Stabilizing the chaotic dynamics of the Lü system

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Abstract. The chaotic dynamics of the Lü system introduced in [5] is stabilized. First, we suppress the chaos stabilizing the dynamics of the system to the unstable equilibria. Then, using a simple linear controller, the system is driven to a stable state. The Lyapunov function method is employed. Numerical illustrations are presented to support the analytical results.

M.S.C. 2000: 93C10, 93B52.
Key words: controlling chaos, linear feedback control.

§1. Introduction

In general, the chaotic oscillations in a physical system are harmful and sometimes they can lead even to disasters. Chaotic behavior is observed in practical applications of many fields, from engineering to biology and economics. In Physics it is studied the control of turbulence, control of lasers, control of chaos in plasma, control of the dipole domains. In Medicine study and treatment of cardiac arrythmia was among the most promising among the early applications of chaos control. In Mechanics it is studied the control of pendulums, beams, plates and so on. A survey on the applications of the control of chaos is [1] which cites about 200 papers. Various mathematical definitions of chaos are known, but all of them contain the property of "sensitivity to the initial conditions", that is, roughly speaking, even small perturbations to the initial conditions lead to divergent trajectories. Chaos can be suppressed using linear or nonlinear feedback methods [2], [8], [6], [7], [3].

Consider the nonlinear $n-$dimensional differential system [3], $n > 0$,

$$\dot{x} = f(x), x \in M \subseteq \mathbb{R}^n, f \in C^1(\mathbb{R}^n), \tag{1.1.1}$$

assumed dissipative and having an unstable equilibrium point x_0. Suppose that (1.1.1) displays a chaotic dynamics. To control the chaotic dynamics to a stable state, consider a control function $u(t) := A(x - x_0) + g(x - x_0, t)$, where A is a constant control matrix, and g is a nonlinear vectorial control function. Define the controlled system associated to the system (1.1.1) by:

$$\dot{x} = f(x) + A(x - x_0) + g(x - x_0, t). \tag{1.1.2}$$

Applied Sciences, Vol.9, 2007, pp. 174-180.

Assume that $g \in C^1(\mathbb{R}^n)$ and $g(0,t) = 0$ to ensure the existence and uniqueness of the solution of the systems (1.1.1) and (1.1.2). It is clear that x_0 is an equilibrium point of the system (1.1.2). The control $u(t)$ is chosen such that the trajectories of the system (1.1.1) to converge to the unstable equilibrium point x_0, i.e. $\lim_{t\to+\infty} \|x(t)-x_0\| = 0$. Applying the Taylor expansion on the right part in (1.1.2) around the point x_0, we get:

$$\dot{X} = J(x_0)X + h(X, A, t), \tag{1.1.3}$$

where

$X = x - x_0$, $J(x_0) = \frac{\partial f(X)}{\partial X} |_{X=0}$, and $h(X, A, t)$ is the Taylor rest. Suppose that $h(X, A, t)$ satisfies:

a) $h(0, A, t) = 0$;

b) $h(X, A, t)$ and $\frac{\partial h(X,A,t)}{\partial X}$ are continue in a bounded neighborhood $\|X\| < +\infty$;

c) $\lim_{\|X\|\to+\infty} \|h(X, A, t)\|/\|X\| = 0$ uniformly in $t \in [0, +\infty)$.

Definition 1.1. *The system (1.1.1) is said to be* **controlled to the unstable equilibrium point** x_0*, if the matrix A is chosen such that $X = 0$ is an equilibrium stable point of the linearized system of the system (1.1.3).*

The present work is organized as follows. In Section 1 we record some basic details of the system under study. Section 2 describes the methods to control chaos to the unstable fixed points, while Section 3 presents a linear feedback control of the system. Numerical illustrations are presented in all three sections.

§2. Description of the system

The Lü system is a three-dimensional differential system given by:

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= cy - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{2.2.4}$$

where $a, b, c \in \mathbb{R}$, $a, b > 0$, and it was introduced recently in [5]. In Fig.1 is presented a chaotic attractor of the Lü system [13]. In the present work, we study the Lü system in order to drive the chaotic dynamics of the system to a stable state.

If $c > 0$ the Lü system has three isolated equilibria $O(0, 0, 0)$, $A_1\,(x_0, y_0, z_0)$, $A_2\,(-x_0, -y_0, z_0)$,where $x_0 = y_0 = \sqrt{bc}$, $z_0 = c$ and it has only one isolated equilibrium $O(0, 0, 0)$ if $c \leq 0$.

