On a seawater intrusion problem

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Abstract. The aim of this paper is to give an existence and uniqueness result of weak solution for a coupled quasilinear elliptic system describing the flow of groundwater in a coastal confined aquifer. The seawater intrusion phenomenon is modelled using the sharp interface approach.

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§1. Introduction

Coastal aquifers serve as major sources for freshwater supply in many countries around the world, especially in arid and semi-arid zones. The fact that coastal zones contain some of the densely populated areas in the world makes the need for freshwater even more acute. The intensive extraction of groundwater from coastal aquifers reduces freshwater outflow to the sea and creates local water table depression, causing seawater to migrate inland and rising toward the wells. This phenomenon, called seawater intrusion, has become one of the major constraints imposed on groundwater utilization. As seawater intrusion progresses, existing pumping wells, especially those close to the coast, become saline and have to be abandoned. Also the area above the intruding seawater wedge is lost as source of natural replenishment to the aquifer. To avert this trend, which is unsustainable in the long-term, it is necessary to use mathematical models to simulate the phenomenon and to make quantitative estimations of the effects of various management decisions should be taken to remedy the problem.

Freshwater and saltwater are miscible fluids and therefore, the zone separating them takes the form of a transition zone caused by hydrodynamic dispersion. For certain problems, the simulation can be simplified by assuming that each liquid is confined to a well defined portion of the flow domain with an abrupt interface separating the two domains (cf. [2, 3]). This modelling approach, best known as sharp interface, does not give information concerning the nature of the transition zone but does reproduce the regional flow dynamics of the system and the response of the interface to applied stresses [3].

In this paper, we address the seawater intrusion problem with sharp interface approach in a confined aquifer. The model to be presented herein is formulated in terms of a two-dimensional coupled quasilinear elliptic system. The main difficulties

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related to the analysis of this system are the coupling between system equations and the degeneracy due to the possibility to have no saltwater or no freshwater in some zones of the aquifer. We prove, by mean of a fixed point argument, the existence of at least one weak solution. The necessary estimates are established using the Stampacchia trunction technique. To prove the uniqueness of the solution, we make use of the transposition method [1]. This method consists in studying the existence of the solution of a dual problem, in order to prove the uniqueness of the primal one.

The outline of the paper is as follows. In section 2 the governing system as well as it's variational formulation is presented. In the third section we prove the existence of at least one weak solution of the problem using the Schauder fixed point theorem. The forth section is devoted to prove better regularity result for this solution which enables us to obtain, using the transposition method, the uniqueness result in the last section.

§2. Governing system

Let us consider a stationnary flow, with predominantly horizontal components, of fresh and salt groundwater, separated by sharp interface, in a heterogeneous confined aquifer. Substituting Darcy's law into continuity equations of the two fluids (fresh and salt), using the continuity of flux and the pressure through the interfacial boundary and integrating equations over the vertical lead to the following system of coupled differential equations [3, 2]:

$$\left\{\begin{array}{lcll} -div(K(x)B_f(x,\phi_f,\phi_s)\nabla\phi_f) & = & Q_f & \text{on } \Omega, \\ -div(K(x)B_s(x,\phi_f,\phi_s)\nabla\phi_s) & = & Q_s & \text{on } \Omega, \end{array}\right. \tag{2.1}$$

where Ω is an open bounded domain of $\mathbb{R}^2$, with piecewise smooth boundary Γ, describing the projection of the porous medium on the horizontal plane; ϕ_f is the freshwater head; ϕ_s is the saltwater head; Q_f and Q_s are the freshwater and saltwater inflow/outflow terms; K is the hydraulic conductivity, assumed the same for both freshwater and saltwater. $B_f(x,\phi_f,\phi_s)$ and $B_s(x,\phi_f,\phi_s)$ represent the thicknesses of freshwater zone and saltwater zone respectively and they are given by:

$$B_f(x,\phi_f,\phi_s) = \xi_t(x) - \xi_i(\phi_f,\phi_s), \tag{2.2}$$

$$B_s(x,\phi_f,\phi_s) = \xi_i(\phi_f,\phi_s) - \xi_b(x), \tag{2.3}$$

where ξ_b and ξ_t are the elevations of the lower and the upper surfaces of the aquifer, respectively (figure 1). The interface elevation, ξ_i, is given by the following relation:

$$\xi_i(\phi_f,\phi_s) = (1+\delta)\phi_s - \delta\phi_f, \tag{2.4}$$

where $\delta = \frac{\rho_f}{\rho_s-\rho_f}$ with ρ_f and ρ_s are two positive constants representing freshwater and saltwater densities.

