors q (price-vectors on the space of values $\mathbb{R}^m$) such that $p = A^*q$. The price of x in p can be viewed as the price of its a-representation ax in such q.

Now we can found the **$\mathcal{S}$-linear state preference model**.

Economic model. We consider a situation in which there are n-goods and an m-dimensional continuous infinity of states of the world. Without loss of generality, we assume that, in our model, the set of the states of the world is $\mathbb{R}^m$, and the set of the goods is $\underline{n} = \{k \in \mathbb{N} : k \leq n\}$.

In these conditions, we give the following

Definition (of $\mathcal{S}$-linear state preference model). We define $\mathcal{S}$-linear state preference model a system $(\underline{n}, \mathbb{R}^m, A)$, where $A : \mathbb{R}^n \to \mathcal{S}'_m$ is a linear operator. We call A the values-operator of the model, every n-vector x a portfolio of the model and, for every portfolio x, we call the tempered distribution $A(x)$ the A-representation of x.

Remark. Note that, if x is a portfolio, then it is a linear combination of the canonical basis e of $\mathbb{R}^n$, $x = \sum xe$. Then, $A(x) = \sum xA(e)$, and consequently $\dim A(\mathbb{R}^n) \leq n$.

Definition (of regular portfolio). If x is an n-portfolio, we call x as being A-regular, if its representation $A(x)$ is a regular distribution. If s is a state of the world (in our model s is a real m-vector) and $A(x)$ is a regular tempered distribution generated by a continuous function f_x, we say that $f_x(s)$ is the value of the portfolio x in the state s.

Economic interpretation. The first goal is the presence, in our model, of the analogous of the Arrow-Debreu "contingent claims". Using the canonical $\mathcal{S}$-basis δ, we have $A(x) = \int A(x)\delta$. So we can argue that, for every portfolio x, the A-representation of x is an $\mathcal{S}$-linear combination of the "elementary securities" represented by the elements of the canonical $\mathcal{S}$-basis. This prove that the Dirac $\mathcal{S}$-basis represents the analogous of the family of the Arrow-Debreu contingent claims. In particular, for every state of the world s, the delta centered at s, δ_s, represents the elementary security whose value is 1 in the state of the world s and 0 in every other state of the world.

Example. Assume that the state of a portfolio x is the tempered distribution generated by sin: $A(x) = [\sin]$ then x has value 0 in every state $s = k\pi$, with k an integer.

Example. Let the values of a portfolio x be 3, 7.5 and 4.8 in three distinct states of the world s, t and u, and let the value of x be 0 in every other state of the world. Then the values-representation of x is $3\delta_s + 7.5\delta_t + 4.8\delta_u$.

Definition (of system of prices). A system of prices in the space $\mathbb{R}^n$ is an n-tuple. A system of prices in $\mathcal{S}'_m$ is a smooth function of class $\mathcal{S}$ defined on the set of the state of the world. If p is a system of prices in $\mathbb{R}^n$, then we define the price of a portfolio x in p, as usual, as the product $(p \mid x)_{\mathbb{R}^n} = \sum xp$. If q is a system of prices in $\mathcal{S}'_m$, we define the price of a tempered distribution y as the following product

$$(q \mid y)_{\mathcal{S}'_m} := y(q).$$

Remark (the price of a tempered distribution as superposition). Concerning the preceding definition, note that $y(q) = \int_{\mathbb{R}^m} qydl_m$. In fact, applying, first the definition of integral of an integrable distribution and then the definition of the product by a smooth function of a distribution we have

$$\int_{\mathbb{R}^m} qydl_m = (qy)(1_{\mathbb{R}^n}) = y(1_{\mathbb{R}^n}q) = y(q).$$

