# Information denoising and quantization by diffusion reaction model

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**Abstract.** We present a new diffusion reaction model for signal denoising and quantization. We first discuss on classical quantization methods and present the most popular denoising models. Then we construct our method as a combination of quantization and denoising terms and show its efficiency on noisy signals.

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## 1 Introduction

Generally, the degradation of the original signal $z$ occurs during the acquisition process and can be modeled by a linear tranformation and additive noise, as the following equation

$$u_0 = Az + \eta \tag{1.1}$$

where $A$ is a convolution operator and $\eta$ is an additive white noise of standard deviation $\sigma$ that can be considered as gaussian. As known, the inverse problem of recovering $z$ from $u_0$ is ill posed, since the operator $A$ is not be invertible. In practice $A$ is a band limited filter while generally signal $z$ contains jumps and then high frequencies. Then Gibbs effect is introduced. Indeed, the fourier transform of $Az$ can be modeled by

$$\widehat{Az}(\xi) = \mathbb{I}_{[-\xi_0,\xi_0]}(\xi) \cdot \hat{z}(\xi)$$

which corresponds to the cutoff frequencies satisfying $|\xi| \geq \xi_0$. For simplicity let us set $\xi_0 = 1/2$. Then, since we know that the invers Fourier transform of $\mathbb{I}_{[-1/2,1/2]}$ is nothing but the sinus cardinal ($\text{sinc}(x) := \sin(\pi x)/\pi x$) function, the new reconstruction of the signal is given by :

$$Az(x) = \text{sinc}(x) * z(x).$$

Consequently, taking acount of the oscillatory behavior of the sinc function, artifactual ripples occur near a discontinuities since only a finite portion of the Fourier transformation is considered.

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The presence of noise worsens the situation. Generally, noise contains mostly high frequencies information, whereas a signal is distributed over all its frequency domain : low frequencies correspond to its tendency and high frequencies correspond to its discontinuities. Hence, the signal is dominant for low frequencies and its tendency is kept by applying a simple lowpass filter, while the effect of the noise is reduced. Indeed, a low-pass filter passes relatively low frequency components in the signal but stops the high frequency components. Unfortunately, with a such kind of filter, the high frequencies of the signal are also destroyed. This implies that the output signal is smooth and discontinuities are blured.

The combination of this two degradations (the cut-off given by $A$ and the additive noise $\eta$) in an initial signal $z$ makes its restauration difficult, since we must reconstruct high frequencies destoried by the cut-off operator $A$ and in the same time eliminate those introduced by the noise. Taking the complexity of the problem into account, we restrict our self in this paper on the class of piecewice constant signals. This signal type arises in several areas and many industrial and medical applications. More precisely, we assume that $z$ is a piecewice constant signal taking a finite and known values $V_N = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$. Then the problem that we modelise here is how to quantify $u_0$ into the values $V_N$ taking acount the degradations given by (1.1). This is clearly a quantization and denoising problem.

Let us now discussing on quantization process.

## 2 Quantization process

The classical method to quantify $u_0$ into the values $V_N$ is to consider a potential function $H(s)$ satisfying $H(+\infty) = H(-\infty) = +\infty$ having $\{\lambda_i\}$ as local minima and to solve the minimum problem : "$\min W(u) = \int H(u)dx$", with $u = u_0$ as initial data. The minimum is interpreted as the steady state of the following ordinary differential equation :

$$\frac{\partial u}{\partial t} = -h(u) \qquad u(\cdot, 0) = u_0, \tag{2.1}$$

where $h = H'$. In the case where the values of signal $z$ are unknown, we can fixe $V_N$ by the classical Lloyd method [8] which consists to estimate the quadratic error of $u_0$ with respect to $V_N$ by minimizing the energy :

$$E(V_N \cup W_N) = \sum_{i=1}^{N} \int_{\beta_i}^{\beta_{i+1}} (\lambda - \lambda_i)^2 df(\lambda),$$

where $f(\lambda) = P(u_0 < \lambda)$ is the probability distribution associated to the signal $u_0$, and $W_N = \beta_0, \dots, \beta_{N+1}$ represents the set of separator such that $\beta_i < \lambda_i < \beta_{i+1}$. Alvarez and Esclarin [1] have improved the Lloyd's method by the introduction of two terms in the energy $E(V_N)$ in order to penalize quantization values for being too close. They propose the following :

$$E(V_N \cup W_N) = \sum_{i=1}^{N} \int_{\beta_i}^{\beta_{i+1}} (\lambda - \lambda_i)^2 df(\lambda) + |\beta_{i+1} - \beta_i|^{-1} + (\beta_i + \beta_{i+1} - 2\lambda_i)^2.$$

The first added term means that $|\beta_{i+1} - \beta_i|$ must be sufficiently large and the last term implies that $\beta_i$ and $\beta_{i+1}$ are symmetrical with respect to $\lambda_i$. This improvement removes some non-uniqueness problems that can appear when the Lloyd energy is used.

