Optimal control of deteriorating production inventory systems

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Abstract

We are using in this paper an optimal control approach to determine the optimal production rate in a production inventory system where items are subject to deterioration. Necessary and sufficient optimality conditions are obtained. The optimal solutions are given in detail for specific exogenous functions and illustrative examples are provided.

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1 Introduction

Inventory theory is one of the main areas of Operations Research and Management Science, and a variety of mathematical models have been introduced for the control of inventory systems. Of particular interest in inventory theory are deteriorating inventory systems in which the product deteriorates with time. Three surveys are available (Nahmias (1982), Raafat (1991), and Goyal and Giri (2001)), which shows the importance of the subject.

We will mention first that inventory models are classified as deterministic (static or dynamic) or stochastic, according to whether the demand rate is known (constant or dynamic) or random. Standard calculus is used to deal with static models (Harris (1913)) while filtering techniques (Kalman (1960a, 1960b) are used on stochastic models.

The model we are interested in is dynamic. Dynamic programming has been used to solve some dynamic models (Wagner and Whitin (1958)). In the absence of deterioration, some dynamic models have been studied using an optimal control approach. For example, Salama (2000) considered the optimal control of an unreliable manufacturing system with restarting costs. Riddalls and Bennett (2001) used an optimal control algorithm to a differential equation model of a production inventory system to cater for batch production costs which, usually, are not modelled in aggregate production problems. Zhang *et al.* (2001) were concerned with the scheduling of a marketing

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production system with a demand dependent on the marketing status. Khemlnitsky and Gerchak (2002) used an optimal control approach to solve a production system where demand depends on the inventory level. Kiesmüller (2003) was interested in the optimal control of recovery systems, where attention is given to recycling and remanufacturing of used products in order to reduce waste. Dobos (2003) were interested in the optimal control of reverse logistics systems, where reusable materials returned from the market are remanufactured. Finally, multi-echelon inventory systems have been considered by Diks and de Kok (1998). We are not aware of any papers dealing with the optimal control of deteriorating inventory systems.

We are thus considering in the present paper the optimal control of a deteriorating inventory system. The goal is to determine the production rate that minimizes some cost function. The layout of the paper is as follows. We start in Section 2 by introducing the notation. Then we build the mathematical model and derive the necessary and sufficient optimality conditions in the case where the planning horizon is finite. We also give explicit solutions as functions of time for some specific exogenous functions. The results obtained are extended in Section 3 to the case of an infinite planning horizon. In Section 4 we illustrate the results obtained by performing a complete analysis of a system with linear demand rate. Finally we formulate our conclusions in Section 5.

2 Model Formulation

Let us consider a manufacturing firm producing a single product. To state the model we use the following notation:

$I(t)$	:	the inventory level at time t,
$P(t)$	:	the production rate at time t,
$D(t)$	:	the demand rate at time t,
$\theta(t, I(t))$	:	the deterioration rate at time t corresponding to $I(t)$,
$h(I(t))$	:	the holding cost rate corresponding to $I(t)$,
$K(P(t))$	:	the production cost rate corresponding to $P(t)$,
T	:	the length of the planning horizon,
ρ	:	the constant nonnegative discount rate,
I_0	:	the initial inventory level,
$\hat{\theta}$	:	the deterioration goal rate,
$\hat{I}$	:	the inventory goal level,
$\hat{P}$	:	the production goal rate,
c	:	the positive unit production cost.

The interpretation of the goal rates are as follows:

- We are assuming that product deterioration can be somehow controlled by the firm. This can be done by keeping the product in adequate conditions through control of such elements as pressure, temperature, etc. The firm wants to keep the product deterioration close to the deterioration goal rate $\hat{\theta}$. For example, $\hat{\theta}$ could be 10 units of the finished product, during the time interval $[0, T]$.

- The inventory goal level $\hat{I}$ is a safety stock that the company wants to keep on hand. For example, $\hat{I}$ could be 200 units of the finished product.
- The production goal rate $\hat{P}$ is the most efficient rate desired by the firm.

All functions are assumed to be non-negative, continuous and differentiable functions. Since demand occurs at rate D, production occurs at the controllable rate P, and deterioration occurs at rate θ, it follows that the inventory level $I(t)$ evolves according to the dynamics (or state equation)

$$\frac{d}{dt}I(t) = P(t) - D(t) - \theta(t, I(t)). \tag{2.1}$$

The model is represented as an optimal control problem with one state variable (inventory level) and one control variable (rate of manufacturing). The problem $(\mathcal{P})$ associated to this model is to minimize the following objective function

$$\min_{P(t)\geq 0} J(P, I) = \int_0^T F(t, I(t), P(t))dt,$$

subject to the state equation (2.1) where

$$F(t, I(t), P(t)) = e^{-\rho t}\left(\frac{1}{2}\left[h(I(t)) - h(\hat{I})\right]^2 + \frac{1}{2}\left[K(P(t)) - K(\hat{P})\right]^2 + \frac{c}{2}\left[\theta(t, I(t)) - \hat{\theta}\right]^2\right). \tag{2.2}$$