To resolve system (2.1)-(2.4), we consider nonhomogeneous Dirichlet boundary conditions:

$$\left\{\begin{array}{ll} \phi_f = g_f & \text{on } \Gamma \\ \phi_s = g_s & \text{on } \Gamma \end{array}\right. \tag{2.5}$$

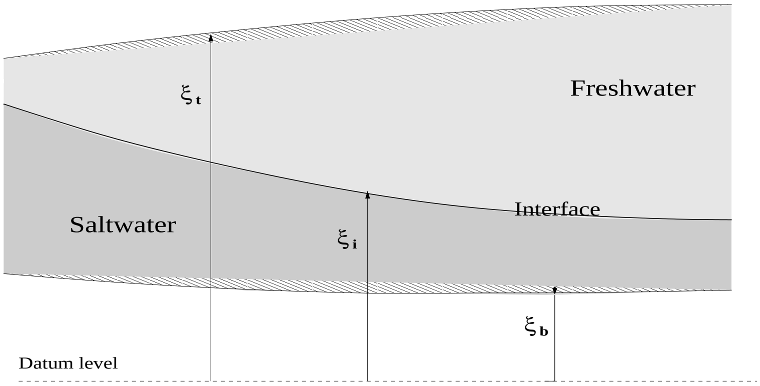

Figure 1: freshwater/saltwater sharp interface in confined aquifer

where g_f and g_s are given functions.

Hence, the variational formulation of (2.1)-(2.5) reads:

$$(2.6)\qquad \begin{cases} \text{Find } \phi=(\phi_f,\phi_s)\in V_g \text{ such that} \\ \int_\Omega K(x)B_f(x,\phi_f,\phi_s)\nabla\phi_f\nabla\varphi_f = \int_\Omega Q_f\varphi_f \quad \forall\varphi_f\in H_0^1(\Omega) \\ \int_\Omega K(x)B_s(x,\phi_f,\phi_s)\nabla\phi_s\nabla\varphi_s = \int_\Omega Q_s\varphi_s \quad \forall\varphi_s\in H_0^1(\Omega) \end{cases}$$

where $V_g = V_{g_f}\times V_{g_s}$, with $V_{g_e} = H_0^1(\Omega)+g_e$ $(e=f,s)$.

§3. Existence result

In this section we prove the existence of at least one solution of (2.6) using a fixed point theorem. Our hypothesis are as follows:

$(H1)$ There exists $K_m\in\mathbb{R}$ and $K_M\in\mathbb{R}$ such that

$$K(x)\in L^\infty(\Omega) \qquad \text{and} \qquad 0<K_m\le K(x)\le K_M \quad \text{a.e. } x\in\Omega.$$

$(H2)$ $Q=(Q_f,Q_s)\in L^4(\Omega)^2$, $(\xi_b,\xi_t)\in L^\infty(\Omega)^2$ and $g=(g_f,g_s)\in H^1(\Omega)^2$.

$(H3)$ There exists $\sigma>0$, $R=(R_f,R_s)\in\mathbb{R}^2$ and $R'=(R'_f,R'_s)\in\mathbb{R}^2$ such that:

$$R'>R \quad (\text{ie. } R'_f>R_f \text{ and } R'_s>R_s),$$

$$((1+\delta)R'_s-\delta R_f)\le \operatorname*{infess}_\Omega \xi_t-\sigma,$$

$$((1+\delta)R_s-\delta R'_f)\ge \operatorname*{supess}_\Omega \xi_b+\sigma.$$

$(H4)$ There exists a real $r > 2$ such that

$$C(r)\frac{\|Q_f\|_{L^4(\Omega)}}{K_m\sigma} \leq min\{R_f^{'} - g_f^M, g_f^m - R_f\},$$

$$C(r)\frac{\|Q_s\|_{L^4(\Omega)}}{(1+\alpha)K_m\sigma} \leq min\{R_s^{'} - g_s^M, g_s^m - R_s\},$$

where g_e^M and g_e^m are respectively the maximum and the minimal value of g_e on Γ $(e = f, s)$ and $C(r) = c(4)c(r)mes(\Omega)^{1/2-1/r}2^{r/(r-2)}$, with $c(r)$ denoting the best constant associated with the inequality:

$$\|\varphi\|_{L^r(\Omega)} \leq c(r)\|\nabla\varphi\|_{L^2(\Omega)} \quad \forall\varphi \in H_0^1(\Omega).$$

