On Space-Times with commutative parings of vector fields

Giuseppe Caristi and Massimiliano Ferrara

Abstract

Let *(M,g)* be general space-time and let S be a Sachs frame over *M.* In this paper we use the complex vectoríal formalism constructed in [2]. If h_A , *(A =* 1, 2, 3, 4) are the null vector fields which define S, we assume in this paper that (h_1, h_2) and (h_3, h_4) define commutative parings. If $D_{12} = \{h_1, h_2\}$ and $D_{34} = \{h_3, h_4\}$ are the distributions defined by (h_1, h_2) and (h_3, h_4) then it is shown that M is of type D in Petrov's classification, and M is the local Riemannian product $M = M_{12} \times M_{34}$, where M_{12} is a surface tangent to D_{12} and M_{34} is a surface tangent to D_{34} . If the simple unit forms corresponding to M_{12} and M_{34} have the same recurrence form, then M moves to a Schwarzchild space-time of type 1, and M is endowed with a symplectic structure.

M.S.C. 2000: 53C15, 53C80, 83C75.
Key words: Commutative parings, sachs frame, Schwarzchild space-time.

1 The main result

Let (M, g) be a general space-time satisfying the usual integrability conditions and let $S\{h_A\}$ be a Sachs frame (on a null frame) over M. It is assumed that, the vector fields h_1, h_4 are real null vector field whereas h_2, h_3 are complex conjugate vector fields.

We shall make use of the complex vectorial formalism (abv. C.V.F.) constructed in [2]. This formalism is based on the local isomorphism $\tilde{A} : L(4) \to S0^3C$ where L (4) and $S0^3C$ are the 4-dimensional Lorentz group and the 3-dimensional rotation group.

If $\{\theta^A\}$ is the dual basis of h_A , A = 1,2,3,4 then there exists an isomorphism between the 2-forms $\theta^A \wedge \theta^B$ of the 6-dimensional L_6^* and the antidual forms Z^α (α=1,2,3), which form a basis of the complex space C^3 . That is in compact form (1.1) $dZ = \sigma \wedge Z$.

The 1-forms $\sigma_\alpha, \overline{\sigma}_\alpha$ (which are complex conjugate) depend on θ^A via

$$\sigma_\alpha = \sigma_{\alpha A}\theta^A, \overline{\sigma}_\alpha = \overline{\sigma}_{\alpha A}\overline{\theta}^A. \tag{1.1}$$

$$\theta^1 = \overline{\theta}^1, \theta^4 = \overline{\theta}^4,\ \theta^2 = \overline{\theta}^3. \tag{1.2}$$

Applied Sciences, Vol.7, 2005, pp. 58-61.

The coefficients $\sigma_{\alpha A}, \overline{\sigma}_{\alpha A}$ correspond in [2] to the spinorial coefficients of Newmann and Penrose [5]. In terms of σ the manifold (M, g) is structured by the connection

$$(1.3)\qquad \begin{cases} \nabla h_1 = -\frac{1}{4}(\sigma_3 + \overline{\sigma}_3) \otimes h_1 + \frac{\overline{\sigma}}{2} \otimes h_2 + \frac{\sigma_2}{2} \otimes h_3 \\ \nabla h_2 = -\frac{1}{2}\overline{\sigma}_1 \otimes h_1 + \frac{1}{4}(\sigma_3 - \overline{\sigma}_3) \otimes h_2 + \frac{\sigma_2}{2} \otimes h_4 \\ \nabla h_3 = \frac{1}{2}\sigma_1 \otimes h_1 - \frac{1}{4}(\sigma_3 - \overline{\sigma}_3) \otimes h_3 + \frac{\overline{\sigma}_2}{2} \otimes h_4 \\ \nabla h_3 = -\frac{1}{2}\sigma_1 \otimes h_1 - \frac{\overline{\sigma}_1}{2} \otimes h_3 + \frac{1}{4}(\sigma_3 + \overline{\sigma}_3) \otimes h_4 \end{cases}$$

and the first group of structure equations is expressed by

$$(1.4)\qquad \begin{cases} d\theta^1 = \frac{1}{2}(\sigma_3 + \overline{\sigma}_3) \wedge \theta^1 + \frac{\overline{\sigma}_1}{2} \wedge \theta^2 + \frac{\sigma_1}{2} \wedge \theta^3 \\ d\theta^2 = -\frac{1}{2}\overline{\sigma}_2 \wedge \theta^1 + \frac{1}{4}(\sigma_3 - \overline{\sigma}_3) \wedge \theta^2 + \frac{\sigma_1}{2} \wedge \theta^4 \\ d\theta^3 = -\frac{1}{2}\sigma_2 \wedge \theta^1 - \frac{1}{4}(\sigma_3 - \overline{\sigma}_3) \wedge \theta^3 + \frac{\overline{\sigma}_1}{2} \wedge \theta^4 \\ d\theta^4 = -\frac{1}{2}\sigma_2 \wedge \theta^2 - \frac{1}{2}\overline{\sigma}_2 \wedge \theta^3 - \frac{1}{4}(\sigma_3 + \overline{\sigma}_3) \wedge \theta^4 \end{cases}$$

(see also [4]).

We assume in this paper that the parings (h_1, h_2) and (h_3, h_4) of null vector fields, are commutative (see also [4]); thus we have

$$(1.5)\qquad [h_1, h_2] = 0.$$

and

$$(1.6)\qquad [h_3, h_4] = 0;$$

where [.] denotes a Lie braket.