The main tool in the study of problems $(\mathcal{P})$ involves looking for the necessary optimality conditions in the form of the Pontryagin maximum principle (see Pontryagin *et al.* (1962)). The theory involves the Hamiltonian function

$$H(t, I(t), P(t), \lambda(t)) = -F(t, I(t), P(t)) + \lambda(t)f(t, I(t), P(t)), \tag{2.3}$$

where $f(t, I(t), P(t)) = P(t) - D(t) - \theta(t, I(t))$ and λ is the adjoint function associated with the differential equation constraint. The next result is a specialized version of the Pontryagin maximum principle for problem $(\mathcal{P})$

Theorem 2.1 *A necessary condition for (P^*, I^*) to be an optimal solution of problem $(\mathcal{P})$ is*

$$\begin{aligned}&\left[K(P^*(t)) - K(\hat{P})\right]\left(\frac{d}{dt}P^*(t)\frac{d^2}{dP^2}K(P^*(t)) - \left(\rho + \frac{\partial}{\partial I}\theta(t, I^*(t))\right)\frac{d}{dP}K(P^*(t))\right) = \\ &-\frac{d}{dt}P^*(t)\left(\frac{d}{dP}K(P^*(t))\right)^2 + \left[h(I^*(t)) - h(\hat{I})\right]\frac{d}{dI}h(I^*(t)) + c\left[\theta(t, I^*(t)) - \hat{\theta}\right]\frac{\partial}{\partial I}\theta(t, I^*(t)),\end{aligned} \tag{2.4}$$

and

$$I^*(0) = I_0, \quad \left[K(P^*(T)) - K(\hat{P})\right]\frac{d}{dP}K(P^*(T)) = 0, \quad P^*(t) \geq 0.$$

Proof. Assume that (P^*, I^*) is an optimal solution to problem $(\mathcal{P})$. By the maximum principle, there exists an adjoint function λ satisfying

$$H(t, I^*(t), P^*(t), \lambda(t)) \geq H(t, I^*(t), P(t), \lambda(t)), \text{ for all } P(t) \geq 0 \tag{2.5}$$

$$-\frac{d}{dt}\lambda(t) = \frac{\partial}{\partial I}H(t, I^*(t), P^*(t), \lambda(t)), \tag{2.6}$$

$$I^*(0) = I_0, \qquad \lambda(T) = 0. \tag{2.7}$$

Equation (2.5) is equivalent to

$$\frac{\partial}{\partial P} H(t, I^*(t), P^*(t), \lambda(t)) = 0,$$

which is equivalent to

$$\lambda(t) = e^{-\rho t}\Big[K(P(t)) - K(\hat{P})\Big]\frac{d}{dP}K(P(t)). \tag{2.8}$$

Equation (2.6) is equivalent to

(2.9)
$$\frac{d}{dt}\lambda(t) = e^{-\rho t}\Big\{\Big[h(I(t)) - h(\hat{I})\Big]\frac{d}{dI}h(I(t)) + c\Big[\theta(t, I(t)) - \hat{\theta}\Big]\frac{\partial}{\partial I}\theta(t, I(t))\Big\} + \lambda(t)\frac{\partial}{\partial I}\theta(t, I(t)).$$

Now, combining Equation (2.8) and Equation (2.9) completes the proof. □

The following lemma is a standard result from convex analysis, needed in the proof of the next theorem.

Lemma 2.1 *Let X be a normed vector space and let $f : X \to \mathbb{R}$ be a convex function, differentiable at some point $\bar{x} \in X$. Then the gradient $\nabla f(\bar{x})$ of f at $\bar{x}$ satisfies the following inequality:*

$$\nabla f(\bar{x})(\bar{x} - x) \geq f(\bar{x}) - f(x), \text{ for all } x \in X.$$

Theorem 2.2 *Assume the functions $F(t, \cdot, P)$ and $\theta(t, \cdot)$ are convex. Then, the necessary conditions* (2.5)-(2.7) *are sufficient for (P^*, I^*) to be an optimal solution for problem $(\mathcal{P})$.*

Proof. By Lemma 2.1 and the convexity of the function $F(t, \cdot, P)$, we have the following inequality:

$$\frac{\partial}{\partial I}F(t, I^*(t), P(t))\,[I^*(t) - I(t)] \geq F(t, I^*(t), P(t)) - F(t, I(t), P(t)), \quad \text{for all } t, I, P.$$

Equation (2.6) is equivalent to

$$\frac{\partial}{\partial I}F(t, I^*(t), P(t)) = \frac{d}{dt}\lambda(t) - \lambda(t)\frac{\partial}{\partial I}\theta(t, I^*(t)).$$

Substituting the value of $\frac{\partial}{\partial I}F(t, I^*(t), P(t))$ in the last inequality we obtain

(2.10)
$$\frac{d}{dt}\lambda(t)(I^*(t) - I(t)) - \lambda(t)\frac{\partial}{\partial I}\theta(t, I^*(t))\,[I^*(t) - I(t)] \geq$$
$$e^{-\rho t}\Big[[h(I^*(t)) - h(\hat{I})]^2 - [h(I(t)) - h(\hat{I})]^2 + \frac{c}{2}[\theta(t, I^*(t)) - \hat{\theta}]^2 - [\theta(t, I(t)) - \hat{\theta}]^2\Big], \ \forall\ t, I, P.$$