But the value $y(q)$ can be interpreted in a more impressive way: it is the superposition of the family q under the system of coefficients y, as defined by Carfì:

$$\int_{\mathbb{R}^m} yq := y(q).$$

Remark (the system of prices as $\mathcal{S}$**-linear functional).** Classically the price-systems in an infinite-dimensional vector space X are the linear functionals on X. Our definition of price-vector in $\mathcal{S}'_m$, although more adherent to the finite dimensional case, returns in the classical definition of system of prices. In fact, let q be a price-system, in our acceptation, we can associate the functional $(q \mid \cdot)_{\mathcal{S}'_m}$, it is linear by definition of addition and multiplication by scalar for tempered distributions. But we can say more: it is $\mathcal{S}$-linear. In the sense of the following

Definition (of $\mathcal{S}$-linear functional). Let $L : \mathcal{S}'_m \to \mathbb{R}$ be a functional. We say that L is an $\mathcal{S}$-linear functional if, for every tempered distribution a on $\mathbb{R}^k$ and for every family v of tempered distributions on $\mathbb{R}^m$ indexed by $\mathbb{R}^k$, we have that the family $L(v) = (L(v_i))_{i\in\mathbb{R}^k}$ is of class $\mathcal{S}$ (that is the function $\mathbb{R}^k \to \mathbb{R} : i \mapsto L(v_i)$ is of class $\mathcal{S}$) and moreover

$$L(\int_{\mathbb{R}^k} av) = \int_{\mathbb{R}^k} aL(v).$$

Theorem. *Let $q \in \mathcal{S}_m$. Then the functional $(q \mid \cdot)_{\mathcal{S}'_m}$ is an $\mathcal{S}$ -linear functional.*

Proof. We have to prove that, for every family v of tempered distributions on $\mathbb{R}^m$ indexed by $\mathbb{R}^k$, the family

$$(q \mid v)_{\mathcal{S}'_m} := ((q \mid v_i)_{\mathcal{S}'_m})_{i\in\mathbb{R}^k}$$

is a of class $\mathcal{S}$ and moreover that

$$(q \mid \int_{\mathbb{R}^k} av)_{\mathcal{S}'_m} = \int_{\mathbb{R}^k} a(q \mid v)_{\mathcal{S}'_m}.$$

Concerning the first point, the function

$$\mathbb{R}^k \to \mathbb{R} : i \mapsto (q \mid v_i)_{\mathcal{S}'_m} = v_i(q),$$

is the function $v(q)$, that is of class $\mathcal{S}$ since v is of class $\mathcal{S}$ (and q is of class $\mathcal{S}$).

For the second point, we have

$$\begin{aligned}(q \mid \int_{\mathbb{R}^k} av)_{\mathcal{S}'_m} &= \left(\int_{\mathbb{R}^k} av\right)(q) = a(\widehat{v}(q)) = \int_{\mathbb{R}^k} a(v_i(q))_{i\in\mathbb{R}^k} = \\ &= \int_{\mathbb{R}^k} a((q \mid v_i)_{\mathcal{S}'_m})_{i\in\ \mathbb{R}^k} = \int_{\mathbb{R}^k} a(q \mid v)_{\mathcal{S}'_m}. \blacksquare\end{aligned}$$

We are searching for a system of prices in the space of values-representations, such that, in this system of prices, the price of the A-representation of x is the price $\sum xp$.

More precisely, we give the following

Definition (the A-representation of a system of prices). Let $(\underline{n}, \mathbb{R}^m, A)$ be an $\mathcal{S}$-linear state preference model a system. Let p be an n-vector. A smooth function of class $\mathcal{S}$ on $\mathbb{R}^m$ q, such that, for every portfolio x, we have

$$(p \mid x)_{\mathbb{R}^n} = (q \mid Ax)_{\mathcal{S}'_m},$$

is called an A-representation of p in $\mathcal{S}'_m$.

We shall prove a characterization of the representation of a system of prices analogous to that presented in the case of the classical state preference model. To this end, we have to examine more deeply the $\mathcal{S}$-linear functionals on $\mathcal{S}'_m$.

First of all, we shall prove that our model covers totally the standard infinite dimensional state preference model (in the case presented in the paper). As it is shown in the following remark.

