Availability analysis of series systems with cold standby components and general repair time

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Abstract. We study the availability analysis of two different series system configurations with cold standby components and general repair times. The time-to-failure for each of the primary components is assumed to be exponentially distributed with parameter λ. This article gives a recursive method using the supplementary variable technique and treating the supplementary variable as the remaining repair time, to develop the steady-state probability distribution of the number of working components in the system. We obtain the explicit expressions for the steady-state availability for two configurations and perform comparisons. For the two configurations comparisons are made for specific values of distribution parameters. The configurations are ranked based on availability (Av) for two various repair time distributions: exponential and deterministic where benefit is availability.

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§1. Introduction

In this article, we use a supplementary variable technique to study the availability analysis of two different series system configurations with cold standby components. The steady-state availability (Av) has widely been analyzed in the literature because of its prevalence in power plants, manufacturing systems and industrial systems. Maintaining a high or required level of availability is often an essential requisite. A standby component is called a cold standby if its failure is zero. Primary components can be considered to be repairable.

The supplementary variable technique was first proposed by Cox (1955) and it had been widely applied to the M/G/1 queueing system by Cohen (1969) Hokstad (1975), Keilson and Kooharian (1960), Takacs (1963) and many others.

The problem considered in this paper is to find explicit expressions for the availability to series system with repair time distribution of the general type. We give a recursive method using the supplementary variable technique and treating the supplementary variable as the remaining repair time and develop the Av_i for configuration

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i where $i = 1, 2$. Next, for each configuration the explicit expressions for the Av for two different repair time distributions such as exponential (M) and deterministic (D) are provided. Finally, we rank the two configurations for the Av based on assumed numerical values given to the system parameters.

§2. Description of the system

We consider the requirements of a 30 MW power plant for the sake of discussions. We assume that generators are available in units of 30 MW and 15 MW. We also assume that standby generators are not allowed to fail while inactive before they put into full operations. It is assumed that all switchover times are instantaneous and switching is perfect, e.g. primary components can be considered to be repairable. Suppose that each of the primary components fails independently of the state of the others and has an exponential time-to-failure distribution with parameter λ. Whenever one of these components fails, it is immediately replaced by a cold standby component. It is assumed that the time-to-repair of the components are independent and identically distributed (i.i.d.) random variables having a distribution $B(u)$ $(u \geq 0)$, a probability density function $b(u)$ $(u \geq 0)$ and mean service time.

If one component is in repair, then arriving failed components have to wait in the queue until the server is available. Let us assume that failed components arriving at the server form a single waiting line and are served in the order of their arrivals, i.e. according to the first-come-first-served basis. Once a component is repaired, it is as good as new. Suppose that the server can serve only one primary component at a time, and that the server is independent of the arrival of components. We give below the two configurations of the system.

We consider two configurations as follows. The first configuration is a series system of one primary 30 MW component with one cold standby 30 MW component. The second configuration is a series system of two primary 15 MW components and one cold standby 15 MW component. The standby unit can replace either one of the initially working units in case of failure.

§3. Availability analysis of the system

We use the following supplementary variable:
U = remaining repair time for the component under repair.
The state of the system at time t is given by
$N(t)$ = number of working components in the system and
$U(t)$ = remaining repair time for the component being repaired.
Let us define

$$P_n(u,t)du = Pr[N(t) = n, u < U(t) < u + du], u \geq 0.$$

$$P_n(t) = \int_0^\infty P_n(u,t)du$$

3.1 Availability for configuration

Relating the state of the system at time t and $t + dt$, we get

$$\frac{d}{dt}P_2(t) = -\lambda P_2 + P_1(0,t) \tag{3.1}$$

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial u})P_1(u,t) = -\lambda P_1(u,t) + \lambda P_2(u,t) + b(u)P_0(0,t) \tag{3.2}$$

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial u})P_0(u,t) = \lambda P_1(u,t). \tag{3.3}$$

In steady-state let us define

$$P_n = \lim_{t\to\infty} P_n(t), n = 0,1,2\dots$$

$$P_n(u) = \lim_{t\to\infty} P_n(u,t), n = 0,1,2\dots$$

and further define

$$P_2(u) = P_2 b(u) \tag{3.4}$$

From (1)-(4), the steady-state equations are given by

$$0 = -\lambda P_2 + P_1(0) \tag{3.5}$$

$$-\frac{d}{du}P_1(u) = -\lambda P_1(u) + \lambda P_2(u)b(u) + P_0(0)b(u) \tag{3.6}$$

$$-\frac{d}{du}P_1(u) = \lambda P_1(u). \tag{3.7}$$

From (1) it follows that

$$P_1(0) = \lambda P_2. \tag{3.8}$$

Define

$$\begin{aligned} B^*(s) &= \textstyle\int_0^\infty e^{-su}dB(u) = \int_0^\infty e^{-su}b(u)du \\ P^*(s) &= \textstyle\int_0^\infty e^{-su}P_n(u)du \\ P_n &= P_n^*(0) = \textstyle\int_0^\infty P_n(u)du \end{aligned}$$