Proposition 2.1. *([13]) The following statements are true :*
a) if $a > 0, b > 0$ and $c < 0$, then $O(0, 0, 0)$ is asymptotically stable,
b) if $a > 0, b > 0$ and $c > 0$, then $O(0, 0, 0)$ is saddle,
c) if $a > 0, b > 0$ and $0 < c < (a+b)/3$, then the equilibrium points $A_{1,2}\,(\pm x_0, \pm y_0, z_0)$ are stable.

Observe that for some values of the parameters of the system the equilibrium points can be unstable. In the next section we deal only with these cases.

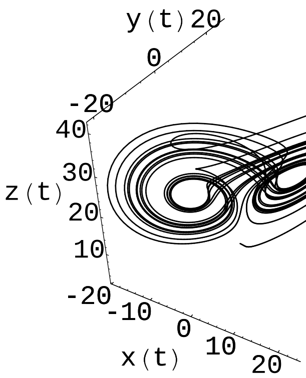

Figure 1: Chaotic attractor of the Lü system (2.2.4), corresponding to the parameters $(a, b, c) = (36, 3, 20)$ and the initial values $(-1, 0.1, 4)$.

PROPOSITION 2.2. *([13]) If $c = (a+b)/3$ the Lü system displays a Hopf bifurcation at points $A_{1,2}$.*

A generalization of the Lü system has been recently considered in [10],[11],[12].

§3. Controlling chaos to unstable fixed points

In this section, using a feedback linear control we drive the chaotic trajectory $(x(t), y(t), z(t))$ of the Lü system to a desired unstable equilibrium point (x_0, y_0, z_0).

Assume that the controlled system associated to the Lü system is given by:

$$\begin{aligned} \dot{x} &= a(y-x) - u_1 \\ \dot{y} &= cy - xz - u_2 \\ \dot{z} &= xy - bz - u_3 \end{aligned} \tag{3.3.5}$$

where u_1, u_2 and u_3 are external laws of input and $a, b, c > 0$ are chosen such that the all equilibrium points are unstable. It is known that in applications is more desirable a simple control. So we consider here: $u_1 = k(x - x_0), u_2 = m(y - y_0)$ and $u_3 = n(z - z_0)$, where $(k, m, n) \in \mathbb{R}^3$. Therefore, the system (3.3.5) leads to:

$$\begin{aligned} \dot{x} &= a(y-x) - k(x - x_0) \\ \dot{y} &= cy - xz - m(y - y_0) \\ \dot{z} &= xy - bz - n(z - z_0). \end{aligned} \tag{3.3.6}$$

The controlled system (3.3.6) has one equilibrium point (x_0, y_0, z_0). The linear system associated to the controlled system (3.3.6) about this equilibrium point is:

$$\begin{aligned} \dot{X} &= -(a+k)X + aY \\ \dot{Y} &= -z_0 X + (c-m)Y - x_0 Z \\ \dot{Z} &= y_0 X + x_0 Y - (b+n)Z. \end{aligned} \tag{3.3.7}$$

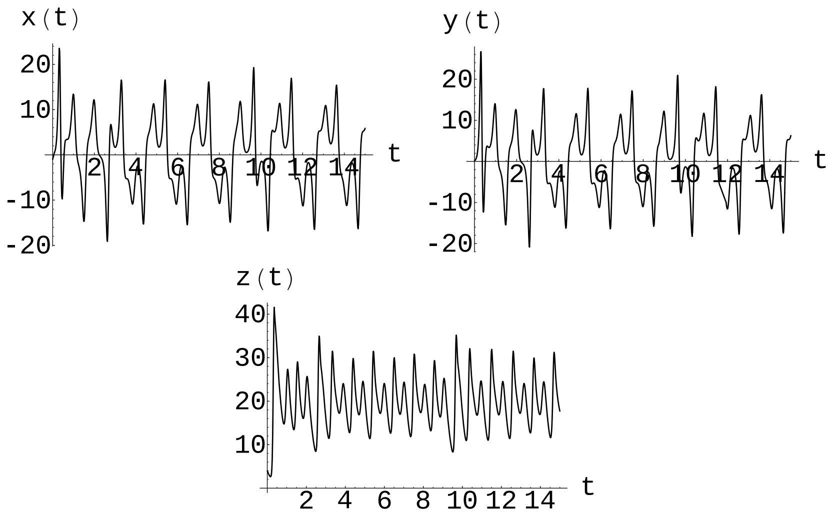

Figure 2: The time series $x(t)$ (left), $y(t)$ (right) and $z(t)$ (bellow) of the uncontrolled system (2.2.4), corresponding to the parameters $(a, b, c) = (2.1, 0.6, 30)$ and the initial values $(0.1, -0.3, 0.2)$. One can observe the irregular oscillations of the system.