Remark (confrontation with the classic infinite dimensional state preference model). Let q be an A-representation of a system of prices p. Denote by L_q the $\mathcal{S}$-linear functional $(q \mid \cdot)_{\mathcal{S}'_m}$. We have

$$L_q(A(x)) = (q \mid A(x))_{\mathcal{S}'_m} = (p \mid x)_{\mathbb{R}^n},$$

and then

$$L_q \circ A = (p \mid \cdot)_{\mathbb{R}^n}.$$

We desire to prove a more deep circumstance, first of all it is possible to consider the transpose of A. In fact, A is continuous because it is linear and defined on a finite dimensional vector space. So, it is transposable and his transpose is from $\mathcal{S}''_m$ to $(\mathbb{R}^m)^*$ (actually, in order that $L \circ A$ be in $(\mathbb{R}^m)^*$, is enough the linearity of A). Concluding A is transposable. The transpose of A is defined as follows ${}^tA(L) = L \circ A$, for every strongly-continuous functional on $\mathcal{S}'_m$.

Moreover, an $\mathcal{S}$-linear functional l is a strongly-continuous linear functional. In fact, for every $\mathcal{S}$-family v, the image $l(v)$ is of class $\mathcal{S}$. Moreover, considering the function

$$f_l : \mathbb{R}^m \to \mathbb{R} : f_l(i) = l(\delta_i),$$

we have that f_l is of class $\mathcal{S}$, since l is of class $\mathcal{S}$. Moreover, for every $w \in \mathcal{S}'_m$, we have

$$l(w) = l(\int w\delta) = \int wl(\delta) = \int f_l w d\mu_m = w(f_l) = (f_l)^{**}(w),$$

where by $(\cdot)^{**}$ we denoted the canonical injection of $\mathcal{S}_m$ in its bidual. Then $l = (f_l)^{**} \in \mathcal{S}''_m$, so we can write

$${}^tA(L_q) = (p \mid \cdot)_{\mathbb{R}^n}.$$

Concluding, if q is an A-representation in our model of a system of prices p, then L_q is an A-representation of p in the sense of the classic infinite dimensional state preference model. The converse is wholly analogous.

Economic interpretation. The system of prices q, on the space of the A-representations, is an A-representation of the system of prices p if and only if satisfies the following transformation rule

$${}^tA(L_q) = L_q \circ A = (p \mid \cdot)_{\mathbb{R}^n}.$$

Note that, we can also write, in a more impressive way

$$\sum_n xp = (p \mid x)_{\mathbb{R}^n} = L_q(A(x)) = \int_{\mathbb{R}^m} A(x)L_q(\delta) = (A(x) \mid L_q(\delta))_{\mathcal{S}'_m}.$$

Remark. Note that $\mathcal{S}_n$ is a reflexive space, and then, for every $L \in \mathcal{S}''_m$, there is an $\mathcal{S}$ -function f such that $f^{**} = L$. Then, every element of the bidual is an $\mathcal{S}$-linear functional and then, it is an A-representation of the system p if and only if we have

$$\sum_n xp = \int_{\mathbb{R}^m} A(x)L(\delta).$$

Theorem (on the existence of the Euclidean adjoint). *Let A be a linear operator. Then, there exists a unique operator $A^* : \mathcal{S}_m \to \mathbb{R}^n$ such that*

$$(q \mid Ax)_{\mathcal{S}'_m} = (A^* q \mid x)_{\mathbb{R}^n}.$$

Proof. Let q be in $\mathcal{S}_m$, the functional on $\mathbb{R}^n$, defined by $x \mapsto (q \mid A(x))_{\mathcal{S}'_m}$ is a linear functional and then, by the representation theorem, there is a n -vector q^* such that $(q \mid A(x))_{\mathcal{S}'_m} = (q^* \mid x)_{\mathbb{R}^n}$ for every n-tuple x. Moreover, q^* is unique, putting $A^*(q) = q^*$ we conclude.

Definition (of the Euclidean adjoint). Let A be a linear operator. The unique operator $A^* : \mathcal{S}_m \to \mathbb{R}^n$ such that

$$(f \mid Ax)_{\mathcal{S}'_m} = (A^* f \mid x)_{\mathbb{R}^n},$$

is called the Euclidean-adjoint of A.

