Hochschild and cyclic homology of ρ-algebras

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Abstract. In this paper we present our construction of the Hochschild and cyclic homology of a ρ-algebra. We show that these constructions are generalizations of the Hochschild homology and cyclic homology of algebras and of the superalgebras. Finally we calculate some Hochschild homology groups of the quaternionic algebra $\mathbb{H}$.

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1 Introduction

The study of ρ-algebras seems to have many applications in noncommutative geometry. A ρ-algebra A over the field k ($\mathbb{C}$ or $\mathbb{R}$) is a G-graded algebra (G is a commutative group) together with a twisted cocycle $\rho : G \times G \to k$. These algebras where defined for the first time in the paper [1] and are generalizations of usual algebras (the case when G is trivial) and of $\mathbb{Z}$ ($\mathbb{Z}_2$)-superalgebras (the case when G is $\mathbb{Z}$ resp. $\mathbb{Z}_2$).

There are studied some geometrical objects on ρ-algebras: the Lagrangian and Hamiltonian formalism for ρ-algebras in [13], linear connections and distributions, with applications to the quantum hyperplane in [3], [4] and [5] and matrix algebra in [6]. Fields and forms over a ρ-algebra A are presented in [7]. 2-derivations and 2-linear connections on A with applications to the quaternionic algebra $\mathbb{H}$ are studied in [8]. But there are no constructed some basic homologies related to the noncommutative geometry as Hochschild and cyclic homology, and we are focus on these things in this paper.

In this paper we present our construction of the Hochschild and cyclic homology of a ρ-algebra A. This construction is a generalization of the Hochschild homology of an arbitrary associative algebra from [11] and also is a generalization of the Hochschild and cyclic homology of an $\mathbb{Z}_2$-superalgebra from [10]. To do this, we define a new class of differential graded algebras denoted by DG ρ-algebras and we show that DG-algebras and DG-superalgebras are particular cases of our DG ρ-algebras.

First, we review the principal notions about ρ-algebras (see [1]) and we introduce the differential graded ρ-algebras (see [4]). As an example of DG ρ-algebra we define the algebra of noncommutative differential forms $\Omega_\alpha A = \underset{n\geq 0}{\otimes} \Omega^n_\alpha A$ of a ρ-algebra A and

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we show that this construction is a generalization of the algebra of noncommutative differential forms of an associative algebra and of a $\mathbb{Z}_2$-superalgebra. In the third section we define the Hochschild $HH(A)$ and cyclic homology $HC(A)$ of a ρ-algebra A. We also prove that if the algebra A is ρ-commutative then $HH^1(A)=\Omega^1_\alpha A$. Finally we calculate some Hochschild homology groups of the quaternion algebra $\mathbb{H}$.

2 ρ-algebras

In this section we present shortly a class of noncommutative algebras which are ρ-algebras, for more details see [1].

Let G be an abelian group, additively written, and let A be a G-graded algebra over the field k (which is $\mathbb{R}$ or $\mathbb{C}$). This means that as a vector space A has a G-grading $A = \oplus_{a\in G} A_a$, the G-degree of a (nonzero) homogeneous element f of A is denoted as $|f|$. Let $\rho : G \times G \to k$ be a map which satisfies

$$\rho(a,b) = \rho(b,a)^{-1} \text{ and } \rho(a+b,c) = \rho(a,c)\rho(b,c), \tag{2.1}$$

for any $a, b, c \in G$.

A G-graded algebra A with a given cocycle ρ is a ρ-algebra. A ρ-algebra is ρ-*commutative* (or *almost commutative algebra*) if the ρ-*commutator* $[f,g]_\rho = fg - \rho(|f|,|g|)gf$ of f and g is zero, for all homogeneous elements $f, g \in A$.

Example 2.1. *1) Any usual (commutative) algebra is a ρ-algebra with the trivial group G.*

2) Let be the group $G = \mathbb{Z}$ ($\mathbb{Z}_2$) and the cocycle $\rho(a,b) = (-1)^{ab}$,. for any $a, b \in G$. In this case any ρ–(commutative) algebra is a super(commutative) algebra.