Remark 3.1. *Physically, we often have* $\underset{\Omega}{infess}\, \xi_2 > \underset{\Omega}{supess}\, \xi_0$, *and then* $(\underset{\Omega}{infess}\, \xi_2 - \sigma) > \mu(\underset{\Omega}{supess}\, \xi_0 + \sigma)$ *for some* $\mu > 1$ *and* $\sigma > 0$. *Hence, in order to justify* $(H2)$, *it suffices to set* $R_f' = \mu R_f$ *and* $R_s' = \mu R_s$, *with*

$$R_f = \frac{(\underset{\Omega}{infess}\, \xi_2 - \sigma) - \mu(\underset{\Omega}{supess}\, \xi_0 + \sigma)}{\delta(\mu^2 - 1)}$$

and

$$R_s = \frac{\mu(\underset{\Omega}{infess}\, \xi_2 - \sigma) - (\underset{\Omega}{supess}\, \xi_0 + \sigma)}{(1+\delta)(\mu^2 - 1)}.$$

The existence result is the following:

Theorem 3.1. *Under assumptions* $(H1) - (H4)$, *the variational problem (2.6) has at least one solution* $(\phi_f,\, \phi_s)$.
Moreover:

$$\sigma \leq B_f(x, \phi_f, \phi_s) \leq (\xi_t - \xi_b - \sigma), \tag{3.7}$$

$$\sigma \leq B_s(x, \phi_f, \phi_s) \leq (\xi_t - \xi_b - \sigma), \tag{3.8}$$

$$\|\nabla\phi_f\|_{L^2(\Omega)} \leq C(g_f, Q_f, \sigma, K_m, \Omega), \tag{3.9}$$

$$\|\nabla\phi_s\|_{L^2(\Omega)} \leq C(g_s, Q_s, \sigma, K_m, \Omega), \tag{3.10}$$

where $C(g_e, Q_e, \sigma, K_m, \Omega)$ *is a constant depending only on* g_e, Q_e, K_m, σ, *and* Ω $(e = f, s)$.

Proof. For R and R' satisfying $(H3)$, let us introduce the following closed of $L^2(\Omega)^2$:

$$B_{RR'} = \{\phi \in L^2(\Omega)^2;\ R \leq \phi(x) \leq R' \text{ a.e. } x \in \Omega\}.$$

By $(H1)$ and $(H3)$, for all $\phi = (\phi_f, \phi_s) \in B_{RR'}$, we have:

$$K_m\sigma \leq K(x)B_f(x, \phi_f, \phi_s) \leq K_M(\xi_t - \xi_b - \sigma), \tag{3.11}$$

$$K_m\sigma \leq K(x)B_s(x, \phi_f, \phi_s) \leq K_M(\xi_t - \xi_b - \sigma). \tag{3.12}$$

Therefore, the application T defined by:

$$\begin{aligned} T: \qquad & B_{RR'} \longrightarrow L^2(\Omega)^2 \\ & \phi = (\phi_f, \phi_s) \longrightarrow T\,\phi = u = (u_f, u_s), \end{aligned}$$

where u is a solution of the following linear problem:

$$\begin{cases} \text{Find } \ u = (u_f, u_s) \in V_g \text{ s.t.} \\ \int_\Omega K(x)B_f(x, \phi_f, \phi_s)\nabla u_f \nabla\varphi_f = \int_\Omega Q_f\varphi_f \quad \forall\varphi_f \in H_0^1(\Omega), \\ \int_\Omega K(x)B_s(x, \phi_f, \phi_s)\nabla u_s \nabla\varphi_s = \int_\Omega Q_s\varphi_s \quad \forall\varphi_s \in H_0^1(\Omega), \end{cases} \tag{3.13}$$

is well defined. Moreover, $u = T\,\phi$ satisfy the following estimates:

$$\|\nabla u_f\|_{L^2(\Omega)} \leq C(g_f, Q_f, \sigma, K_m, \Omega) \tag{3.14}$$

$$\|\nabla u_s\|_{L^2(\Omega)} \leq C(g_s, Q_s, \sigma, K_m, \Omega), \tag{3.15}$$

where $C(g_e, Q_e, \sigma, K_m, \Omega)$ is a constant depending only on g_e, Q_e, σ, K_m, and Ω ($e = f, s$).
Hence, any fixed point of T will be a solution of (2.6) satisfying (3.7)-(3.10).

