About some Riemann surfaces and plane tilings

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Abstract. In this paper, after a short introduction, we focus on some properties of Riemann surfaces that are connected with aspects from classical and modern crystallography. The third part of the paper contain our personal results. We define a discrete type generalized metrics for a plane tiling **T**and certain types of isometries and we develop a discrete type geometry associated to a tiling. These topics may be useful for investigating plane tilings with the trivial symmetry group and construction of new "surfaces".

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1 Introduction

Intuitively, a tiling of a plane is a way in which the entire space of the plane can be covered with a few tiles repeatedly in such a way that the tiles do not overlap. A periodic tiling is one in which the pattern of tiles repeats. i.e. its symmetry group contains a lattice. In 1966 Robert Berger proved that no fixed algorithm will determine if a given set of arbitrary tiles will tile the plane. Hence Berger's proof that no such algorithm exists implies that there must be sets of tiles that can tile the plane only non-periodically. An aperiodic tiling is a tiling of the plane by a set of prototiles that can only be tiled in a non-repeating (non-periodic) pattern. A tile set with this property is called aperiodic. So, the Penrose tiling is an aperiodic tiling. Aperiodic tiling was first considered only an interesting mathematical structure, but physical materials were later found where the atoms were arranged in the same pattern as a Penrose tiling. This pattern is not periodic (repeating exactly) but it is quasiperiodic (almost repeating), so the materials were named quasicrystals.

In classical crystallography, periodicity was imposed as well, but this restriction is not considered as fundamental in the modern era. With the discovery of intermetallic quasicrystals in the early 1980s, it became clear that there exist aperiodic point sets that share a basic property with periodic point sets. Quasicrystals were discovered by Schectman due to unusual 10-fold rotational symmetry exhibited in certain directions

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by their (electron) diffraction patterns, a symmetry which is well known to be impossible for ordinary crystals (see crystlographical restriction). In a normal crystalline solid the positions of atoms are arranged in a periodic crystal lattice of points, which repeats itself in space. In a quasicrystal, the pattern of atoms is only quasiperiodic. The local arrangements of atoms are fixed, and in a regular pattern, but are not periodic throughout the entire material: each cell has a different configuration of cells surrounding it. Quasicrystals helped to redefine the notion of what makes a crystal, since they do not have a repeating unit cell but do display sharp diffraction peaks. There is a strong analogy between the quasicrystal and the Penrose tiling of Roger Penrose and some Riemann Surfaces, between some Riemann Surfaces and some orbifolds. In fact, some quasicrystals can be sliced such that the atoms on the surface follow the exact pattern of the Penrose tiling, see [3] and [5]. For more results see [12], [13], [14], [15].

Some authors like B. Grünbaum, G. C. Shepherd, C. Radin, R. M. Robinson, R. Penrose, H. Wang have investigated the periodic and aperiodic plane tilings ([7], [12], [13], [15]). We have not found yet any investigation for "pathological" tilings i.e. tilings with trivial symmetry group which are not mentioned above. Also, we have not found yet some classification for tilings with trivial symmetry group.

2 Riemann surfaces

We know that a Hausdorff connected topological space R is an (abstract) *Riemann surface* if there exists an atlas $\{(\varphi_j, U_j), j \in J\}$ such that:

1. $\{U_j : j \in J\}$ is an open cover of R;
2. each $\varphi_j : U_j \longrightarrow C_j$, where C_j is an open subset of the complex plane, is a homeomorphism;
3. if $U = U_i \cap U_j \neq \varnothing$ then $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U) \longrightarrow \varphi_i(U)$ is an analytic map between the plane sets $\varphi_j(U)$ and $\varphi_i(U)$.