3) The N-dimensional quantum hyperplane ([1], [4]) S^q_N.

4) The algebra of matrix $M_n(\mathbb{C})$ ([7]).

5) The quaternionic algebra $\mathbb{H}$ ([8]).

Definition 2.2. *Let $\alpha, \beta \in G$. A ρ-*derivation of the order (α,β) *is a linear map $X : A \to A$, which fulfils the properties:*

1) $X : A_* \to A_{*+\beta}$,

2) $X(fg) = (Xf)g + \rho(\alpha, |f|)f(Xg)$,for any $f \in A_{|f|}$ and $g \in A$.

The basic properties of ρ-derivations are in [6] and [8].

Let M be a G-graded left module over a ρ-commutative algebra A, with the usual properties, in particular $|f\psi| = |f| + |\psi|$ for $f \in A$, $\psi \in M$. Then M is also a right A ρ-module with the right action on M defined by $\psi f = \rho(|\psi|,|f|) f\psi$ for any $f \in A_{|f|}$ and $\psi \in M_{|\psi|}$. In fact M is a ρ-bimodule over A, i.e. $f(\psi g) = (f\psi)g$ for any $f, g \in A$, $\psi \in M$.

2.1 Differential graded ρ-algebras

Next we give a generalization of the classical differential graded algebra and of the differential graded superalgebras by defining so called *differential graded ρ-algebras*. We denote by $G'=\mathbb{Z} \times G$ and we define the cocycle $\rho' : G' \times G' \to k$ as follows $\rho'((n,\alpha),(m,\beta)) = (-1)^{nm} \rho(\alpha,\beta)$. Obviously the function ρ' satisfy the properties (2.1).

Definition 2.3. $\Omega = \underset{(n,\beta)\in G'}{\oplus} \Omega_\beta^n$ *is a* ρ-differential graded algebra *(*DG ρ-algebra*) if there is an element* $\alpha \in G$ *and a* ρ'*-derivation of the order* $((1,\alpha),(1,0))$ $d:\Omega\to\Omega$ *such that* $d^2=0$*, i.e.* $d:\Omega_*^n \to \Omega_*^{n+1}$ *and*

$$d(\omega\theta) = (d\omega)\,\theta + (-1)^n\,\rho\,(\alpha,|\omega|)\,\omega d\theta \tag{2.2}$$

for any $\omega \in \Omega_{|\omega|}^n$ *and* $\theta\in\Omega$.

If we denote by $|\omega|' = (n,|\omega|)$ the G'-degree of $\omega\in\Omega_{|\omega|}^n$, then the relation (2.2) becomes: $d(\omega\theta) = (d\omega)\theta + \rho'(|d|',|\omega|')\omega d\theta$ and Ω is a ρ'-algebra.

Example 2.4. *1) If the group* G *is trivial then* Ω *is the classical differential graded algebra.*

2) If the group G *is* $\mathbb{Z}_2$ *and the map* ρ *is given by* $\rho(a,b) = (-1)^{ab}$ *then* Ω *is a differential graded superalgebra ([10]).*

Definition 2.5. *Let* A *be a* ρ*-algebra. The pair* $\left(\Omega(A) = \underset{(n,\alpha)\in G'}{\oplus}\Omega_\alpha^n(A), d\right)$ *is a* ρ*-differential calculus over* A *if* $\Omega(A)$ *is a* ρ*-differential graded algebra,* $\Omega(A)$ *is an* A*-bimodule and* $\Omega^0(A) = A$.

The first example of a ρ-differential calculus over the ρ-commutative algebra A is the algebra of forms $(\Omega(A), d)$ of A from [1].

The second example of a ρ-differential calculus over a ρ-algebra is the universal differential calculus of A from the next section.