In lemmas 3.2-3.3, we shall prove that the hypothesis of the well-known Schaudre fixed point theorem [6] are fulfilled for T. □ □

Lemma 3.1. *[4, Lemma B.1.] Let $f(t)$, $k_0 \leq t < \infty$, be nonnegative and nonincreasing function such that*

$$f(s) \leq [C/(s-k)^\gamma]|f(k)|^\beta, \qquad s > k > k_0,$$

where C, γ, β are positive constants with $\beta > 1$. Then

$$f(k_0 + d) = 0,$$

where

$$d^\gamma = C|f(k_0)|^{\beta-1}2^{\gamma\beta/(\beta-1)}.$$

Lemma 3.2. $T(B_{RR'}) \subset B_{RR'}$.

Proof. Since $V_g \subset L^2(\Omega)^2$, it suffices to show that, for any u solution of (3.13), we have:

$$R \leq u(x) \leq R' \qquad \text{a.e. } x \in \Omega.$$

For any real constant $N \geq g_f^M$ we set

$$\psi_f^N = (u_f - N)^+ \text{ and } \Omega(N) = \{x \in \Omega : \, u > N\}.$$

Taking $\varphi_f = \psi_f^N \in H_0^1(\Omega)$ as test function in the first equation of (3.13) we obtain:

$$\int_\Omega K(x) B_f(x, \phi_f, \phi_s) \left|\nabla \psi_f^N\right|^2 = \int_\Omega Q_f \psi_f^N.$$

Then

$$\sigma K_m \int_\Omega \left|\nabla \psi_f^N\right|^2 dx \leq \|Q_f\|_{L^4(\Omega)} \|\psi_f^N\|_{L^4(\Omega)} mes(\Omega(N))^{\frac{1}{2}},$$

or

$$\left(\int_\Omega \left|\nabla \psi_f^N\right|^2 dx\right)^{\frac{1}{2}} \leq \frac{1}{\sigma K_m} c(4) \|Q_f\|_{L^4(\Omega)} mes(\Omega(N))^{\frac{1}{2}}, \tag{3.16}$$

where $mes(\Omega(N))$ denotes the Lebesgue measure of $\Omega(N)$.

On the other hand, for any $r > 2$ and $l \geq N \geq g_f^M$, we have

$$\begin{array}{rcl} (l-N)^r mes(\Omega(l)) & = & \int_{\Omega(l)} (l-N)^r dx \quad \leq \quad \int_{\Omega(l)} (u_f - N)^r dx \\ & \leq & \int_{\Omega(l)} \left|\psi_f^N\right|^r dx \quad \leq \quad c(r)^r \|\nabla \psi_f^N\|^r_{L^2(\Omega)}. \end{array} \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$mes(\Omega(l)) \leq \left(\frac{c(4)c(r)\|Q_f\|_{L^4(\Omega)}}{\sigma K_m (l-N)}\right)^r mes(\Omega(N))^{\frac{r}{2}}.$$

Using the previous lemma, we find

$$\Omega(d + g_f^M) = 0,$$

where $d = \frac{1}{\sigma K_m} c(4)c(r)\|Q_f\|_{L^4(\Omega)} mes(\Omega(g_f^M))^{1/2-1/r} 2^{r/(r-2)}$.
Thus

$$u_f(x) \leq g_f^M + \frac{C(r)\|Q_f\|_{L^4(\Omega)}}{\sigma K_m}, \tag{3.18}$$

and thanks to assumption $(H4)$, we deduce

$$u_f(x) \leq R'_f.$$

In order to prove the other inequality, we set $\hat{u_f} = R''_f - u_f$ and $\hat{g_f} = R''_f - g_f$ with $R''_f \geq g_f^M$. So $\hat{u_f} \in V_{\hat{g_f}} = H_0^1(\Omega) + \hat{g_f}$ is a solution of the following variational problem

$$\int_\Omega K(x) B_f(x, \phi_f, \phi_s) \nabla \hat{u_f} \nabla \varphi_f = -\int_\Omega Q_f \varphi_f \qquad \forall \varphi_f \in V_0.$$

Repeating the above estimates for $\hat{u_f}$ and $N \geq R''_f - g_f^m$, leads to

$$\hat{u_f}(x) \leq R''_f - g_f^m + \frac{C(r)\|Q_f\|_{L^4(\Omega)}}{\sigma K_m}.$$

Therefore, by $(H4)$, we deduce

$$u_f(x) \geq R_f \quad \text{a.e. } x \in \Omega.$$

By the same way and using the second inequality of $(H4)$, we get

$$R_s \leq u_s(x) \leq R'_s.$$

Then

$$T(B_{RR'}) \subset B_{RR'}.$$

□

□

Lemma 3.3. *T is continuous from $B_{RR'}$ to $L^2(\Omega)^2$ and $T(B_{RR'})$ is relatively compact in $L^2(\Omega)^2$.*

Proof. Let $p \in B_{RR'}$ be fixed and $(\phi_n)_{n\geq 0}$ a sequence such that $\phi_n \to \phi$ in $L^2(\Omega)^2$ for $n \to \infty$.