One method of constructing a Riemann surface is described by the following theorem ([1], pg. 118). *Let D be a subdomain of $\mathbf{C} \cup \infty$ and G be a group of Möbius transformation which leaves D invariant and which acts discontinuously in D. Then D/G is a Riemann surface.*

Also, we consider Riemann surfaces as analytically continued holomorphic functions, interpreted geometrically as surfaces immersed in $\mathbf{C}^2$ ([5]). First, let F=(U, f) be a *function element*, where U is a domain in $\mathbf{C}$ and f is a holomorphic function $f : U \longrightarrow \mathbf{C}$. The function elements $F_1 = (U_1, f_1), F = (U_2, f_2)$ are *direct continuation of each other* if $V = U_1 \cap U_2 \neq \varnothing$ and $f_1 = f_2$ when restricted to V. We consider now the *complete, global analytic function determined by the function element* $F_0 = (U_0, f_0)$ being the maximal collection of function elements $\mathbf{F}$ such that $(\forall) F_i \in \mathbf{F}$ $(\exists) F_0 \longrightarrow ... \longrightarrow F_i$ a chain of function elements all in $\mathbf{F}$, with every link in the chain a direct analytic continuation. The immersed Riemann surface R corresponding to $\mathbf{F}$ is the image of the immersion $\psi : \mathbf{F} \longrightarrow \mathbf{C}^2$ given by $\psi : (U, f) \longmapsto \{(x, f(x)) : x \in U\}$.

The vertices of the Penrose tiling of the plane is a familiar example of a quasiperiodic set. In [5] is given a standard construction for a quasiperiodic set. Moreover, in [5] is proved that up to linear transformations, there are seven *discrete* Riemann surfaces generated by conformal maps of right triangles (3 periodic and 4 quasiperiodic). The surfaces quasiperiodic form a quasiperiodic point sets closely related to the vertex sets of quasiperiodic tilings.

For the definition of orbifolds we use [11].

A geometric *n-orbifold M* with model the Riemannian manifold X is a metric space $|M|$ together with an atlas of charts $\left\{\tilde{U}_i,\ f_i\right\}$, such that

1. $f_i : \tilde{U}_i \longrightarrow U_i$, where $\tilde{U}_i$ is an open subset of X and where the $U_i's$ form an open cover of $|M|$;

2. the $f_i's$ are folding maps, i.e. the group G_i of diffeomorphisms $\tilde{U}_i$ preserving the fibers of f_i must be finite and the induced map $\tilde{U}_i/G_i \longrightarrow U_i$ must be a homeomorphism;

3. the charts are compatible, meaning that, whenever $f_i\left(x_i\right) = f_j\left(x_j\right)$, there exist some open neighborhoods $\tilde{V}_i, \tilde{V}_j$ *of* $x_i,\ x_j$ *in* $\tilde{U}_i,\ \tilde{U}_i$, and a diffeomorphism that preserve the Riemannian metric $v_{ij} : \tilde{V}_i \longrightarrow \tilde{V}_j$, such that $(f_j \circ v_{ij})(x) = f_i, (\forall)\, x \in \tilde{V}_i$.

If X is the Euclidean, spherical or hyperbolic n-space, the orbifold will be called Euclidean, spherical or hyperbolic.

We consider now the category of hyperbolic orbifolds. Let G be a Fuchsian group, that is a discrete subgroup of conformal automorphisms of the unit disc D; we do not require G to be torsion-free or finitely generated. Then the quotient $M = D/G$ has a structure of a hyperbolic orbifold; each point of M has a neighborhood which is modeled on the quotient of a disk by a finite group of rotations. The projection $\pi : D \longrightarrow M = D/G$ is the universal covering of the orbifold M. Let $B \in M$ denote the discrete set of branch points of X, i. e. the points in the quotient corresponding to a fixed points of elliptic elements of G. Then *M-B* has the structure of an ordinary Riemann surface, see [4]. In [11] are given very interesting properties of classical tessellations and its manifolds, that are, in fact, orbifolds. For more results see also [3], [3], [5], [6], [8], [9], [9], [18].

3 The geometry of discrete type associated to plane tilings

Let E^2 be the Euclidean plane.

Definition 1 *A* plane tiling *(see also [7]) is a countable family* $\mathbf{T}$ *of closed sets* T_i*,* $i \in I$*, called* tiles*, which satisfies the axioms:*
$GS1 : \bigcup_{i\in I} T_i = E^2,\ |I| = \aleph_0;$
$GS2 : int(T_i) \cap int(T_j) = \varnothing$.