2.2 The algebra of universal differential forms of a ρ-algebra

In this section A is a ρ-algebra (not necessarily ρ-commutative). Next we present the construction of the algebra of universal differential forms $\Omega_\alpha A$ of A from the paper [6].

Let α be an arbitrary element of G. By definition the algebra of *universal differential forms* (also called the *algebra of noncommutative differential forms*) of the ρ-algebra A is the algebra $\Omega_\alpha A$ spanned by the algebra A and the symbols da, $a\in A$ which satisfy the following relations:

1. da is linear in a.
2. the ρ-Leibniz rule: $d(ab) = d(a)b + \rho(\alpha,|a|)adb$.
3. $d(1) = 0$.

Let $\Omega_\alpha^n A$ the space of n-forms $a_0da_1...da_n$, $a_i\in A$ for any $0\le i\le n$. $\Omega_\alpha^n A$ is an A-bimodule with the left multiplication

$$a(a_0da_1...da_n) = aa_0da_1...da_n, \tag{2.3}$$

and the right multiplication is given by: $(a_0da_1...da_n)a_{n+1} =$

$$=\sum_{i=1}^n(-1)^{n-i}\rho(\alpha,\sum_{j=i+1}^n|a_j|)(a_0da_1...d(a_ia_{i+1})...da_{n+1})+$$
$$+(-1)^n\rho(\alpha,\sum_{i=1}^n|a_j|)a_0a_1da_2...da_{n+1}.$$

$\Omega_\alpha A = \underset{n\in\mathbb{Z}}{\oplus}\Omega_\alpha^n A$ is a $\mathbb{Z}$-graded algebra with the multiplication $\Omega_\alpha^n A\cdot\Omega_\alpha^m A\subset\Omega_\alpha^{n+m}A$ given by: $(a_0da_1...da_n)(a_{n+1}da_{n+2}...da_{m+n}) = ((a_0da_1...da_n)a_{n+1})da_{n+2}...da_{m+n})$, for any $a_i\in A$, $0\leq i\leq n+m$, $n,m\in\mathbb{N}$.

Theorem 2.6. *([6]) The map d:$\Omega_\alpha^*\to\Omega_\alpha^{*+1}$ is a derivation of the order $((1,0),(1,\alpha))$ and $(\Omega_\alpha A, d)$ is a ρ-differential calculus over A.*

Example 2.7. *1) In the case when the group G is trivial then A is the usual associative algebra and $\Omega_\alpha A$ is the algebra of universal differential forms of A.*

2) If the group G is $\mathbb{Z}_2$ and the cocycle ρ is $\rho(a,b)=(-1)^{ab}$ then A is a superalgebra. In the case when $\alpha=1$ $\Omega_\alpha A$ is the superalgebra of universal differential forms of A from [10].

3 Hochschild and cyclic homology of $\rho-$algebras

We define the Hochschild and cyclic homology using the method from ([2]). First we introduce the corresponding operators from the usual associative algebras or superalgebras in our case of ρ-algebras.

3.1 Operators on $\Omega_\alpha^n A$

Let α be an element from the group $\dot{G}$. We introduce the following operators on the algebra $\Omega_\alpha A$: b' : $\Omega_\alpha^{n+1}A\to\Omega_\alpha^n A$ by $b'(\omega dx)=\rho(|\omega|,\alpha)\omega x$ and the **Hochschild boundary**: $b:\Omega_\alpha^{n+1}A\to\Omega_\alpha^n A$ defined by

$$b(\omega dx) = (-1)^n\rho(|\omega|,\alpha)\omega x - (-1)^n\rho(|\omega|,|x|+\alpha)\rho(n|\alpha|,|x|)x\omega \tag{3.1}$$

for any $\omega\in\Omega_\alpha^n A$ and any homogeneous element $x\in A$.