On the other hand, the convexity of the functions $\theta(t, \cdot)$ and Lemma 2.1 give

$$\frac{\partial}{\partial I}\theta(t, I^*(t))\,[I^*(t) - I(t)] \leq \theta(t, I^*(t)) - \theta(t, I(t)), \quad \text{for all } t, I.$$

Substituting this inequality in (2.10) we get

$$\frac{d}{dt}\lambda(t)\,[I^*(t)-I(t)]-\lambda(t)\Big[\theta(t,I^*(t))-\theta(t,I(t))\Big]\geq$$
$$e^{-\rho t}\Big[[h(I^*(t))-h(\hat{I})]^2-[h(I(t))-h(\hat{I})]^2+\frac{c}{2}[\theta(t,I^*(t))-\hat{\theta}]^2-[\theta(t,I(t))-\hat{\theta}]^2\Big],\quad \forall t,I.$$

Now, by the state equation (2.1) we have

$$\theta(t,I^*(t))=P^*(t)-D(t)-\frac{d}{dt}I^*(t) \text{ and } \theta(t,I(t))=P(t)-D(t)-\frac{d}{dt}I(t).$$

Therefore, substituting these two equalities in the last inequality we obtain

$$\frac{d}{dt}\lambda(t)\,[I^*(t)-I(t)]-\lambda(t)\Big[[P^*(t)-P(t)]-[\frac{d}{dt}I^*(t)-\frac{d}{dt}I(t)]\Big]\geq$$
$$e^{-\rho t}\Big[[h(I^*(t))-h(\hat{I})]^2-[h(I(t))-h(\hat{I})]^2+\frac{c}{2}[\theta(t,I^*(t))-\hat{\theta}]^2-[\theta(t,I(t))-\hat{\theta}]^2\Big],\quad \forall t,I.$$

It follows from equation (2.5)

$$\lambda(t)\,[P^*(t)-P(t)]\geq e^{-\rho t}\Big[[K(P^*(t))-K(\hat{P})]^2-[K(P(t))-K(\hat{P})]^2\Big],\quad \text{for all } t,P.$$

Finally, simple computations give

$$\frac{d}{dt}\Big[\lambda(t)(I^*(t)-I(t))\Big]\geq F(t,I^*(t),P^*(t))-F(t,I(t),P(t)),\quad \text{for all } t,I,P.$$

By integrating this inequality on the interval $[0,T]$ and by using the initial and the terminal conditions (2.13) we obtain

$$0\geq\int_0^T F(t,I^*(t),P^*(t))dt-\int_0^T F(t,I(t),P(t))dt,\quad \text{for all } I,P,$$

which ensures that

$$J(t,I^*(t),P^*(t))\leq J(t,I(t),P(t)),\quad \text{for all } I,P.$$

This completes the proof. □

Remark 2.1 The convexity of $F(t,\cdot,P)$ is ensured provided some hypothesis are made on h and $\theta(t,\cdot)$. For example, it suffices that

1. both functions be affine or,
2. both functions be concave with the two natural constraints $h(I(t))\leq h(\hat{I})$ and $\theta(t,I(t))\leq\hat{\theta}$, (these inequalities mean that we don't want to spend more than the holding cost goal rate and we don't want the deterioration rate to be larger than the deterioration goal rate).

Up to this point, all exogenous functions were taken as very general functions. An explicit solution is then difficult to obtain for the problem we want to study. To exploit the intrinsic properties of the system, we determine the explicit solution in the case where exogenous functions have explicit forms. The obtained results will enable us to develop optimal strategies for more complex systems by considering other exogenous functions. Other exogenous functions are not considered here though, in order not to involve complex notation and excessive technical detail. For illustration purposes, let us assume the following forms for the exogenous functions

$$\begin{aligned} K(P(t)) &= K_1 P(t) + K_2, \\ h(I(t)) &= h_1 I(t) + h_2, \\ \theta(t, I(t)) &= \theta_1 I(t) + \theta_2, \end{aligned}$$

where K_i, θ_i, and h_i $(i = 1, 2)$ are real constants with K_1, θ_1, and h_1 nonzero. In this case, since I is close to $\hat{I}$ and $\theta(t, I(t))$ is close to $\hat{\theta}$, then we may take $\hat{\theta} = \theta_1 \hat{I} + \theta_2$ in problem $(\mathcal{P})$. Thus, the objective function (2.2) becomes

$$F(t, I(t), P(t)) = e^{-\rho t} \left\{ \frac{h}{2} \left[I(t) - \hat{I} \right]^2 + \frac{K}{2} \left[P(t) - \hat{P} \right]^2 \right\}, \tag{2.11}$$

with $h = h_1^2 + c\theta_1^2$ and $K = K_1^2$. Under the above forms of the exogenous functions Theorem 2.1 has the following simpler form.