Let's consider a plane tiling $\mathbf{T}$. Then we have:

$$\bigcup_{i\in I} T_i = \bigcup_{i\in I} int\,(T_i) \bigcup \left(E^2 - \bigcup_{i\in I} int\,(T_i)\right).$$

We denote $\mathbf{B}(\mathbf{T}) = E^2 - \bigcup_{i\in I} int\,(T_i)$. and we call it, conventionally, *boundary tile* ([17]) of the tiling T. Then we can write:

$$\bigcup_{i\in I} T_i = \bigcup_{i\in I} int(T_i) \cup \mathbf{B}(\mathbf{T}).$$

In other words, to any tiling $\mathbf{T}$ we can associate the *canonical plane partition* of the Euclidean plane formed by the interiors of the tiles and the boundary tile.

A plane tiling is *locally finite* ([7]) if every circular disc, with the center at any point, meets just a number of tiles. It what follows, we suppose that we have a locally finite plane tiling with closed topological discs. We fix two tiles T and $T^{'}$ so that $d(T,\ T^{'}) > 0$ (where d is the usual Euclidean metric). Under these conditions, we see that we can have a finite chain of tiles which joins the given tiles without passing two times through the same tile. The chain is not unique, but it can have a minimal number of tiles, say k, and we call it *the minimal chain* which joins the given tiles. We call k the *length* of the chain ([16], [17]).

Let now $y,\ w \in E^2$.

Definition 2 *We define a* map

$$d_{\mathbf{T}} : E^2 \times E^2 \longrightarrow \mathbf{R} \cup \infty,$$

by

1. $d_{\mathbf{T}}(y, w) = 0$, if $(\exists)\ T_i \in \mathbf{T}$ such that $y,\ w \in int(T_i)$ or $y,\ w \in \mathbf{B}(\mathbf{T})$;
2. $d_{\mathbf{T}}(y, w) = k$, (k positive integer) if

   $$y \in int\,(T_i)\,,\ w \in int\,(T_j)\,,\ T_i,\ T_j \in \mathbf{T},\ i \neq j,$$

   and the minimal chain which joins these tiles has length k;
3. $d_{\mathbf{T}}\,(y,\ w) = \infty$, if $y \in int\,(T_i)$ and $w \in \mathbf{B}(\mathbf{T})$ or $y \in \mathbf{B}(\mathbf{T})$ and $w \in int\,(T_i)$;
4. $d_{\mathbf{T}}(y, w) = \frac{3}{2}$, if $y \in int\,(T_i)\,,\ w \in int\,(T_j)$ and $d(T_i,\ T_j) = 0$.

We define the relation " $\equiv$ " on set $E^2 = \bigcup_{i\in I} T_i$ as follows:

$$y \equiv w \Leftrightarrow (\exists) T_i \in \mathbf{T} \text{ such that } y,\ w \in int\,(T_i) \text{ or } y,\ w \in \mathbf{B}(\mathbf{T}).$$

Obviously, the above relation is *an equivalence relation* on E^2. We denote the quotient space under this relation with Ω.

A map $\delta : M \times M \longrightarrow \mathbf{R} \cup \infty$ which satisfies the properties:

1. $\delta\,(x,\ y) = 0 \Leftrightarrow\ x = y$;
2. $\delta\,(x,\ y) \geq 0, (\forall)\ x,\ y \in M$;
3. $\delta\,(x,\ y) = \delta(y,\ x), (\forall) x,\ y \in M$;
4. $\delta\,(x,\ y) \leq \delta\,(x,\ z) + \delta\,(z,\ y)\,, (\forall) x,\ y,\ z \in M$

is called a *generalized metric*. The couple (M, δ) it is called a *generalized metric space* (see, also [19]).

With the obvious notation for the equivalence class, we have our main result:

Theorem 1 *The map* $\mathbf{d_T} : \Omega \times \Omega \longrightarrow \mathbf{R} \cup \infty$, *defined by*

$$\mathbf{d_T}(\hat{y},\ \hat{w}) = d_{\mathbf{T}}(y,\ w),$$

is a generalized metric.