## 3 Denoising process

Since the presence of the noise, we must include a denoising process in the diffusion equation (2.1). The classical method is to use the isotropic diffusion

$$\frac{\partial u}{\partial t} = u'' \qquad u(\cdot,0) = u_0, \tag{3.1}$$

which is nothing but the known heat equation in one dimension. The solution of this equation for an initial data with bounded quadratic norm, is given by $u(x,t) = G_t * u_0$ where $G_t(x) = (1/2\pi t)\exp(-x^2/2t)$ is the gaussian function. This diffusion process kind clearly contradict the requierement of the quantization effect. Indeed, in ordre to creat discontinuities, the quantization effect introduces some high frequencies in the signal. But the convolution of a signal with the gaussian function with variance $t$, is equivalent to multiply its Fourier transform by a gaussian function with variance $1/t$. Then for large $t$ we find two opposite and contradictory requirements : to creat and in the same time to destroy high frequencies.

Consequently, the denoising process must avoid all introduction of blur effect or any other artifact in order to preserve the local tendancy of the initial signal by "farizing" the creation of high freqiencies. Following this viewpoint, one of the popular model arising in signal and image restoration has been proposed by Malik and Perona [9], by the partial differential equation :

$$\frac{\partial u}{\partial t} = \big(g(|u'|^2)u'\big)' \qquad u(\cdot,0) = u_0, \tag{3.2}$$

where $g$ is a smooth non-increasing positive function with $g(0) = 1$ and $sg(s^2) \to 0$ at infinity. The idea of the equation (3.2) is that the restoration process obtained is conditional : if $|u'(x)|$ is large then $g(|u'|^2) \simeq 0$ and the diffusion will be stopped and if $|u'(x)|$ is small then $g(|u'|^2) \simeq 1$ and the diffusion will tend to smooth around $x$ as the isotropic heat equation. This model (3.2) has been considered as an important improvement of the signal restoration and the edge detection theory [10].

Unfortunately, the Malik and Perona model is ill posed. Indeed, by writing the equation in the form :

$$\frac{\partial u}{\partial t} = (g(|u'|^2) + 2|u'|g'(|u'|^2))u'',$$

we observe that the diffusion is inverted in the regions where $|u'|$ is large and the process can be interpreted as a backward heat equation which is known to be ill posed. The ill posedness means that very close initial signals could produce divergent solutions.

For all that, there are many attempts at extracting information from the Malik and Perona equation and understanding whether (3.2) can be given a "well-posedness" theory. We can refer to the papers of Kawohl and Kutev [6] and Kichenassamy [7] in which we find the confirmation that in general case, we have non existence of a weak solution and non uniqueness results.

An other approach is to slightly modify the equation (3.2) by putting a regularized term in place of $|u'|^2$ in order to have a well-posed equation. There are essentially two propositions which we consider as a "direct derivation" from the Malik-Perona Model. The first one, proposed by Catté, Lions, Morel and Coll, consists in special regularization whereby $|u'|^2$ is replaced by $|\rho * u'|^2$, where $\rho$ is a smooth kernel, yielding the following model :

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g(|\rho * u'|^2)u'\right)' \qquad u(\cdot, 0) = u_0. \tag{3.3}$$

The well posedness of this model is proved in [5], and the choice $\rho = G_\sigma$, a Gaussian with variance $\sigma$, is motived by the classical theory of edge detection [12]. The second proposition is time-delay regularization, where one replaces $|u'|^2$ by an average of its values from 0 to $t$. The idea of Nitzberg and Shiota [11] is to use an exponential kernel such that $|u'|^2$ is replaced by :

$$v(x,t) = e^{-t} v_0(x) + \int_0^t e^{(s-t)} |u'(x,s)|^2 \, ds, \tag{3.4}$$

where $v_0$ is an initial data (for example $v_0 = |u_0'|^2$). The avantage of this model is that the inhibition term $v$ acts as a memory of the denoising process and introduces a certain stability with respect to the variation of the scale.

In [2] the author of the present paper have shown that in any dimension, this model admits an unique classical solution $(u, v)$ which can blow up in finite time. With A. Chambolle [4], we also have studied the model from a numerical viewpoint. Unfortunately this model, as the Malik-Perona equation, is enable to reduce noise with large slop since there is no spatial smoothing.