Corollary 2.1 *A necessary condition for (P^*, I^*) to be an optimal solution of problem $(\mathcal{P})$ is*

$$\frac{d^2}{dt^2} I(t) - \rho \frac{d}{dt} I(t) - \left[(\rho + \theta_1)\theta_1 + \frac{h}{K} \right] I(t) = \beta(t), \tag{2.12}$$

with

$$I^*(0) = I_0, \quad P^*(T) = \hat{P}, \quad P^*(t) \geq 0, \tag{2.13}$$

where

$$\beta(t) = (\rho + \theta_1)[D(t) - \hat{P} + \theta_2] - \frac{d}{dt} D(t) - \frac{h}{K} \hat{I}. \tag{2.14}$$

The problem stated in Corollary 2.1 is a two-point boundary value problem. This is because it is both an initial value problem and a terminal value problem. It is solved in the next corollary.

Corollary 2.2 *The optimal solution (P^*, I^*) to problem $(\mathcal{P})$ is given by*

$$P^*(t) = \max \left\{ P(t), 0 \right\},$$

$$I^*(t) = I_0 e^{-\theta_1 t} + \int_0^t \left[P^*(s) - D(s) - \theta_2 \right] e^{\theta_1 (s - t)} ds, \tag{2.15}$$

where

$$P(t) = a_1(m_1 + \theta_1)e^{m_1 t} + a_2(m_2 + \theta_1)e^{m_2 t} + \frac{d}{dt} Q(t) + \theta_1 Q(t) + D(t) + \theta_2,$$

the constants a_1, a_2, m_1, and m_2 are given in the proof below, and $Q(t)$ is a particular solution of Equation (2.12).

Proof. We solve Equation (2.12) by the standard method. The characteristic equation is

$$m^2 - \rho m - (\rho + \theta_1)\theta_1 + \frac{h}{K} = 0.$$

It has two real roots of opposite signs, given by

$$\begin{aligned} m_1 &= \frac{1}{2}\Big(\rho - \sqrt{\rho^2 + 4\Big[(\rho + \theta_1)\theta_1 + \frac{h}{K}\Big]}\Big) < 0, \\ m_2 &= \frac{1}{2}\Big(\rho + \sqrt{\rho^2 + 4\Big[(\rho + \theta_1)\theta_1 + \frac{h}{K}\Big]}\Big) > 0, \end{aligned}$$

and therefore,

$$I(t) = a_1 e^{m_1 t} + a_2 e^{m_2 t} + Q(t), \tag{2.16}$$

where $Q(t)$ is a particular solution of (2.12). The initial and terminal conditions (2.13) are used to determine the constants a_1 and a_2 as follows. From the initial condition we have

$$a_1 + a_2 + Q(0) = I_0,$$

and from the terminal condition we have

$$a_1(m_1 + \theta_1)e^{m_1 T} + a_2(m_2 + \theta_1)e^{m_2 T} + \left[\frac{c}{K} + \frac{d}{dt}Q(T) + \theta_1 Q(T) + D(T)\right] = 0.$$

By putting

$$\begin{aligned} b_1 &= I_0 - Q(0), \\ b_2 &= -\left[\frac{c}{K} + \frac{d}{dt}Q(T) + \theta_1 Q(T) + D(T)\right], \end{aligned}$$

we obtain the following system of two linear equations in two unknowns

$$\begin{aligned} a_1 + a_2 &= b_1 \\ a_1(m_1 + \theta_1)e^{m_1 T} + a_2(m_2 + \theta_1)e^{m_2 T} &= b_2, \end{aligned}$$

which has the following unique solution

$$\begin{aligned} a_1 &= \frac{b_2 - (m_2 + \theta_1)e^{m_2 T} b_1}{(m_1 + \theta_1)e^{m_1 T} - (m_2 + \theta_1)e^{m_2 T}}, \\ a_2 &= \frac{b_1(m_1 + \theta_1)e^{m_1 T} - b_2}{(m_1 + \theta_1)e^{m_1 T} - (m_2 + \theta_1)e^{m_2 T}}. \end{aligned}$$

The expression of P is deduced using the optimal expression of I along with the state equation (2.1). Recalling that the optimal control P^* has to be nonnegative, so P^* will be

$$P^*(t) = \max\Big\{P(t), 0\Big\},$$

and then I^* will be deduced directly from (2.1). □

Remark 2.2

1. Note that if we had chosen the deterioration function θ depending on both variables t and I instead of I only (for example: $\theta(t, I(t)) = \theta(t)I(t)$), then the differential equation (2.12) would become with variable coefficients. The procedure used in the proof of Corollary 2.2 is no longer appropriate. However, the differential equation (2.12) can be solved by more technical methods (for instance power series methods).

2. Observe that Equation (2.8) becomes

$$\lambda(t) = K_1^2 e^{-\rho t}[P(t) - \hat{P}],$$

from which

$$P(t) = \hat{P} + \frac{e^{\rho t}\lambda(t)}{K_1^2}.$$

Therefore, by choosing $\hat{P}$ large enough so that $P(t)$ is nonnegative, then the optimal control P^* will be equal to P. In this case, Equation (2.15) reduces to Equation (2.16)

3 Extension

The optimal solution of $(\mathcal{P})$ obtained in Corollary 2.2 can be extended under some conditions to the case of an infinite planning horizon. We will show that these solutions make sense when $T \to \infty$. Assume $\rho > 0$. We show that the limit of the finite horizon problem optimal solution solves the following corresponding infinite horizon problem:

$$(\mathcal{P}_\infty) \qquad \begin{cases} \displaystyle\min_{P(t)\geq 0} J(P, I) = \int_0^\infty F(t, I(t), P(t))dt \\ \displaystyle\frac{d}{dt}I(t) = f(t, I(t), P(t)), \quad I(0) = I_0 \end{cases}$$

where F and f are given in Section 2.