Proof. We prove first that the above map is correctly defined. So, we have the cases:

1. Let be $\hat{y} = int(T_i),\ \hat{w} = int(T_j)$ and $y' \in \hat{y},\ w' \in \hat{w}$. If $i \neq j$, we have $\mathbf{d_T}(\hat{y},\ \hat{w}) = d_{\mathbf{T}}(y,\ w) = k = d_{\mathbf{T}}(y',\ w') = \mathbf{d_T}(\hat{y'},\ \hat{w'})$ where k is the length of the minimal chain (see the above definitions) or

$$\mathbf{d_T}(\hat{y},\ \hat{w}) = d_{\mathbf{T}}(y,\ w) = \tfrac{3}{2} = d_{\mathbf{T}}(y',\ w') = \mathbf{d_T}(\hat{y'},\ \hat{w'}),$$

according to above definition. If *i=j* , we have

$$\mathbf{d_T}(\hat{y},\ \hat{w}) = d_{\mathbf{T}}(y,\ w) = 0 = d_{\mathbf{T}}(y',\ w') = \mathbf{d_T}(\hat{y'}, \hat{w'}).$$

In the same way we prove the cases:

2. $\hat{y} = int\,(T_i)\,,\ \hat{w} = \mathbf{B}(\mathbf{T})$;
3. $\hat{y} = \mathbf{B}(\mathbf{T}),\ \hat{w} = \mathbf{B}(\mathbf{T})$.

We check now that $\mathbf{d_T}$ is a generalized metric. It is clear that we have

1. $\mathbf{d_T}(\hat{y},\ \hat{w}) = 0$ iff $\hat{y} = \hat{w}$,
2. $\mathbf{d_T}(\hat{y},\ \hat{w}) \geq 0\ \forall\ \hat{y},\ \hat{w}$ and
3. $\mathbf{d_T}(\hat{y},\ \hat{w}) = \mathbf{d_T}(\hat{w},\ \hat{y}), \forall\ \hat{y},\ \hat{w}$.

We prove now that the following inequality holds:

$$\mathbf{d_T}(\hat{y},\ \hat{w}) \leq \mathbf{d_T}(\hat{y},\ \hat{u}) + \mathbf{d_T}(\hat{u},\ \hat{w}), \forall \hat{y}, \hat{w}, \hat{u} \ (1)$$

We have the following cases:

1. $\hat{y} = \hat{w} = \hat{u} = \mathbf{B}(\mathbf{T})$, then (1) holds;
2. $\hat{y} = \hat{w} = \mathbf{B}(\mathbf{T}),\ \hat{u} = int\,(T_i)$, then $0 \leq \infty + \infty$, so (1) holds;
3. $\hat{w} = \hat{u} = \mathbf{B}(\mathbf{T}),\ \hat{y} = int\,(T_i)$, then $\infty \leq \infty + 0$, so (1) holds;
4. $\hat{y} = \hat{u} = \mathbf{B}(\mathbf{T}),\ \hat{w} = int\,(T_i)$, then $\infty \leq 0 + \infty$, so (1) holds;
5. $\hat{y} = \mathbf{B}(\mathbf{T}),\ \hat{w} = \hat{u} = int\,(T_i)$, then $\infty \leq \infty + 0$, so (1) holds;
6. $\hat{w} = \mathbf{B}(\mathbf{T}),\ \hat{y} = \hat{u} = int\,(T_i)$, then $\infty \leq 0 + \infty$, so (1) holds;

7. $\hat{u} = \mathbf{B}(\mathbf{T}),\ \hat{y} = \hat{w} = int\,(T_i)$, then $0 \leq \infty + \infty$, so (1) holds;

8. $\hat{y} = \mathbf{B}(\mathbf{T}),\ \hat{w} = int\,(T_i),\ \hat{u} = int\,(T_j),\ i \neq j$, then $\infty \leq \infty + k$, where k is the length of the minimal chain which joins the tiles T_i and T_j or $\infty \leq \infty + \frac{3}{2}$ if $d\,(T_i, T_j) = 0$, so (1) holds;

9. $\hat{w} = \mathbf{B}(\mathbf{T}),\ \hat{y} = int\,(T_i),\ \hat{u} = int\,(T_j),\ i \neq j$, then $\infty \leq k + \infty$, where k is the length of the minimal chain which joins the tiles T_i and T_j or $\infty \leq \frac{3}{2} + \infty$ if $d\,(T_i, T_j) = 0$, so (1) holds;

10. $\hat{u} = \mathbf{B}(\mathbf{T}),\ \hat{y} = int\,(T_i),\ \hat{w} = int\,(T_j),\ i \neq j$, then $k \leq \infty + \infty$, where k is the length of the minimal chain which joins the tiles T_i and T_j or $\frac{3}{2} \leq \infty + \infty$ if $d\,(T_i, T_j) = 0$, so (1) holds; (May be obtained three analogues cases by interchanging i with j).