## 4 A New quantization and denoising model

Our idea is to define $v$ as a combination of the above two regularization types, (3.3) and (3.4), by the following :

$$v(x,t) = e^{-t} |\rho * u_0'|^2(x) + \int_0^t e^{(s-t)} |\rho * u'(x,s)|^2 \, ds, \tag{4.1}$$

that we associate to the denoising and quantization equation :

$$\frac{\partial u}{\partial t} = \left(g(v)u'\right)' - \theta(t)h(u)$$

where the term $(g(v)u')'$ is as Malik and Perona, $v$ is given by (4.1), $h$ is a quantization function and $\theta(t)$ is introduced to perform a balance between denoising and quantization. The goal of the introduction of this parameter, $\theta(t)$, is to favour the denoising process for small scales by the inhibition of the quantization effect by choosing

$\theta(t) \simeq 0$ for small $t$ and $\theta(t) \simeq 1$ if $t$ is large. Therefore, the proposed diffusion-reaction process is described by the system :

$$\frac{\partial u}{\partial t} = \big(g(v)u'\big)' - \theta(t)h(u), \qquad u(\cdot,0) = u_0, \tag{4.2}$$

$$\frac{\partial v}{\partial t} = |\rho * u'|^2 - v, \qquad v(\cdot,0) = |\rho * u_0'|^2. \tag{4.3}$$

The function $g$ is as the Malik and Perona Model and we choose : $g(s) = \big(1+s\big)^{-1}$. $h$ is assumed to be continous and Lipschitz function satisfying $h(s) = s - M$ if $s > M$ and $h(s) = s - M$ if $s < m$, with $m = \lambda_1$ and $M = \lambda_N$ (see the figure). $\theta(t)$ is a continous positif function such that $0 \leq \theta(t) \leq 1$. $\rho$ is a positif relguar kernel, for instance a gaussian with variance $\sigma > 0$. In the term $|\rho' * u|$, $u(\cdot, t)$ is assumed be extended linearly and continously in all $\mathbb{R}$.

Now, we construct an iterative scheme that we considere as numerical approximation of system (4.2)-(4.3). For all fixed $\delta > 0$, we define the sequence $(u_n^\delta, v_n^\delta)_n$ by the following interative scheme :

$$(u_0^\delta, v_0^\delta) = (u_0, |\rho * u_0'|^2),$$

$$\frac{u_{n+1}^\delta - u_n^\delta}{\delta} = \big(g(v_n^\delta)(u_{n+1}^\delta)'\big)' - \theta(n\delta)h(u_n^\delta), \tag{4.4}$$

$$\frac{v_{n+1}^\delta - v_n^\delta}{\delta} = |\rho' * u_{n+1}^\delta|^2 - v_{n+1}^\delta. \tag{4.5}$$

Some propreties of the iterative scheme are given by the next proposition :

**Proposition 1.** *Assume that* $0 \leq v_n^\delta \in L^\infty(0,1)$ *and* $\delta \sup |h'| < 1$, *then we have :* $u_{n+1}^\delta \in H^1(0,1)$ *is given by the following minimum problem :*

$$u_{n+1}^\delta := Argmin_{w \in H^1(0,1)} E_n^\delta(w) \tag{4.6}$$

$$\text{where} \quad E_n^\delta(w) := \int_0^1 \Big(g(v_{(\delta,n)})|w'|^2 + \frac{1}{2\delta}|w - u_{(\delta,n)}|^2 + \theta(n\delta)H(w)\Big)\,dx.$$

*and in addition* $u_{n+1}^\delta$ *satisfys the maximum principle given by :*

$$\min\{m, \min u_n^\delta\} \leq u_{n+1}^\delta \leq \max\{M, \max u_n^\delta\}. \tag{4.7}$$

Remark that (4.7) implies that, for all $n$ :

$$\min\{m, \min u_0\} \leq u_n^\delta \leq \max\{M, \max u_0\}.$$

In [3] we can find the proof of the proposition. In the same paper we prove that the iterative scheme converges to the continuous model.