For the infinite horizon case, the terminal condition $\lambda(T) = 0$ must be changed to

$$\lim_{t\to\infty} \lambda(t) = 0, \tag{3.1}$$

where λ is the adjoint function associated to problem $(\mathcal{P})$. In practice, the production rate $P(t)$ is usually bounded and so $K_1^2[P(t) - \hat{P}]$ is bounded. Then, for $\rho > 0$

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} e^{-\rho t} K_1^2 [P(t) - \hat{P}] = 0,$$

so that (3.1) is satisfied. An inspection of our proofs of Theorems 2.1-2.2 shows that the equations (2.5)-(2.7) with (3.1) instead of $\lambda(T) = 0$, are necessary and sufficient conditions for (P_∞^*, I_∞^*) (given below) to be an optimal solution to $(\mathcal{P}_\infty)$:

$$P_\infty^*(t) = \max\Big\{P_\infty(t), 0\Big\},$$

$$I_\infty^*(t) = I_0 e^{-\theta_1 t} + \int_0^t \left[P_\infty^*(s) - D(s) - \theta_2\right] e^{\theta_1(s-t)} ds,$$

where

$$P_\infty(t) = b_1(m_1+\theta_1)e^{m_1 t} + \frac{d}{dt}Q(t) + \theta_1 Q(t) + D(t) + \theta_2,$$

with $b_1 = I_0 - Q(0)$ and $m_1 = \frac{1}{2}\Big(\rho - \sqrt{\rho^2 + 4\Big[(\rho+\theta_1)\theta_1 + \frac{h}{K}\Big]}\Big)$.

4 Illustrative Examples

We want in this section to present some illustrative examples of the results obtained. Computations of problem $(\mathcal{P})$ optimal solution are performed for the constant demand functions $D(t) = d$ and the linear demand functions $D(t) = d_1 t + d_2$. These computations are performed in the infinite horizon case where the formulas are simple, and similar computations can be done in the finite horizon case.

Case 1: Constant Demand. We assume a positive demand function $D(t) = d$ with $d > 0$. The particular solution of Equation (2.12) is first found to be $Q(t) = q$ with

$$q = -\frac{(\rho+\theta_1)(d-\hat{P}) - \frac{h}{K}\hat{I}}{(\rho+\theta_1)\theta_1 + \frac{h}{K}}.$$

The expression of the optimal control $P^*(t)$ and the optimal state $I^*(t)$ depend on the system parameters as follows:

- If $I_0 = q$, then P and I are constant and given by

  $$P(t) = \theta_1 q + d + \theta_2 \text{ and } I(t) = q.$$

  Therefore P^* is constant and equal to $\theta q + d + \theta_2$ or 0 depending on whether $\theta_1 q + d + \theta_2$ is positive or negative, while I^* is given by

  $$I^*(t) = \begin{cases} I_0 e^{-\theta_1 t} - \displaystyle\int_0^t (d+\theta_2) e^{\theta_1(s-t)} ds, & \theta_1 q + d + \theta_2 \leq 0, \\ I_0 e^{-\theta_1 t} + \displaystyle\int_0^t \theta_1 q e^{\theta_1(s-t)} ds, & \theta_1 q + d + \theta_2 > 0. \end{cases}$$

- If $I_0 \neq q$, then, $P(t)$ is given by

  $$P(t) = (I_0 - q)(m_1+\theta_1)e^{m_1 t} + \theta_1 q + d + \theta_2.$$

  At this point, it is preferable to consider separately the cases $I_0 < q$ and $I_0 > q$.

- ⋆ If $I_0 < q$, we note that since

$$\frac{d}{dt}P(t) = (I_0 - q)m_1(m_1 + \theta_1)e^{m_1 t},$$

is negative then, $P(t)$ decreases to $\theta_1 q + d + \theta_2$, starting from the value

$$P(0) = m_1(I_0 - q) + I_0\theta_1 + d + \theta_2.$$

Clearly $P^*(t) = P(t)$ and $I^*(t) = I(t)$ whenever $\theta_1 q + d + \theta_2$ is nonnegative. In case $\theta_1 q + d + \theta_2$ and $P(0)$ are both negative, then $P^*(t) = 0$ and I^* is given by

$$I^*(t) = I_0 e^{-\theta_1 t} - \int_0^t (d + \theta_2)e^{\theta_1(s-t)}ds.$$

Otherwise, there exists some t_0 for which $P(t_0) = 0$. Then,

$$P^*(t) = \begin{cases} 0, & t \le t_0, \\ P(t) & t > t_0. \end{cases} \tag{4.2}$$

and

$$I^*(t) = \begin{cases} I_0 e^{-\theta_1 t} - \int_0^t (d+\theta_2)e^{\theta_1(s-t)}ds, & t \le t_0, \\ I_0 e^{-\theta_1 t} - \int_0^{t_0} d e^{\theta_1(s-t_0)}ds + \int_{t_0}^t (P(s) - d - \theta_2)e^{\theta_1(s-t)}ds, & t > t_0. \end{cases}$$

An explicit form of I^* can easily be obtained.