11. $\hat{y} = \hat{w} = \hat{u} = int\,(T_i)$, then (1) is obviously;

12. $\hat{y} = \hat{w} = int(T_i), \hat{u} = int(T_j),\ i \neq j$, then $0 \leq k + k$, where k is the length of the minimal chain which joins the tiles T_i and T_j or $0 \leq \frac{3}{2} + \frac{3}{2}$ if $d(T_i, T_j) = 0$, so (1) holds;

13. $\hat{y} = \hat{u} = int(T_i), \hat{w} = int(T_j),\ i \neq j$, then we have $k \leq 0 + k$, where k is the length of the minimal chain which joins the tiles T_i and T_j or $\frac{3}{2} \leq 0 + \frac{3}{2}$ if $d(T_i,\ T_j) = 0$, so (1) holds;

14. $\hat{w} = \hat{u} = int(T_i), \hat{y} = int(T_j),\ i \neq j$, then we have $k \leq k + 0$, where k is the length of the minimal chain which joins the tiles T_i and T_j or $\frac{1}{2} \leq \frac{1}{2} + 0$ if $d(T_i,\ T_j) = 0$, so (1) holds;

15. $\hat{y} = int(T_i),\ \hat{w} = int(T_j),\ \hat{u} = int(T_l)$, where $i,\ j,\ l$ are distinct, then, if $\mathbf{d_T}(\hat{y},\ \hat{w}) = k,\ \mathbf{d_T}(\hat{y},\ \hat{u}) = p,\ \mathbf{d_T}(\hat{u},\ \hat{w}) = q$, it follows $k \leq p + q$, otherwise we obtain a contradiction with the the minimality of k; if $d(T_i,\ T_j) = d(T_i,\ T_l) = d(T_l,\ T_j) = 0$, then we have $\frac{3}{2} \leq \frac{3}{2} + \frac{3}{2}$; if $d(T_i,\ T_j) = 0,\ \mathbf{d_T}(\hat{y},\ \hat{u}) = p,\ \mathbf{d_T}(\hat{u},\ \hat{w}) = q$, it follows $\frac{3}{2} \leq p+q$; if $d(T_i,\ T_l) = 0,\ \mathbf{d_T}(\hat{y},\ \hat{w}) = k,\ \mathbf{d_T}(\hat{u},\ \hat{w}) \in \{k-1,\ k,\ k+1\}$ then we obtain $k \leq \frac{3}{2} + k$ or $k \leq \frac{3}{2} + k + 1$ or $k \leq \frac{3}{2} + k - 1$ when $k \geq 2$; if $d(T_j,\ T_l) = 0,\ \mathbf{d_T}(\hat{y},\ \hat{w}) = k,\ \mathbf{d_T}(\hat{y},\ \hat{u}) \in \{k-1,\ k,\ k+1\}$ then we obtain $k \leq k + \frac{3}{2}$ or $k \leq k + 1 + \frac{3}{2}$ or $k \leq k - 1 + \frac{3}{2}$ when $k \geq 2$; if $d(T_i,\ T_j) = d(T_i,\ T_l) = 0,\ \mathbf{d_T}(\hat{u},\ \hat{w}) = q$ then we obtain $\frac{3}{2} \leq \frac{3}{2} + q$; if $d(T_i,\ T_j) = d(T_j,\ T_l) = 0,\ \mathbf{d_T}(\hat{y},\ \hat{u}) = p$ then we obtain $\frac{3}{2} \leq p + \frac{3}{2}$; if $d(T_i,\ T_l) = d(T_l,\ T_j) = 0,\ d(T_i,\ T_j) \neq 0$ then we obtain $1 \leq \frac{3}{2} + \frac{3}{2}$ so (1) holds. In the same way we prove the others five cases that are obtained by permutations q.e.d.