Let us now describe how to discretize the coupled system (4.4)-(4.5). We denote by $u_i^n$ (resp. $v_i^n$) the approximation of $u$ (resp. $v$) at point $(ih)$ $(0 \leq i \leq N)$ and time $t = n\,\delta$, where the size of the initial signal $u_0$ is aqual to $N$ and $h = 1/N$. Using the classical finite-differences, we write the approximation of $(g(v)u')'$ at point $ih$ and at scale $t=(n+1)\,\delta$ by $\frac{1}{h^2}((g(v_i^n)(u_{i+1}^{n+1}-u_i^{n+1}))-(g(v_{i-1}^n)(u_i^{n+1}-u_{i-1}^{n+1})))$, then the equation (4.4) becomes:

$$\frac{u_i^{n+1}-u_i^n}{\delta}=\frac{1}{h^2}\Big(\big(g(v_i^n)(u_{i+1}^{n+1}-u_i^{n+1})\big)-\big(g(v_{i-1}^n)(u_i^{n+1}-u_{i-1}^{n+1})\big)\Big)-\theta(n\delta)\,h(u_i^n).$$

By rearranging the first term of right hand, we can write it in the form $1/h^2A(v^n)u^{n+1}$ where the matrix $A(v^n)$ is tridiagonal and positive defined. By classical arguments we know that $[I+\delta h^{-2}A(v^n)]$ is invertible.

In figure 1, the original signal (Top-Left) is piecewise constant taking the values $\lambda_1=0$ and $\lambda_2=10$. Then we construct the function $h$ by using this values, as the following : $h(x)=x-\lambda_1$ if $x\le\lambda_1, \alpha((x-\lambda_1)(x-\lambda_2)^2+(x-\lambda_2)^2(x-\lambda_2))$ if $x\in[\lambda_1,\lambda_2]$ and $x-\lambda_2$ if $x\ge\lambda_2$, where $\alpha>0$ is choosen such that $\delta\sup|h'|<1$ (Here we use $\delta=0.1$). In the next signal (Top-Right) we represent the result of a cutoff frequencies such that only the frequencies satisfying $|\xi|\le 8$ are preserved. The signal (Middle Left) represents the noise that we added to signal (Top-Right). Then we obtain the signal (Middle Right) that we considere the initial data $u_0$. In the signal (down), we clearly remark that our model performs the required quantization. This experiment have been done at scale $\sigma=15$, which correponds to the evolution time $t=112.5$.

In figure 2 we show the evolution steps of $u_0$ with the following models : (Top) without denoising only with the equation $(dt/dt=-h(u))$, (Middle) with our model without quantization function ie. $h\equiv 0$ and (Down) with our model.

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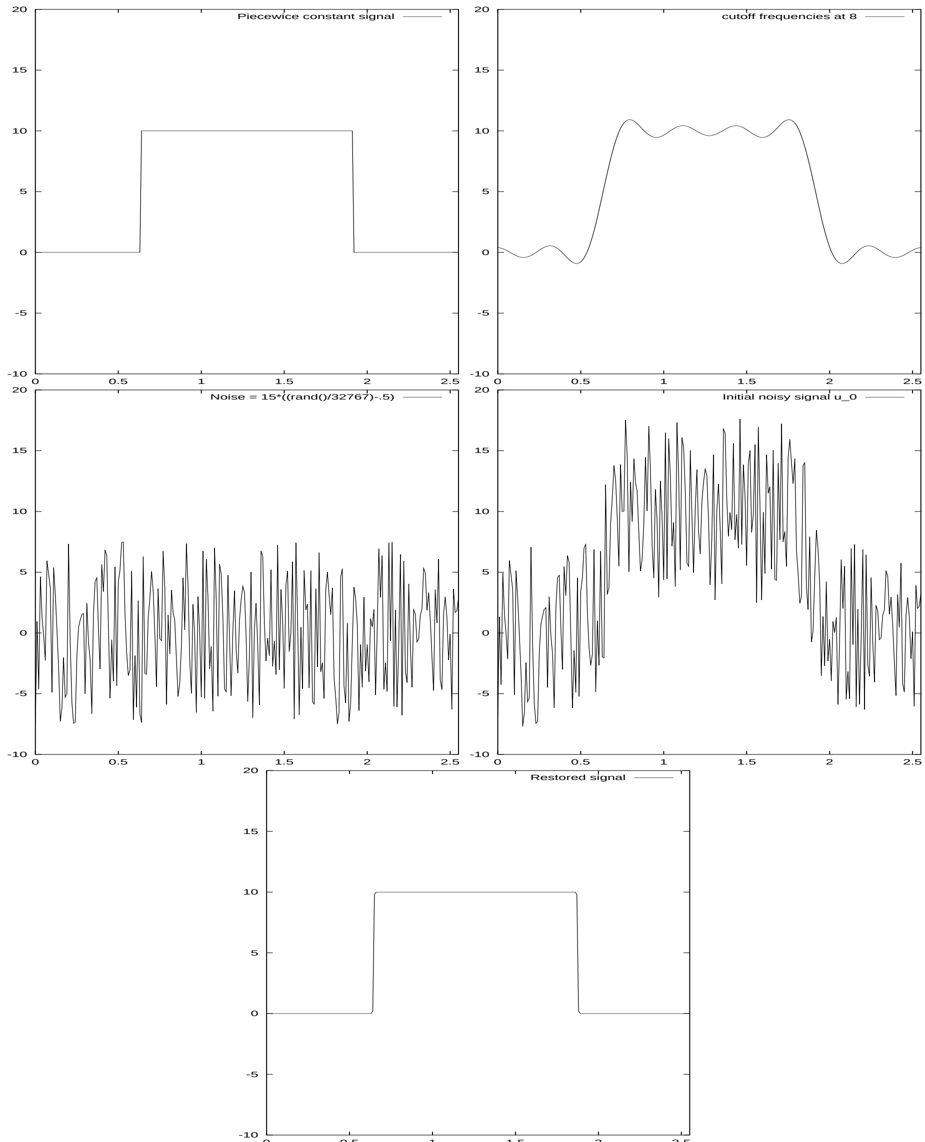