Recall that

$$R \leq \phi(x) \leq R', \qquad R \leq \phi_n(x) \leq R' \qquad \text{a.e.}\, x \in \mathbb{R}^2. \tag{3.19}$$

We set

$$u_n = T\phi_n, \qquad u = T\phi.$$

The sequence $(u_n)_{n\geq 0}$ satisfies:

$$\begin{cases} \int_\Omega K(x)B_f(x,\phi_f^n,\phi_s^n)\nabla u_f^n \nabla\varphi &= \int_\Omega Q_f\varphi, \\ \int_\Omega K(x)B_s(x,\phi_f^n,\phi_s^n)\nabla u_s^n \nabla\varphi &= \int_\Omega Q_s\varphi \qquad \forall\varphi \in H_0^1(\Omega), \end{cases} \tag{3.20}$$

$$\|\nabla u_f^n\|_{L^2(\Omega)} \leq C, \quad \|\nabla u_s^n\|^2_{L^2(\Omega)} \leq C \qquad \forall n \in \mathbb{N}, \tag{3.21}$$

where C is a positive constant not depending on n, which implies the existence of u^* such that, up to a subsequence, u^n converges to u^* weakly in $H^1(\Omega)^2$ and strongly in $L^2(\Omega)^2$.

Moreover, by dominated convergence theorem and taking into account $(H1)$ and (3.19),

$$K(x)B_f(x,\phi_f^n,\phi_s^n)\nabla\varphi \longrightarrow K(x)B_f(x,\phi_f,\phi_s)\nabla\varphi \text{ in } L^2(\Omega) \qquad \forall\varphi \in H_0^1(\Omega),$$

$$K(x)B_s(x,\phi_f^n,\phi_s^n)\nabla\varphi \longrightarrow K(x)B_s(x,\phi_f,\phi_s)\nabla\varphi \text{ in } L^2(\Omega) \qquad \forall\varphi \in H_0^1(\Omega).$$

Passing to the limit in (3.20) and using the uniqueness of solution of (3.13) we have $u^* = u$ and that the entire sequence converges to u. Therefore T is continuous from $B_{RR'}$ to $L^2(\Omega)^2$.

The proof of the relative compactness of the embedding of $T(B_{RR'})$ into $L^2(\Omega)^2$ is elementary. □ □

Finally Theorem 3.1 follows from the Schauder fixed point theorem [6] and Lemmas 3.2-3.3.

§4. Regularity result

In order to prove the uniqueness result, we need to prove better regularity of the solution. That is the first derivatives of $\phi = (\phi_f, \phi_s)$ are in L^p for a $p > 2$.

First, we give a preliminary result which is a consequence of theorem 4.2 page 38 of [5].

Lemma 4.4. *Let $A \in (L^\infty(\Omega))^n$ such that there exists $\gamma > 0$ satisfying*

$$\sum_{i,j=1}^{n} A_{ij}(x) X_i X_j \geq \gamma \|X\|^2 \qquad \forall x \in \Omega \text{ and } X \in \mathbb{R}^n.$$

Then there exists $p = p(\gamma, \bar{A}) > 2$, with $\bar{A} = max_{ij}\|A_{ij}\|_{L^\infty(\Omega)}$, satisfying the following property:
For any $f \in W^{-1,p}(\Omega)$ and $g \in W^{1,p}(\Omega)$, the unique solution y of the problem

$$\begin{cases} \nabla \cdot (A \nabla y) = f & \in \Omega \\ \quad y \in H_0^1(\Omega) + y_D \end{cases}$$

belongs to $W^{1,p}(\Omega)$. Moreover we have the following estimate:

$$\|y\|_{W^{1,p}(\Omega)} \leq c(\Omega,\, f,\, y_D,\, \bar{A},\, \gamma),$$

with $c(\Omega,\, f,\, y_D,\, \bar{A},\, \gamma), \geq 0$ is a constant depending only on Ω, f, y_D, γ and $\bar{A}$.