Consider now the first unstable point $(x_0, y_0, z_0) = (0, 0, 0)$. Then system (3.3.7) leads to:

$$\begin{aligned} \dot{X} &= -(a+k)X + aY \\ \dot{Y} &= (c-m)Y \\ \dot{Z} &= -(b+n)Z. \end{aligned} \tag{3.3.8}$$

Define in the following the Lyapunov function of the system (3.3.8) by:

$$V(X, Y, Z) = \frac{X^2 + Y^2 + Z^2}{2}. \tag{3.3.9}$$

The function V satisfies:
i) $V(0, 0, 0) = 0$
ii) $V(X, Y, Z) > 0$ for X, Y, Z in the neighborhood of the origin, therefore $V(X, Y, Z)$ is positive definite. In addition, the time derivative of the function V is:

$$\begin{aligned} &\tfrac{dV}{dt} = X\dot{X} + Y\dot{Y} + Z\dot{Z} = -(a+k)X^2 + aXY + (c-m)Y^2 - (b+n)Z^2 = \\ &= -\left(\sqrt{a+k}X - \tfrac{a}{2\sqrt{a+k}}Y\right)^2 + Y^2\left(\tfrac{a^2}{4(a+k)} + c - m\right) - (b+n)Z^2. \end{aligned}$$

Hence the derivative $\frac{dV}{dt} < 0$ whenever,

$$a + k > 0, \frac{a^2}{4(a+k)} + c - m < 0, b + n > 0, \tag{3.3.10}$$

i.e, $\frac{dV}{dt}$ is negative definite under condition (3.3.10). Therefore, we have the proposition:

PROPOSITION 3.3. *If the feedbacks k, m, n satisfy $a + k > 0$, $\frac{a^2}{4(a+k)} + c - m < 0$ and $b + n > 0$ then the equilibrium solution $(0, 0, 0)$ of the controlled system (3.3.6) is asymptotically stable.*

For the second unstable point $A_1\left(\sqrt{bc}, \sqrt{bc}, c\right)$, (for A_2 is similar) system (3.3.7) leads to:

$$\begin{aligned} \dot{X} &= -(a+k)X + aY \\ \dot{Y} &= -cX + (c-m)Y - \sqrt{bc}Z \\ \dot{Z} &= \sqrt{bc}X + \sqrt{bc}Y - (b+n)Z. \end{aligned} \tag{3.3.11}$$

We choose the Lyapunov function for the system (3.3.11) given by:

$$V(X, Y, Z) = \frac{\frac{c}{a}X^2 + Y^2 + Z^2}{2} \tag{3.3.12}$$

The function V satisfies:
i) $V(0, 0, 0) = 0$
ii) $V(X, Y, Z) > 0$ for X, Y, Z in the neighborhood of the origin, therefore $V(X, Y, Z)$ is positive definite. In addition, we have that the time orbital derivative of the function V is:

$$\begin{aligned} \tfrac{dV}{dt} &= -\tfrac{c}{a}(a+k)X^2 + (c-m)Y^2 + \sqrt{bc}XZ - (b+n)Z^2 = \\ &= -\left(\sqrt{\tfrac{c}{a}(a+k)}X - \tfrac{\sqrt{ab}}{2\sqrt{a+k}}Z\right)^2 + (c-m)Y^2 + \left(\tfrac{ab}{4(a+k)} - b - n\right)Z^2. \end{aligned}$$

Therefore the derivative $\frac{dV}{dt} < 0$ whenever,

$$a + k > 0, c - m < 0, \frac{ab}{4(a+k)} - b - n < 0, \tag{3.3.13}$$

i.e $\frac{dV}{dt}$ is negative definite under condition (3.3.13). Consequently, we get the second proposition:

PROPOSITION 3.4. *If the feedbacks k, m, n satisfy $a + k > 0, c - m < 0$ and $\frac{ab}{4(a+k)} - b - n < 0$, then the equilibrium solutions $A_1\left(\sqrt{bc}, \sqrt{bc}, c\right)$ of the controlled system (3.3.6) is asymptotically stable.*