and

$$\int_0^\infty e^{-su}\frac{d}{du}P_n(u)du = sP_n^*(s) - P_n(0).$$

Taking Laplace Stieltjes transforms on both sides of (6)-(7) and using (8) we obtain

$$(\lambda - s)P_1^*(s) = P_1(0)B^*(s) + P_0(0)B^*(s) - P_1(0) \tag{3.9}$$

$$-sP_0^*(s) = \lambda P_1^*(s) - P_0(0). \tag{3.10}$$

A recursive method is used to develop the steady-state solutions $P_n^*(0), (n = 0, 1)$. Put $s = \lambda$ and $s = 0$ in (9) respectively to obtain

$$P_0(0) = \frac{\lambda[1 - B^*(s)]}{B^*(\lambda)} P_2 \tag{3.11}$$

and

$$P_1^*(0) = \frac{P_0(0)}{\lambda} = \frac{[1 - B^*(s)]}{B^*(\lambda)} P_2. \tag{3.12}$$

Differentiating (10) with respect to s and putting $s = 0$ finally gives

$$P_0^*(0) = -\lambda P_1^{*(1)}(0). \tag{3.13}$$

Similarly, differentiating (9) with respect to s and putting $s = 0$ gives

$$\lambda P_1^{*(1)}(0) = P_1^*(0) + b_1[P_0(0) + P_1(0)] \tag{3.14}$$

where $b_1 = -B^{*(1)}(0)$ denotes the mean repair time. From (13)-(14) we get

$$P_0^*(0) = -P_1^*(0) - b_1[P_0(0) + P_1(0)] \tag{3.15}$$

and (11)-(12) on (15) gives finally

$$P_0^*(0) = \frac{P_2}{B^*(\lambda)}[b_1\lambda - 1 + B^*(s)] \tag{3.16}$$

To find P_2 we substitute (12) and (16) in the normalizing condition

$$P_2 + P_1^*(0) + P_0^*(0) = 1$$

which yields

$$P_2 = \frac{B^*(\lambda)}{[b_1\lambda + B^*(\lambda)]} \tag{3.17}$$

Thus

$$P_0^*(0) = \frac{\lambda b_1 - 1 + B^*(\lambda)}{b_1\lambda + B^*(\lambda)}. \tag{3.18}$$

For configuration 1, the explicit expressions for the Av_1 is given by

$$Av_1 = 1 - P_0^*(0) = \frac{1}{b_1\lambda + B^*(\lambda)}. \tag{3.19}$$

3.1.1 Special cases

We present two special cases for two different repair time distributions such as exponential (M) and deterministic (D). The explicit expressions for Av_1 for two different repair time distributions such as exponential and deterministic are given in the following cases.

Case 1: The repair time has exponential distribution. We set the mean repair time $b_1 = \frac{1}{\mu}$, where μ is the repair rate. In this case, we have $B^*(s) = \frac{\mu}{\mu+s}$. From (19) the explicit expression for the $Av_1(M)$ is given by

$$Av_1(M) = \frac{\mu(\lambda+\mu)}{(\lambda)^2 + \mu(\lambda+\mu)} \tag{3.20}$$

Case 2: The repair time distribution is deterministic. We set the mean repair time $b_1 = \frac{1}{\mu}$, In this case, we have $B^*(s) = e^{\frac{s}{\mu}}$. Hence it follows from (19) that the explicit expression for the is $Av_1(D)$ is given by

$$Av_1(D) = \frac{\mu}{\lambda + \mu e^{-\frac{\lambda}{\mu}}}. \tag{3.21}$$

3.2 Availability for configuration 2

Following the same procedure given in the section that analyzes the configuration 1 case, we set up the steady-state equations as follows.

$$0 = -2\lambda P_3 + P_2(0) \tag{3.22}$$

$$-\frac{d}{du}P_2(u) = -2\lambda P_2(u) + 2\lambda P_3 b(u) + P_1(0)b(u) \tag{3.23}$$

$$-\frac{d}{du}P_1(u) = 2\lambda P_2(u) \tag{3.24}$$

where we define

$$P_3(u) = P_3 b(u). \tag{3.25}$$

From (22) we have

$$P_2(0) = 2\lambda P_3. \tag{3.26}$$

Taking Laplace Stieltjes transforms on both sides of (23)-(24) and using (26) we have