By (4) one finds that the Neumann-Penrose coefficients $\sigma_{\alpha A}, \overline{\sigma}_{\alpha A}$ satisfy

$$(1.7)\qquad \sigma_{21} = 0, \qquad \sigma_{22} = 0, \qquad \sigma_{13} = 0, \qquad \sigma_{14} = 0.$$

and

$$(1.8)\qquad \begin{cases} \sigma_{32} + \overline{\sigma}_{33} - \frac{1}{2}\overline{\sigma}_{11} = 0, & \sigma_{33} + \overline{\sigma}_{32} - \frac{1}{2}\overline{\sigma}_{24} = 0 \\ \sigma_{31} + \overline{\sigma}_{31} - \frac{1}{2}\overline{\sigma}_{23} = 0 & \sigma_{34} + \overline{\sigma}_{34} - \frac{1}{2}\overline{\sigma}_{12} = 0 \end{cases}$$

Equations (8) prove the significant fact that the space-time under consideration belongs to type D in Petrov's classification.

Furthermore, we derive from (5), taking account of (8), that

$$(1.9)\qquad \begin{cases} d\theta^1 = \left(\frac{1}{4}(\sigma_3 + \overline{\sigma}_3) - \frac{1}{2}\left(\overline{\sigma}_{11}\theta^2 + \sigma_{11}\theta^3\right)\right) \wedge \theta^1 + \frac{1}{2}(\overline{\sigma}_{12} - \sigma_{12})\,\theta^3 \wedge \theta^2 \\ d\theta^2 = \left(\frac{1}{4}(\sigma_3 - \overline{\sigma}_3) + \frac{1}{2}\left(\overline{\sigma}_{23}\theta^1 - \sigma_{12}\theta^4\right)\right) \wedge \theta^2 - \frac{1}{2}\left(\overline{\sigma}_{22} - \left(\theta^3 + \sigma_{11}\theta^4\right)\theta^3 \wedge \theta^1\right) \end{cases}$$

Furthermore setting $\varphi_{12} = \theta^1 \wedge \theta^2$, we derive by (10)

$$(1.10)\qquad d\varphi_{12} = \frac{1}{2}\left(\sigma_3 - \sigma_{11}\theta^3 - \sigma_{12}\theta^4\right) \wedge \varphi_{12},$$

which shows that φ_{12} is an exterior recurrent (abv. E.R.) [3] form.

Similarly, setting $\varphi_{34} = \theta^3 \wedge \theta^4$, we derive, taking account of (8)

$$d\varphi_{34} = \frac{1}{2}\left(\sigma_3 + \sigma_{23}\theta^1 + \sigma_{24}\theta^2\right) \wedge \varphi_{34}. \tag{1.11}$$

Let $D_{12} = \{h_1, h_2\}$ and $D_{34} = \{h_3, h_4\}$ be the two distributions which have the simple unit forms respectively φ_{12} and φ_{34} . Since both φ_{12} and φ_{34} are E.R., it follows by Frobenius theorem that the manifold M under consideration is the local Riemannian product of two surfaces M_{12} and M_{34} tangent to D_{12} and D_{34} respectively, i.e.

$$M = M_{12} \times M_{34}.$$

It is worth noting that if φ_{12} and φ_{34} have the same recurrence form σ_3, then it follows by (11) and (12) that we have

$$\sigma_{11} = 0, \qquad \sigma_{12} = 0, \qquad \sigma_{23} = 0, \qquad \sigma_{24} = 0. \tag{1.12}$$

Then equations (8) and (13) prove the significant fact that the manifold M under consideration is a Schwarzchild space-time of type l.

Moreover, by reference to Rosca [6], the equations (13) show that the almost symplectic form structure

$$Z^3 = \frac{1}{2}\left(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3\right)$$

moves to a symplectic form. Summing up we state the following

Theorem. *Let* (M, g) *be a general space-time and let* $S = \{h_A, A = 1, 2, 3, 4\}$ *be a Sachs frame over M. If* (h_1, h_2) *and* (h_3, h_4) *define commutative parings then the following emerges:*

(i) M belongs to type D in Petrov's classification;

(ii) M is the local Riemannian product

$$M = M_{12} \times M_{34},$$

where M_{12} *is a surface tangent to the distribution* (h_1, h_2) *and* M_{34} *is a surface tangent to the distribution* (h_1, h_2) *. If the simple unit forms corresponding to* M_{12} *and* M_{34} *have the same recurrence form, then M moves to a Schwarzchild Space-Time of type l.*

References

[1] F. Amato, *Espace-temps à champs isotropes commutatifs*, C.R. Acad. Sci. Paris 289 (1979), 533-536.

[2] M. Cahen, R. Debever and L. Defrise, *A complex vectorial formalism in General Relativity*, J. Math. Mech. 16 (1967), 761-785.

[3] D.K. Datta, *Exterior recurrent forms on a manifold*, Tensor N.S. 36 (1982), 115-120.

[4] W. Israel, *Differential forms in General Relativity*, Commun. Dublin Inst. Advanced Stud., Series A, 19 (1970).

[5] E. Neumann and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, Journal Math. Phys. 3 (1962), 566-578.

[6] Gh. Vrănceanu, R. Rosca, *Introduction in Relativity and Pseudo-Riemannian Geometry*, Editura Academiei Rep. Soc. Romania, Bucharest, 1976, 1-127.

Authors' addresses:

Giuseppe Caristi and Massimiliano Ferrara
D.E.S.Ma.S. "V. Pareto", Faculty of Economics, University of Messina, Italy
e-mail addresses: gcaristi@dipmat.unime.it, mferrara@unime.it