- ⋆ If $I_0 > q$, an analysis similar to the previous one can be conducted. A numerical example is presented below.

Example 4.1 Assume, for example, the following constant demand $d = 20$. Also, assume the following targets $\hat{I} = 1, \hat{P} = 30$, the following unit costs $c = 1, h = 1, K = 1$ and the following deterioration and discount rates $\theta_1 = 0.001, \theta_2 = 0, \rho = 0.01$. The initial inventory I_0 is assumed to be equal to 1. This is a case where $I_0 < q$ and $\theta_1 q + d + \theta_2 > 0$. We have seen that in this case, $P(t) > 0$ so that $P^*(t) = P(t) = (I_0 - q)(m_1 + \theta_1)e^{m_1 t} + \theta_1 q + d + \theta_2$. Variations of $P(t)$ are depicted in Figure 1. It starts from the positive value $P(0) = m_1(I_0 - q) + I_0\theta_1 + d + \theta_2$ then it decreases to tend to the value $\theta_1 q + d + \theta_2 > 0$ as t tends to infinity. Also, $I^*(t) = I(t) = I_0 e^{-\theta_1 t} - \int_0^t (P(s) - d + \theta_2)e^{\theta_1(s-t)}ds$ and the variations of $I^*(t)$ are also depicted in Figure 2. It starts from the positive value $I_0 = 1$ then it increases to tend to the value q as t tends to infinity. We also computed in this case the optimal cost and found $J^* = 4992$. Finally, to determine the effect of the deterioration on the system, we computed the optimal cost for various values of the deterioration rate. The results are plotted in Figure 3. We observe that the higher the deterioration rate, the lower the optimal cost. *Case 2: Linear Demand*. We assume a demand function $D(t) = d_1 t + d_2$ with $d_1 > 0$. The particular solution of Equation (2.12) is first found to be $Q(t) = q_1 t + q_2$ with

$$q_1 = -\frac{(\rho + \theta_1)d_1}{(\rho + \theta_1)\theta_1 + \frac{h}{K}},$$

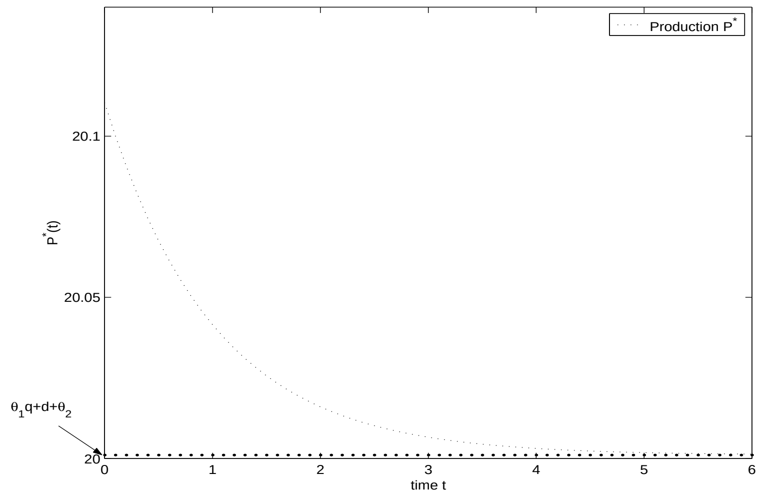

Figure 1: Variations of P^* as function of time t.

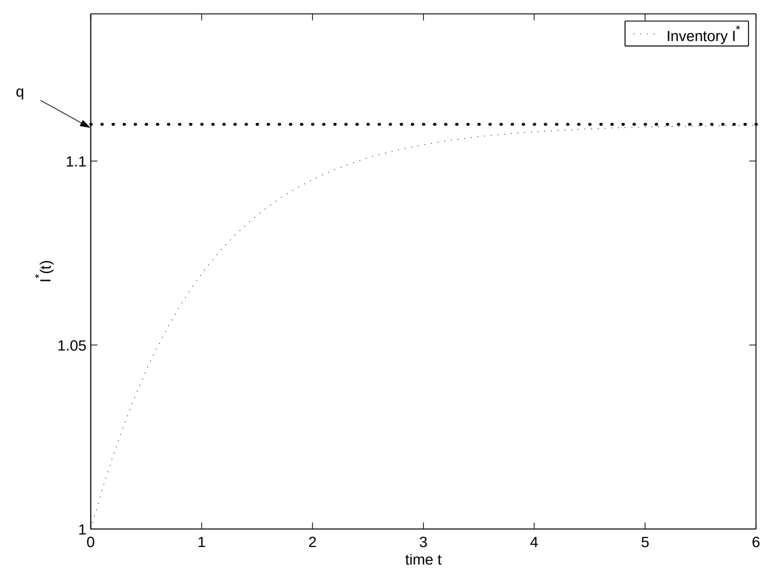

Figure 2: Variations of I^* as function of time t.