Applying the above lemma to (2.6) with $A = K(x)B_f(x, \phi_f, \phi_s)$ (resp. $A = K(x)B_s(x, \phi_f, \phi_s)$), $\gamma = K_m \sigma$ and $\bar{A} = \|K(x)B_f(x, \phi_f, \phi_s)\|_{L^\infty(\Omega)}$ (resp. $\bar{A} = \|K(x)B_s(x, \phi_f, \phi_s)\|_{L^\infty(\Omega)}$), leads to the following regularity result.

Proposition 4.1. *Let $\phi = (\phi_f, \phi_s)$ be a solution of (2.6). There exists $r > 2$ such that if*

$$(g_f, g_s) \in W^{1,r}(\Omega)^2,$$

$\nabla \phi_f$ and $\nabla \phi_s$ are in $L^r(\Omega)$.
Moreover we have:

$$\|\nabla \phi_f\|_{L^r(\Omega)} \leq c_f \qquad \text{and} \qquad \|\nabla \phi_s\|_{L^r(\Omega)} \leq c_s,$$

where c_e $(e = f, s)$ is a positive constant depending on Q_e, g_e, K_m, σ, and Ω.

§5. Uniqueness result

In this section we assume that the hypothesis $(H1) - (H4)$ holds. We also need the following supplementary assumptions:

$(H5)$

$$(g_f, g_s) \in W^{1,r}(\Omega)^2,$$

where r is given by proposition (4.1).

$(H6)$

$$\epsilon := \sigma \frac{K_m}{1+\delta} - 2K_M c(r^*) max(c_f, c_s) > 0,$$

where $r* > 0$ with $\frac{1}{r} + \frac{1}{r*} = \frac{1}{2}$.

Now let $\phi = (\phi_f, \phi_s)$ and $\hat{\phi} = (\hat{\phi}_f, \hat{\phi}_s)$ two solutions of (2.6).

Lemma 5.5. *The following adjoint system:*

$$(5.22) \quad \begin{cases} \textit{Find } u = (u_f, u_s) \in H_0^1(\Omega)^2 \textit{ s.t.} \\ \int_\Omega \frac{K(x)}{\delta} B_f(x, \hat{\phi}_f, \hat{\phi}_s) \nabla u_f \nabla v_f + \int_\Omega K(x) \nabla \phi_f \nabla u_f \, v_f \\ - \int_\Omega K(x) \nabla \phi_s \nabla u_s \, v_f = \int_\Omega (\phi_f - \hat{\phi}_f) v_f, \quad \forall v_f \in H_0^1(\Omega) \\ \int_\Omega \frac{K(x)}{1+\delta} B_s(x, \hat{\phi}_f, \hat{\phi}_s) \nabla u_s \nabla v_s + \int_\Omega K(x) \nabla \phi_s \nabla u_s \, v_s \\ - \int_\Omega K(x) \nabla \phi_f \nabla u_f \, v_s = \int_\Omega (\phi_s - \hat{\phi}_s) v_s, \quad \forall v_s \in H_0^1(\Omega) \end{cases}$$

admits one solution.

Proof. (5.22) can be rewritten in the following form:

$$\begin{cases} \text{Find} \quad u \in V \\ \text{s.t.} \quad a(u, v) = L(v) \qquad \forall v \in V, \end{cases}$$

where $v = (v_f, v_s)$, $u = (u_f, u_s)$,

$$\begin{aligned} a(u, v) = \ & \int_\Omega \frac{K(x)}{\delta} B_f(x, \hat{\phi}_f, \hat{\phi}_s) \nabla u_f \nabla v_f + \int_\Omega K(x) \nabla \phi_f \nabla u_f \, v_f \\ & - \int_\Omega K(x) \nabla \phi_s \nabla u_s \, v_f - \int_\Omega K(x) \nabla \phi_f \nabla u_f \, v_s \\ & \int_\Omega \frac{K(x)}{1+\delta} B_s(x, \hat{\phi}_f, \hat{\phi}_s) \nabla u_s \nabla v_s + \int_\Omega K(x) \nabla \phi_s \nabla u_s \, v_s \end{aligned}$$

and

$$L(v) = \int_\Omega (\phi_f - \hat{\phi}_f) v_f + \int_\Omega (\phi_s - \hat{\phi}_s) v_s.$$