§4. Linear feedback control

Consider in this section a simple controller $u_1(t) = kx$, $k \in \mathbb{R}$. Adding it to the second equation of the system Lü, it leads to:

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= (c-a)x - axz + kx \\ \dot{z} &= xy - bz \end{aligned} \tag{4.4.14}$$

The equilibrium points of the controlled system (4.4.14) are $A(x_0, y_0, z_0)$, where $x_0 = y_0 = z_0 = 0$ or $x_0 = y_0 = \sqrt{b(k+c)}, z_0 = k + c$ if $k + c > 0$. The Jacobian matrix associated to this system in $A(x_0, y_0, z_0)$ is :

$$\begin{pmatrix} -a & a & 0 \\ k - z_0 & c & -x_0 \\ x_0 & x_0 & -b \end{pmatrix} \tag{4.4.15}$$

with the equation associated to the characteristic polynomial given by

(4.4.16)
$$\lambda^3 + (a + b - c)\,\lambda^2 + \left(x_0^2 + ab + az_0 - cb - ac - ak\right)\lambda - acb + 2x_0^2 a - akb + az_0 b = 0.$$

From Routh-Hurwitz conditions, this equation has all roots with negative real parts if and only if $A > 0, C > 0$ and $AB - C > 0$ where $A = a + b - c, B = x_0^2 + ab + az_0 - cb - ac - ak, C = -acb + 2x_0^2 a - akb + az_0 b$. Consider the first equilibrium point $O(0,0,0)$. Then from $C > 0$ and $AB - C > 0$ we get $c + k < 0$ and $AB - C = -a^2k + cak + a^2b - 2acb - a^2c + ab^2 - cb^2 + c^2b + ac^2 > 0$. Remark that if $c + k < 0$ the system does not possess another equilibrium point, so the system is completely controlled to a stable state. Numerical illustrations can be seen in Fig.3.

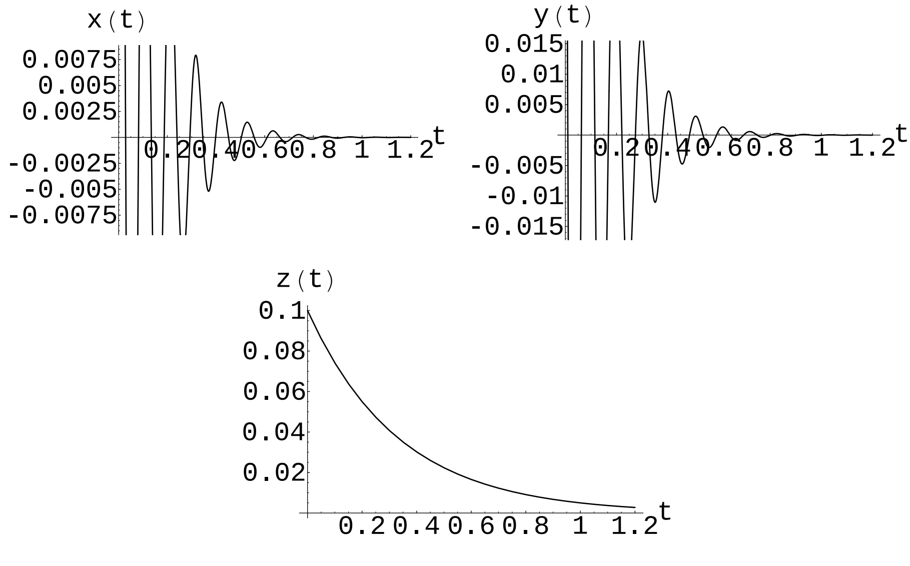

Figure 3: The time series $x(t)$ (left), $y(t)$ (right) and $z(t)$ (bellow) of the controlled system (4.4.14), corresponding to the parameters $(a, b, c) = (36, 3, 20)$, the initial values $(0.1, 0.1, 0.1)$ and $k = -6c$

§5. Conclusions

Methods to suppress chaos in the Lü dynamical system were presented. First, using the Lyapunov function method we stabilized the chaotic trajectories to the unstable fixed points. Then, by a linear control, the system is controlled to a stable state. Analytical results are accompanied by numerical illustrations.

Acknowledgements. This work was (partially) supported through a Research Fellowship offered by CNBSS and a European Community Marie Curie Fellowship, in the framework of the CTS, contract number HPMT-CT-2001-00278.

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