It follows that $(\Omega, \mathbf{d_T})$ is a *generalized metric space*. The *proper points* of the space $(\Omega, \mathbf{d_T})$ correspond to $int(T_i),\ T_i \in \mathbf{T}$, and the *improper point* corresponds to $\mathbf{B(T)}$. For simplicity, we denote A, B, C,..., the proper points.

We can construct, as usually, a basis for a topology on our space induced by the generalized metric and also we can define the neighborhood of a point.

Definition 3 *We say that a* minimal chain of length $k \geq 1$ joins the point A which corresponds to $int(T)$, with the point B, which corresponds to $int(T^{'})$*, and we denote*

$A \longrightarrow B$ if it joins the tile T *with the tile $T^{'}$. The distinct proper points* A*,* B*,* C *are called* collinear *if there is a minimal chain of tiles, which joins two of these and contains the third one. The improper point* **B(T)** *is not collinear with the proper points.*

We suppose that $L(A,C)$ is the minimal chain of tiles which joins A with C. The proper point B will be collinear with A and C if $B \in L(A,C)$. We say in this case, that B is *between* A and C. We denote this by: A-B-C.

Let A, B be two proper points so that $\mathbf{d_T}(A,B) \geq 1$. *The closed segment with the endpoints* A*,* B *and the chain* L, denoted $[AB](L)$ is defined by

$$[AB](L) = \{C \mid C \in L(A,B)\} \cup \{A,B\}$$

If $\mathbf{d_T}(A,B) = 0$, we have the *null segment*, and if $\mathbf{d_T}(A,B) = \frac{3}{2}$, we have the *improper segment*. The *length* of the segment $L(A,B)$ is the length of a minimal chain with the endpoints A, B. The length of the null segment is 0 and the length of improper segment is $\frac{3}{2}$. Two segments are *congruent* if they have the same length. This relation is an equivalence relation on the set $\mathbf{S}$ of all segments.We can easy define a totally ordered relation on $\mathbf{S}$.

Let A and B two proper points. We choose the *infinite arrays of proper points* (A_n) and (B_n) with the following properties:

1. $A_0 = A,\ B_0 = B$;
2. A_0 is between B_0 and A_1, B_0 is between A_0 and B_1;
3. A_n is between A_{n-1} and B_n is between B_{n-1} and B_{n+1}, $\forall n \geq 2$.

The set $(\bigcup[A_n, A_{n+1}])\bigcup(\bigcup[B_n, B_{n+1}])\bigcup[AB]$, $n \in \mathbf{N}$ *is said to be a line passing through* A, B and we denote it by $AB(L^*)$, where L^* represents the *infinite chain* of tiles which is the union of the minimal chains.

Remark 1 *We can define the concurrent lines, the perpendicular lines, the half-planes, parallel lines, half-lines, an order for the set of all points of a line, so an oriented line and a coordinate system. Also we can easy define the angle and the triangle, but we cannot prove yet that there is a unique measure map for the angles. The congruent relations between angles, respectively triangles, raise no problems.*

We say that the proper points A, B, C, D are the vertices of *a generalized parallelogram* if $\mathbf{d_T}(A,B) = \mathbf{d_T}(C,D) = k$ and $\mathbf{d_T}(A,D) = \mathbf{d_T}(B,C) = p$, where $k,p \geq 1$, and the minimal chains which join these points

$$(A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow A)$$

have no common tiles. We denote this generalized parallelogram by $ABCD$.

Definition 4 *The bijective map $t(k) : \mathbf{T}' \longrightarrow \mathbf{T}'$, where $k \in \mathbf{N}$, with the properties:*

1. *there is a minimal chain of length* k *,* $k \geq 1$*, which joins* A *and* $t(k)(A)$*,* $\forall A \in \mathbf{T}'$*;*
2. *for any* $A, B \in \mathbf{T}', A \neq B$*, we have the generalized parallelogram* $AA^{'}B^{'}B$*, where* $A^{'} = t(k)(A)$ *and* $B^{'} = t(k)(B)$*;*

3. $t(k)$*(*$\mathbf{B(T)}$*)=*$\mathbf{B(T)}$*;*

4. $t(k)^p(A) \neq A, (\forall)p > 1,\ p \in \mathbf{N}$*;*

5. *the map* t*(0) is the identity of* $\mathbf{T}'$ *is called* k-translation.