We have

$$\begin{aligned} |a(u, v)| \leq \ & \frac{K_M}{\delta} \|\xi_t - \xi_b - \sigma\|_{L^\infty(\Omega)} \|\nabla u_f\|_{L^2(\Omega)} \|\nabla v_f\|_{L^2(\Omega)} \\ & + K_M \|\nabla \phi_f\|_{L^r(\Omega)} \|\nabla u_f\|_{L^2(\Omega)} \|v_f\|_{L^{r*}(\Omega)} \\ & + K_M \|\nabla \phi_s\|_{L^r(\Omega)} \|\nabla u_s\|_{L^2(\Omega)} \|v_f\|_{L^{r*}(\Omega)} \\ & + K_M \|\nabla \phi_f\|_{L^r(\Omega)} \|\nabla u_f\|_{L^2(\Omega)} \|v_s\|_{L^{r*}(\Omega)} \\ & + \frac{K_M}{1+\delta} \|\xi_t - \xi_b - \sigma\|_{L^\infty(\Omega)} \|\nabla u_s\|_{L^2(\Omega)} \|\nabla v_s\|_{L^2(\Omega)} \\ & + K_M \|\nabla \phi_s\|_{L^r(\Omega)} \|\nabla u_s\|_{L^2(\Omega)} \|v_s\|_{L^{r*}(\Omega)}. \end{aligned}$$

Hence, using Sobolev injections, we find:

$$|a(u, v)| \leq C \|\nabla u\|_{L^2(\Omega)^2} \|\nabla v\|_{L^2(\Omega)^2},$$

where C is a constant not depending on u or v.
In the same way we have:

$$\begin{array}{rl}|a(u,u)| \geq & \sigma\frac{K_m}{\delta}\|\nabla u_f\|^2_{L^2(\Omega)} \\ & -K_M\|\nabla\phi_f\|_{L^r(\Omega)}\|\nabla u_f\|_{L^2(\Omega)}\|u_f\|_{L^{r*}(\Omega)} \\ & -K_M\|\nabla\phi_s\|_{L^r(\Omega)}\|\nabla u_s\|_{L^2(\Omega)}\|u_f\|_{L^{r*}(\Omega)} \\ & -K_M\|\nabla\phi_f\|_{L^r(\Omega)}\|\nabla u_f\|_{L^2(\Omega)}\|u_s\|_{L^{r*}(\Omega)} \\ & +\sigma\frac{K_M}{1+\delta}\|\nabla u_s\|_{L^2(\Omega)}\|\nabla u_s\|_{L^2(\Omega)} \\ & -K_M\|\nabla\phi_s\|_{L^r(\Omega)}\|\nabla u_s\|_{L^2(\Omega)}\|u_s\|_{L^{r*}(\Omega)},\end{array}$$

then

$$\begin{array}{rl}|a(u,u)| \geq & \sigma\frac{K_m}{\delta}\|\nabla u_f\|^2_{L^2(\Omega)} + \sigma\frac{K_m}{1+\delta}\|\nabla u_s\|^2_{L^2(\Omega)} \\ & -K_Mc(r^*)\|\nabla\phi_f\|_{L^r(\Omega)}\|\nabla u_f\|^2_{L^2(\Omega)} \\ & -K_Mc(r^*)\|\nabla\phi_s\|_{L^r(\Omega)}\|\nabla u_s\|_{L^2(\Omega)}\|\nabla u_f\|_{L^2(\Omega)} \\ & -K_Mc(r^*)\|\nabla\phi_f\|_{L^r(\Omega)}\|\nabla u_s\|_{L^2(\Omega)}\|\nabla u_f\|_{L^2(\Omega)} \\ & -K_Mc(r^*)\|\nabla\phi_s\|_{L^r(\Omega)}\|\nabla u_s\|^2_{L^2(\Omega)}.\end{array}$$

Hence

$$\begin{array}{rl}|a(u,u)| \geq & \sigma\frac{K_m}{\delta}\|\nabla u_f\|^2_{L^2(\Omega)} + \sigma\frac{K_m}{1+\delta}\|\nabla u_s\|^2_{L^2(\Omega)} \\ & -K_Mc(r^*)\|\nabla\phi_f\|_{L^r(\Omega)}\|\nabla u_f\|^2_{L^2(\Omega)} \\ & -2K_Mc(r^*)max(\|\nabla\phi_s\|_{L^r(\Omega)},\|\nabla\phi_s\|_{L^r(\Omega)})\|\nabla u_f\|_{L^2(\Omega)}\|\nabla u_s\|_{L^2(\Omega)} \\ & -K_Mc(r^*)\|\nabla\phi_s\|_{L^r(\Omega)}\|\nabla u_s\|^2_{L^2(\Omega)},\end{array}$$