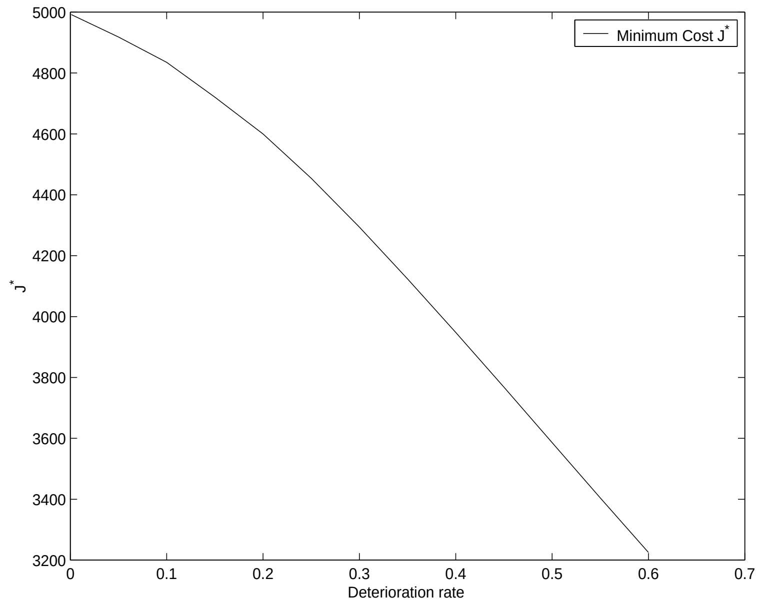

Figure 3: Variations of optimal cost J^* as function of deterioration rate θ.

and

$$q_2 = -\frac{1}{(\rho+\theta_1)\theta_1 + \frac{h}{K}} \left[(\rho+\theta_1)(d_2 - \hat{P}) - d_1 - \frac{h}{K}\hat{I} + \rho q_1 \right].$$

The expression of the optimal control $P^*(t)$ and the optimal state $I^*(t)$ depend on the system parameters as follows:

- If $I_0 = q_2$, then

$$P(t) = (\theta_1 q_1 + d_1)t + (q_1 + \theta_1 q_2 + d_2 + \theta_2) \text{ and } I(t) = Q(t).$$

  We note that $\theta_1 q_1 + d_1$ is always positive, so that if we choose, for example, d_2 so that $q_1 + \theta_1 q_2 + d_2 + \theta_2$ is positive, then $P^*(t) = P(t)$ and $I^*(t) = I(t)$.

- If $I_0 \neq q_2$, then, $P(t)$ is given by

$$P(t) = (I_0 - q_2)(m_1 + \theta_1)e^{m_1 t} + (\theta_1 q_1 + d_1)t + (q_1 + \theta_1 q_2 + d_2 + \theta_2).$$

  ⋆ If $I_0 > q_2$, we note that since

$$\frac{d}{dt}P(t) = (I_0 - q_2)m_1(m_1 + \theta_1)e^{m_1 t} + (\theta_1 q_1 + d_1),$$

  is positive then, $P(t)$ is an increasing function, starting from the value

$$P(0) = (I_0 - q_2)(m_1 + \theta_1) + q_1 + \theta_1 q_2 + d_2 + \theta_2.$$

Clearly $P^*(t) = P(t)$ and $I^*(t) = I(t)$ whenever $P(0)$ is nonnegative. In case $P(0)$ is negative, then there exists some t_0 for which $P(t_0) = 0$, since $\lim_{t\to\infty} P(t) = \infty$. Then,

$$P^*(t) = \begin{cases} 0, & t \le t_0, \\ P(t) & t > t_0. \end{cases}$$

and

$$I^*(t) = \begin{cases} I_0 e^{-\theta_1 t} - \int_0^t (D(s) + \theta_2) e^{\theta_1 (s-t)} ds, & t \le t_0, \\ I_0 e^{-\theta_1 t} - \int_0^{t_0} D(s) e^{\theta_1 (s-t_0)} ds + \int_{t_0}^t (P(s) - D(s) - \theta_2) e^{\theta_1 (s-t)} ds, & t > t_0. \end{cases}$$

An explicit form of I^* can easily be obtained.

⋆ If $I_0 < q_2$, then depending on the values of the system parameter, $P(t)$ is not necessarily monotone. However, it is not hard to check that Equation $P(t) = 0$ may have no roots but at most two roots, and these roots need to be determined in order to determine the intervals on which $P(t)$ is positive. Numerical examples are presented below.

Example 4.2 Assume, for example, the following affine demand, $d_1 = 1$ and $d_2 = 0$. Let $I_0 = 5$ and let all other parameters be the same as in Example 4.1. This is a case where $I_0 > q_2$ and $P(0) < 0$. In this case, there exists $t_0 = 0.997$ for which $P(t_0) = 0$ and

$$P^*(t) = \begin{cases} 0, & t \le t_0, \\ P(t) & t > t_0. \end{cases}$$

Variations of $P^*(t)$ are depicted in Figure 4. The production rate P starts from the negative value $P(0) = -2.663$ then it increases asymptotically to the line $(\theta_1 q_1 + d_1)t + q_1 + \theta_1 q_2 + d_2 + \theta_2 = 0$. Also, the variations of $I^*(t)$ are depicted in Figure 5. It starts from the positive value $I_0 = 1$ then it decreases as t tends to infinity. Finally, to determine the effect of the deterioration on the starting time of the production, we computed the time t_0 for various values of the deterioration rate. The results are plotted in Figure 6. As can be seen and as one would expect, the higher the deterioration rate, the earlier starting time of the production.