In this way we obtain, in the infinite-dimensional case, the same result, of the finite dimensional one, concerning the representation of a price-system, as show the below considerations.

Let p be a system of prices in $\mathbb{R}^n$, p is said A-representable if and only if there is a system of prices q on $\mathcal{S}'_m$ such that $(q \mid Ax)_{\mathcal{S}'_m} = (p \mid x)_{\mathbb{R}^n}$ for every portfolio x.

We have proved as well the following

Theorem. *A price-system p is A -representable if and only if $p \in \mathrm{imA}^*$, i.e., if there exists at least a smooth function q of class $\mathcal{S}$, such that $p = A^*(q)$.*

Concerning the preference relation of a state preference model we have the following.

Definition . Let $A : \mathbb{R}^n \to \mathcal{S}'_m$ be a linear operator. We define preference relation generated by A, the relation on $\mathbb{R}^n$ defined by

$$x \succeq_A x' \Leftrightarrow A\,(x - x') \geq 0_{\mathcal{S}'_m},$$

where, we recall that, a tempered distribution u is said non-negative if and only, for every non-negative $\phi \in \mathcal{S}_m$, we have $u\,(\phi) \geq 0$.

Remark. If v is a linear functional on $\mathbb{R}^n$, there exists only a $p \in \mathbb{R}^n$ such that $v = (p \mid \cdot)_{\mathbb{R}^n}$. Analogously, if L is an $\mathcal{S}$-linear functional on $\mathcal{S}'_m$, by reflexivity of $\mathcal{S}_m$, there exists only a $q \in \mathcal{S}_m$ such that $L = (q \mid \cdot)_{\mathcal{S}'_m}$.

Definition. We say that a functional v on $\mathbb{R}^n$ preserves $\succeq_A$ if and only if

$$x \succeq_A x' \Leftrightarrow v\,(x) \geq v\,(x')\,.$$

Proposition. *Let v be a linear functional on $\mathbb{R}^n$. Then v preserves $\succeq_A$ if and only if*

$$x \succeq_A 0_{\mathbb{R}_m} \Leftrightarrow v\,(x) \geq 0.$$

Proof. ($\Rightarrow$) Obvious. ($\Leftarrow$) Let $x \succeq_A y$ this implies $x - y \succeq_A 0_{\mathbb{R}_m}$, (note that, by definition, $x \succeq_A y$ if and only if $A\,(x - y - 0_{\mathbb{R}^n}) \geq 0_{\mathcal{S}'_m}$,and this is , once more by definition, equivalent to $x - y \succeq_A 0_{\mathbb{R}_m}$), and thus $v\,(x - y) \geq 0$, that is $v\,(x) \geq v(y)$.

Theorem. *Let $p \in \mathbb{R}^n$ be such that $p \in \mathrm{im}\,(\mathrm{A}^*)$. Then, $(p \mid \cdot)_{\mathbb{R}^n}$ preserves $\succeq_A$ if and only if $q \in \mathcal{S}_m$ and $p = A^*(q)$ implies $q \geq 0$ on the subspace* $\mathrm{im}(\mathrm{A})$.

Proof. ($\Rightarrow$). If $(p \mid \cdot)_{\mathbb{R}^n}$ preserves $\succeq_A$ then, for every $x \in \mathbb{R}^n$ such that $x \succeq_A 0_{\mathbb{R}_m}$ we have $(p \mid x)_{\mathbb{R}^n} \geq 0$. Since $p \in \mathrm{im}\,(\mathrm{A}^*)$, there is a smooth function q such that $p = A^*(q)$, hence $(A^*q \mid x)_{\mathbb{R}^n} \geq 0$. By definition of Euclidean adjoint, we have $(q \mid Ax)_{\mathcal{S}'_m} \geq 0$; and then, we have $q \geq 0$ on $\mathrm{im}(\mathrm{A})$, in the sense that $(q \mid y)_{\mathcal{S}'_m} \geq 0$, for every $y \in \mathrm{im}(\mathrm{A})$. The converse is wholly analogous.

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