Figure 1: (Top-Left) Initial piecewice signal. (Top-Right) Cutoff frequencies at $\xi \leq 8$. (Middle-Left) noise. (Middle-Right) $u_0$ =(Top-Right)+ noise. (Down) Signal $u_0$ restored and quantized by our model.

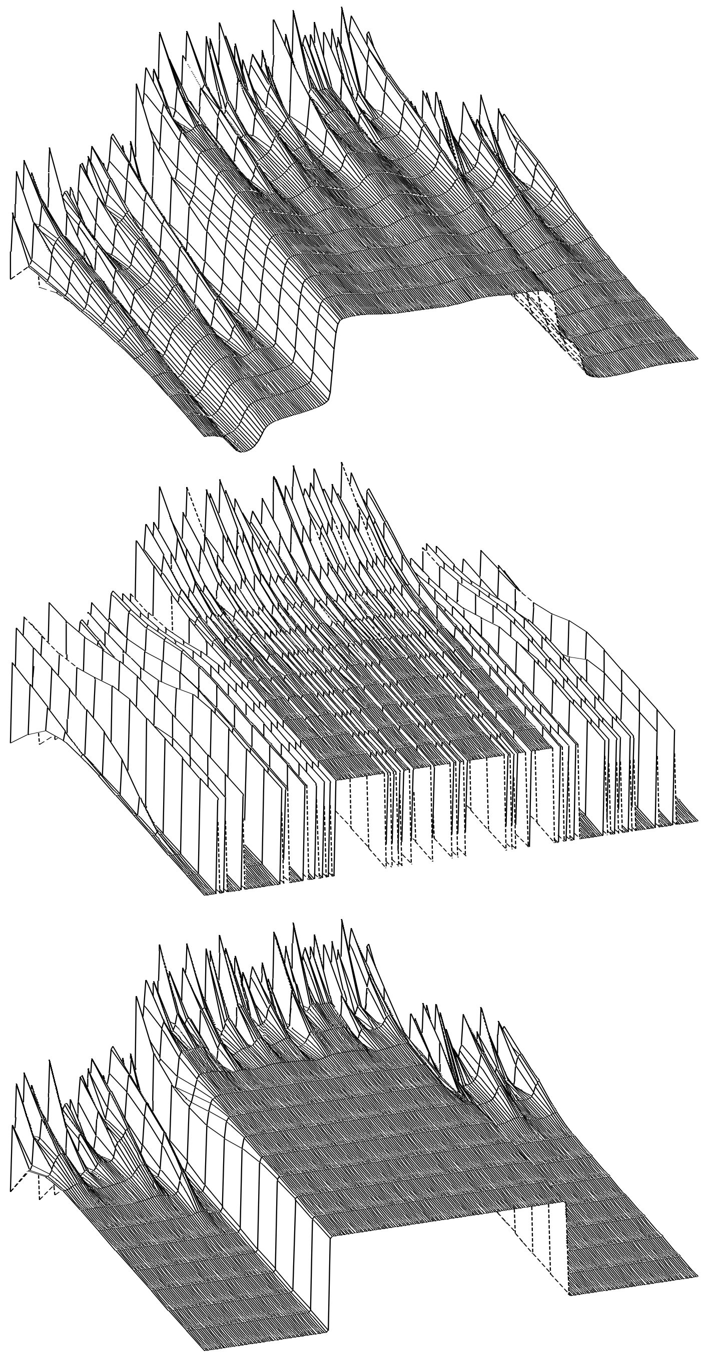

Figure 2: Evolution steps of $u_0$. (Top) without quantization function ie. $h \equiv 0$. (Middle) without denoising process ($du/dt = -h(u)$). (Down) with our model.

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