$$(2\lambda - s)P_2^*(s) = P_2(0)B^*(s) + P_1^*(s)B^*(s) - P_2(0) \tag{3.27}$$

$$-sP_1^*(s) = 2\lambda P_2^*(s) - P_1(0). \tag{3.28}$$

Put $s = 2\lambda$ and $s = 0$ in (27) we get

$$P_1^*(0) = \frac{2\lambda(1 - B^*(2\lambda))}{B^*(2\lambda)} P_3 \tag{3.29}$$

and

$$P_2^*(0) = \frac{(1 - B^*(2\lambda))}{B^*(2\lambda)} P_3. \tag{3.30}$$

Differentiating (28) with respect to s and putting $s = 0$ we get

$$P_1^*(0) = -2\lambda P_2^{*(1)}(0). \tag{3.31}$$

Similarly differentiating (27) with respect to s and putting $s = 0$ we obtain

$$2\lambda P_2^{*(1)} = P_2^*(0) - b_1[P_2(0) + P_1(0)]. \tag{3.32}$$

As well, from (31)-(32) we infer that

$$P_1^{*(0)} = -P_2^*(0) + b_1[P_2(0) + P_1(0)]. \tag{3.33}$$

Using (26) and (29)-(30) we yield

$$P_2^*(0) = \frac{(1 - B^*(2\lambda))}{B^*(2\lambda)} P_3$$

with

$$P_1^*(0) = \frac{2\lambda b_1 + B^*(2\lambda) - 1}{B^*(2\lambda)} P_3. \tag{3.34}$$

In order to find P_3, we use (30) and (34) in the normalizing condition

$$P_3 + P_2^*(0) + P_1^*(0) = 1$$

to give

$$P_3 = \frac{B^*(2\lambda)}{2\lambda b_1 + B^*(2\lambda)}. \tag{3.35}$$

Hence from (34) we obtain

$$P_1^*(0) = \frac{2\lambda b_1 + B^*(2\lambda) - 1}{B^*(2\lambda)}. \tag{3.36}$$

Thus for configuration 2, the explicit expression for the Av_2 is given by

$$Av_1 = 1 - P_0^*(0) = \frac{1}{2b_1\lambda + B^*(2\lambda)}. \tag{3.37}$$

3.2.1 Special cases

For configuration 2, we consider two special cases for two different repair time distribution such as exponential (M) and deterministic (D). We give the explicit expression for the Av_2(M) and Av_2(D) for two different repair time distributions: exponential and deterministic respectively.

(3.38) $$Av_2(M) = \frac{2\mu\lambda + (\mu)^2}{4\lambda^2 + 2\lambda\mu + (\mu)^2}$$

(3.39) $$Av_2(D) = \frac{\mu}{2\lambda + \mu e^{-\frac{2\lambda}{\mu}}}.$$

§4. Comparison of the two configurations

4.1 Comparison of the availability

In this section, the computer software, e.g. MATLAB is used to compare the two configurations in terms of their Av_i (i=1, 2) for two different repair time distributions: exponential and deterministic.

We first perform a comparison for the Av of the configuration 1 and 2 when the repair time distribution is exponential or deterministic. We choose $\mu = 1.5$ and vary the values of λ from 0.2 to 1.6. Numerical results of the $Av_i(M)$ and $Av_i(D)$ for configuration $i(i = 1, 2)$ are shown for cases 1 and 2 respectively. Next, we perform a comparison for the Av of the configurations 1 and 2 when the repair time distributions is exponential or deterministic. We choose $\lambda = 0.2$ and vary the values of μ from 0.5 to 3. Numerical results of the $Av_i(M)$ and $Av_i(D)$ for configuration $i(i = 1, 2)$ are shown for cases 1 and 2 respectively.

For the case of exponential distribution with $0.2 < \lambda < 1.6$, $Av_1(M) > Av_2(M)$. Here in both the configurations availability decreases.

For the deterministic case with $0.2 < \lambda < 1.6$, $Av_1(D) > Av_2(D)$. Here also in both the configurations availability decreases.

Similarly, for the case of exponential distribution with $0.5 < \mu < 3$, $Av_1(M) > Av_2(M)$. Here in both the configurations availability increases.

Again, in the case of deterministic distribution with $0.5 < \mu < 3$, $Av_1(D) > Av_2(D)$, with the same result as in case 1.

§5. Conclusion

In this article, we have first used the supplementary variable technique to develop the steady-state availability, Av of two different series system configurations with cold standby components and general repair times. Next, for each configuration, we present the explicit expressions for the Av for two various repair time distributions such as exponential (M) and deterministic (D). Finally we rank the two configurations based on the availability Av for two various repair time distributions.

References

[1] J.W. Cohen, *The single server queue*, North-Holland, Amsterdam, 1969.

[2] D.R. Cox, *The analysis of non-Markovian stochastic processes by the inclusion of supplementary variable*, Proceedings of the Cambridge Philosophical Society 51 (1955), 433-441.

[3] P. Hokstad, *A supplementary variable technique applied to the $M/G/1$ queue*, Scandinavian Journal of Statistics 2 (1975), 95-98.

[4] J. Keilson and A. Kooharian, *On time-dependent queueing processes*, Annals of Mathematical Statistics 31 (1960), 104-112.

[5] L. Takacs, *Delay distributions for one line with Poisson input, general holding times and various orders of service*, Bell System Technique Journal 42 (1963), 487-504.

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