We can prove easily the following theorem:

Theorem 2 *Any k-translation $t(k)$ is an isometry, i.e.*

$$\mathbf{d_T}\left((t(k)(\widehat{y}), (t(k)(\widehat{w})\right) = \mathbf{d_T}(\widehat{y}, \widehat{w}), (\forall)(\widehat{y}), (\widehat{w}) \in \mathbf{T}'.$$

We can define the *k*-rotation and the *k*-reflexion so that they are isometries.

Remark 2 *Let A an exterior point for a given line $d(L^*)$. Then we* conjecture *that our tiling* $\mathbf{T}$ *can be globally or locally of the types:*

1. *space of quasi-parabolic type if there exists an unique parallel trough A by our line;*

2. *space of quasi-elliptic type if there does not exist a parallel with the given line;*

3. *space of quasi-hyperbolic type if there are two parallel with the given line. There exists a tiling such that the space are locally quasi-parabolic type, quasi-elliptic type and quasi-hyperbolic type (see fig. 7 in [17]).*

For more results see [16], [17].

We know that the Euclidean plane is a Riemann surface. These topics may be used for obtaining a new "'surfaces or a classification for the tilings with the trivial symmetry group? I don't know yet. This is one of the subjects for my further research.
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References

[1] A. Beardon, *The Geometry of Discrete Groups*, Springer, 1983.

[2] A. Costa, *On anticonformal automorphisms of Riemann Surfaces with nonembedable square*, Proc.AMS 124 (1996),601-605.

[3] D. P. DiVicenzo, P. J. Steinhardt Eds., *Quasicrystals: The State of the Art*, Directions in Condensed Matter Physics, Vol. 11, World Scientific, Singapore, 1991.

[4] C. J. Earle, C. McMullen *Quasiconformal Isotropies*, Math. Sciences Reasearch Institute, Berkley, 1986.

[5] V. Esler, *Crystallograohy and Rieman Surfaces*, math.DG/9904016.

[6] G. Gromadki, M. Izquierdo, *Real forms of a Riemann Surface of even genus* Proc. AMS 126 (1988), 3475-3479.

[7] B. Grünbaum and G. C. Schepard, *Tilings and Patterns: An Introduction*, Freemann, New York, 1987.

[8] S. Fujimori, *Minimal Surfaces in Euclidean 3-Space and their mean curvature 1 cousins in hyperbolic 3-space*, Annals of Brazilian Academy of Sciences 75(3) (2003), 271-278.

[9] J. Jost, *Compact Riemann Surfaces*, Springer 1997.

[10] F. Lopez, F. Martin, *A note on the Gauss map of complete nonorientable minimal surfaces*, Pacific J. Math. 194 (2000), 129-136.

[11] J. M. Montesinos, *Classical Tesselations and Three-Manifolds*, Springer, 1987.

[12] C. Radin, *Orbits of orbs: Sphere packings meets Penrose tilings*, Math. Assoc. Monthly 111 (2004), 137-149.

[13] C. Radin, *Tiling, periodicity and crystals*, J. Math. Phys. 43 (1986), 707-712.

[14] C. Radin, *Symmetries of Quasicrystals*, J. Statist. Phys. 95 (1999), 827-833.

[15] R. M. Robinson, *Undecidability and nonperiodicity for tilings of the plane*, Invent. Math. 12 (1971), 177-2009.

[16] A. Sirbu, *The geometry of the tilings (Romanian)*, Doctoral Thesis, University of West, Timisoara, 2001.

[17] A. Sirbu, *On discrete type of geometries of tilings*, Analele Universitatii din Timisoara XXXIX (2001), 401-409.

[18] G. Springer, *Introduction to Riemann surfaces*, Adison-Wesley , 1957.

[19] S. Vickers, *Localic completion of generalized metric spaces I*, Theory and Application of Categories 14 (2005), 328-356.

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