The detailed formula for the Hochschild boundary is:
$b(a_0da_1...da_nda_{n+1}) = \rho(|a_0|,\alpha)\,a_0a_1...da_nda_{n+1}+$

$$\sum_{i=0}^{n}(-1)^i\rho(\sum_{j=0}^{i}|a_j|,\alpha)a_0da_1...d(a_ia_{i+1})...da_nda_{n+1}+$$

$$(-1)^{n+1}\rho(\sum_{j=0}^{n}|a_j|,|a_{n+1}|+\alpha)\rho(n\alpha,|a_{n+1}|)a_{n+1}a_0da_1...da_n$$

for any homogeneous elements $a_i\in A$, $i\in\{1,...,n+1\}$.

Theorem 3.1. $b'^2=0$ *and* $b^2=0$.

Proof. Let $\omega\in\Omega_\alpha^n A$,

$$b'^2(\omega dxdy) = (-1)^{n+1}\rho(|\omega|+|x|,\alpha)b'(\omega dx)y =$$

$$= (-1)^{n+1}\rho(|\omega|+|x|,\alpha)b'(\omega d(xy)-\rho(\alpha,|x|)\omega xdy) =$$

$$= (-1)^{n+1}\rho(|\omega|+|x|,\alpha)(-1)^n\rho(|\omega|,\alpha)\omega xy-$$

$$-(-1)^{n+1}\rho(|\omega|+|x|,\alpha)(-1)^n\rho(\alpha,|x|)\rho(|\omega|+|x|,\alpha)\omega xy) = 0$$

The other relation can be proved in the same way. ∎

Remark 3.2. *If A is ρ-commutative then $b\left(a_0da_1da_2\right)=0$ for any $a_0da_1da_2\in\Omega_\alpha^2A$.*

We introduce the *cyclic operator* $\lambda:\Omega_\alpha^nA\to\Omega_\alpha^nA$ by $\lambda=\beta d$, where $\beta:\Omega_\alpha^{n+1}A\to\Omega_\alpha^nA$ is

$$\beta(\omega dx)=(-1)^n\rho(|\omega|,|x|+\alpha)\rho(n\alpha,|x|)x\omega.$$

It has the following compact expression: $\lambda(\omega dx)=(-1)^n\rho(|\omega|,|x|+\alpha)\rho(n\alpha,|x|)xd\omega$ or the detailed expression

$\lambda(a_0da_1...da_{n+1})=(-1)^n\rho(\sum\limits_{j=0}^{n}|a_j|,|a_{n+1}|+\alpha)\rho(n\alpha,|a_{n+1}|)a_{n+1}da_0da_1...da_n$

The relations between d, b', b, λ, β are given in the following proposition:

Proposition 3.3. *1)* $b\beta+\beta b'=0$,
2) $b'\beta+\beta b=0$,
3) $b'd+db'=id$,
4) $b\lambda-\lambda b'+\beta=0$,
5) $b(1-\lambda)=(1-\lambda)b'$.

The proof is easy but tedious.

3.2 Hochschild and cyclic homology of ρ-algebras

In this subsection we introduce the Hochschild and cyclic homology of a ρ-algebra A.

Definition 3.4. *The* Hochschild homology $HH_\alpha(A)$ *of the ρ-algebra A is the homology of the complex:*

$$...\to\Omega_\alpha^{n+1}A\xrightarrow{b}\Omega_\alpha^nA\to...\ .$$

Remark that $b:\Omega_\alpha\left(A\right)\to\Omega_\alpha\left(A\right)$ is a k-linear morphisms of G' rank $\left(1,0\right)$. It results that $HH_\alpha\left(A\right)=\bigoplus\limits_{\beta\in G}HH_\alpha\left(A_\beta\right)$, where $HH_\alpha\left(A_\beta\right)$ is the homology of the following complex

$$...\to\Omega_\alpha^{n+1}\left(A_\beta\right)\xrightarrow{b}\Omega_\alpha^n\left(A_\beta\right)\to...\ .$$

Form the remark 1 we deduce the following proposition:

Proposition 3.5. *If A is a ρ-commutative algebra then $HH_\alpha^1\left(A\right)=\Omega_\alpha^1\left(A\right)$.*

Definition 3.6. *The* cyclic homology *of a ρ-algebra A is the homology of the cyclic complex:*

$$...\to\Omega_\alpha^{n+1}\left(A\right)_\lambda\xrightarrow{b}\Omega_\alpha^n\left(A\right)_\lambda\to...$$

where $\Omega_\alpha^n\left(A_\beta\right)_\lambda$ is $\Omega_\alpha^n\left(A_\beta\right)/Im(1\text{-}\lambda)$ and its homology groups are denoted by $HC_\alpha(A)$.

Remark 3.7. *If $A=A^{(0)}\oplus A^{(1)}$ is a $\mathbb{Z}_2$-superalgebra and $\alpha=1$ then the previous operators d, b', b, β, λ are identical with the similar one from the book [10]. It follows that our construction of Hochschild and cyclic homology of ρ-algebras is a generalization of those homologies for $\mathbb{Z}_2$-superalgebras.*

Example 3.8. *Let be $\mathbb{H}$ the quaternionic algebra. $\mathbb{H}$ is the algebra over the field $\mathbb{R}$ generated by the elements e_1, e_2, e_3, e_4 with the relations*

$$\left\{ \begin{array}{c} e_i^2 = -e_1 for\ i = 2,3,4, \\ e_ie_1 = e_1e_i = e_i, \\ e_ie_j = -e_je_i, \\ e_2e_3 = e_4, e_2e_4 = -e_3 \text{ and } e_3e_4 = e_2. \end{array} \right. \tag{3.2}$$

$\mathbb{H}$ is a ρ-commutative algebra with the cocycle $\rho : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R}$ given by

$$\rho\left((\alpha_1,\beta_1),(\alpha_2,\beta_2)\right) = (-1)^{\alpha_1\beta_2-\alpha_2\beta_1}.$$

The $\mathbb{Z}_2$ degree of e_1 is $|e_1| = (0,0)$, of e_2 is $|e_2| = (1,0)$, of e_3 is $|e_3| = (0,1)$ and of the element e_4 is $|e_4| = (1,1)$.

Let α be an element of $\mathbb{Z}_2$. The algebra $\Omega_\alpha\mathbb{H}$ is generated by the elements e_1, e_2, e_3, e_4 and their differentials de_1, de_2, de_3, de_4 with the following relations:
1) the relations (3.2)
2) $\left\{ \begin{array}{c} e_ide_j = \rho\left(|e_i|,\alpha\right)\rho\left(|e_i|,|e_j|\right)\left(de_j\right)e_i \text{ for } i \neq j \\ e_ide_i = -\rho\left(|e_i|,\alpha\right)\left(de_i\right)e_i \end{array} \right.$
3) $\left\{ \begin{array}{c} de_ide_i = 0 \\ de_ide_j = -\rho\left(|e_i|,\alpha\right)\rho\left(\alpha,|e_j|\right)de_jde_i fori \neq j. \end{array} \right.$

From the relations 1) and 3) we deduce that $\Omega_\alpha\mathbb{H} = \Omega_\alpha^0\mathbb{H} \oplus \Omega_\alpha^1\mathbb{H} \oplus \Omega_\alpha^2\mathbb{H}$ and we get that $HH_\alpha^n(\mathbb{H}) = 0$ for $n \geq 3$.

From the proposition 4 we have that $HH_\alpha^1(\mathbb{H}) = \Omega_\alpha^1(\mathbb{H})$ and is the space generated by relations 1) and 2).

Concluding and remarks In this paper we had defined the Hochschild and cyclic homology of ρ-algebras. This constructions are natural generalizations of Hochschild and cyclic homology of associative resp. superalgebras and allows us to study in a new way these homologies of several kinds of ρ-algebras quantum hyperplane, matrix algebra, quaternionic algebra.

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