5 Conclusion

We have used in this paper an optimal control approach to determine the optimal production rate in a production inventory system where items are subject to deterioration. The optimal solutions have been given in detail for specific exogenous functions. It may be worth investigating the effect of other exogenous functions on the solution. We expect the solution procedure to be more or less difficult, depending on the shape of these functions.

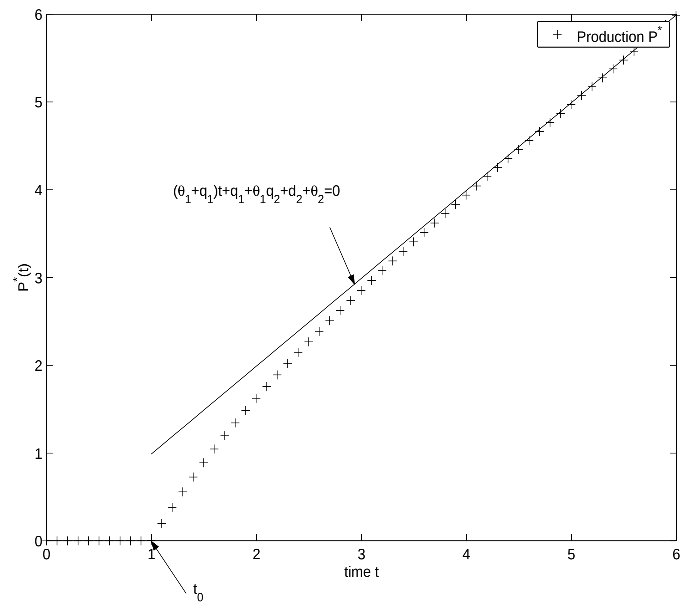

Figure 4: Variations of optimal solution P^* as function of time t.

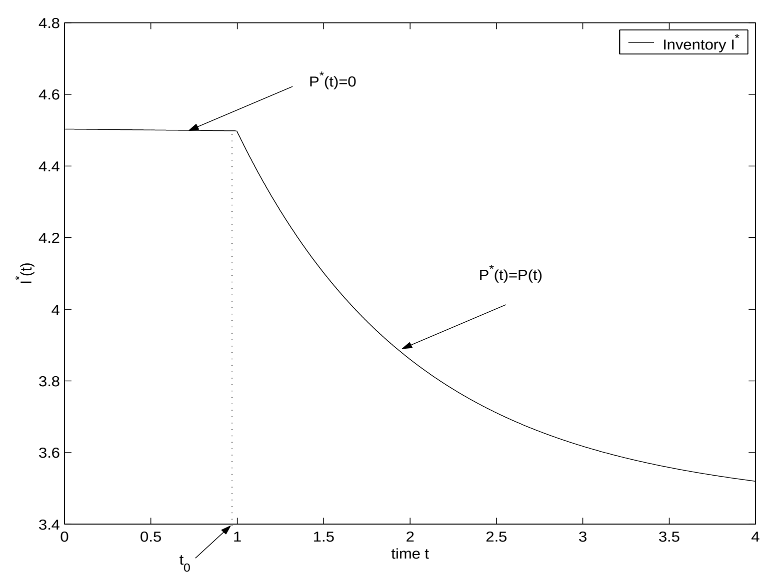

Figure 5: Variations of optimal solution I^* as function of time t.

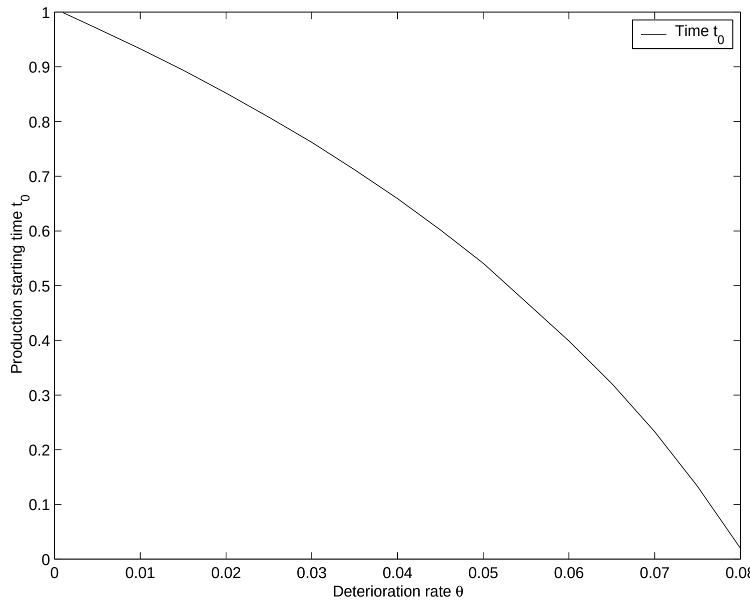

Figure 6: Variations of time t_0 as function of deterioration rate θ.

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