$$\begin{array}{rl}|a(u,u)| \geq & \sigma\frac{K_m}{\delta}\|\nabla u_f\|^2_{L^2(\Omega)} + \sigma\frac{K_m}{1+\delta}\|\nabla u_s\|^2_{L^2(\Omega)} \\ & -2K_Mc(r^*)max(\|\nabla\phi_s\|_{L^r(\Omega)},\|\nabla\phi_s\|_{L^r(\Omega)})\|\nabla u_f\|^2_{L^2(\Omega)} \\ & -2K_Mc(r^*)max(\|\nabla\phi_s\|_{L^r(\Omega)},\|\nabla\phi_s\|_{L^r(\Omega)})\|\nabla u_s\|^2_{L^2(\Omega)}.\end{array}$$

$$\begin{array}{rl}|a(u,u)| \geq & [\sigma\frac{K_m}{1+\delta} - 2K_Mc(r^*)max(\|\nabla\phi_s\|_{L^r(\Omega)},\|\nabla\phi_s\|_{L^r(\Omega)})] \\ & *(\|\nabla u_f\|^2_{L^2(\Omega)} + \|\nabla u_s\|^2_{L^2(\Omega)}).\end{array}$$

Therefore, by Lax-Milgram theorem and taking into account $(H6)$, the result is deduced. □

□

On the other hand, for any $u = (u_f, u_s) \in H^1_0(\Omega)^2$, we have:

$$\int_\Omega K(x)B_f(x,\phi_f,\phi_s)\nabla\phi_f\nabla u_f = \int_\Omega K(x)B_f(x,\hat{\phi}_f,\hat{\phi}_s)\nabla\hat{\phi}_f\nabla u_f,$$

$$\int_\Omega K(x)B_s(x,\phi_f,\phi_s)\nabla\phi_s\nabla u_s = \int_\Omega K(x)B_s(x,\hat{\phi}_f,\hat{\phi}_s)\nabla\hat{\phi}_s\nabla u_s,$$

then

$$\begin{array}{l}\int_\Omega K(x)[B_f(x,\phi_f,\phi_s) - B_f(x,\hat{\phi}_f,\hat{\phi}_s)]\nabla\phi_f\nabla u_f \\ +\int_\Omega K(x)B_f(x,\hat{\phi}_f,\hat{\phi}_s)\nabla(\phi_f - \hat{\phi}_f)\nabla u_f = 0,\end{array}$$

$$\int_\Omega K(x)[B_s(x,\phi_f,\phi_s)-B_s(x,\hat\phi_f,\hat\phi_s)]\nabla\phi_s\nabla u_s$$
$$\int_\Omega K(x)B_s(x,\hat\phi_f,\hat\phi_s)\nabla(\phi_s-\hat\phi_s)\nabla u_s=0,$$

and since

$$B_f(x,\phi_f,\phi_s)-B_f(x,\hat\phi_f,\hat\phi_s)=\delta(\phi_f-\hat\phi_f)-(1+\delta)(\phi_s-\hat\phi_s),$$

$$B_s(x,\phi_f,\phi_s)-B_s(x,\hat\phi_f,\hat\phi_s)=(1+\delta)(\phi_s-\hat\phi_s)-\delta(\phi_f-\hat\phi_f).$$

we obtain

$$\begin{aligned}&\int_\Omega \tfrac{K(x)}{\delta}B_f(x,\hat\phi_f,\hat\phi_s)\nabla(\phi_f-\hat\phi_f)\nabla u_f+\int_\Omega K(x)(\phi_f-\hat\phi_f)\nabla\phi_f\nabla u_f\\&+\int_\Omega \tfrac{K(x)}{1+\delta}B_s(x,\hat\phi_f,\hat\phi_s)\nabla(\phi_s-\hat\phi_s)\nabla u_s+\int_\Omega K(x)(\phi_s-\hat\phi_s)\nabla\phi_s\nabla u_s\\&-\int_\Omega K(x)(\phi_f-\hat\phi_f)\nabla\phi_s\nabla u_s-\int_\Omega K(x)(\phi_s-\hat\phi_s)\nabla\phi_f\nabla u_f=0.\end{aligned}$$

Using $u=(u_f,u_s)$ solution of (3.13) as test function, leads to:

$$\int_\Omega|\phi_f-\hat\phi_f|^2+\int_\Omega|\phi_s-\hat\phi_s|^2=0.$$

Consequently, we have the following uniqueness result:

Theorem 5.2. *Under the hypothesis* $(H1)-(H6)$, *the variational problem (2.6) has one and only one solution.*

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