On the closed-form solution of the improved labor augmented Solow-Swan model

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Abstract

In this paper we consider the Solow-Swan model with purely labor-augmenting technological progress in which the labor growth rate follows the generalized logistic equation. We prove this model to have a dynamic equation of growth whose solution can be expressed in closed-form via the hypergeometric function ${}_2F_1$.

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1 Introduction

The earliest mathematical formulation of economic growth theory was made by Domar [2] and Harrod [3]. In these models input coefficients are fixed and full employment equilibrium growth is not guaranteed. This undesiderable feature led Solow [5] and Swan [6] to the formulation of the so-called neoclassical model which allows for factor substitution and hence full employment equilibrium. The aim of their papers was to provide a theoretical framework for understanding world-wide growth of output and the persistence of geographical differences in per capita output. In the Solow-Swan model, each individual of the population is a member of the labor force and the growth rate of the population is constant. This assumption was relaxed by Cai [1], who considered an increasing bounded labor force whose growth rate decreases monotonically to zero. When the rate of population is given by the logistic equation, Mingari Scarpello and Ritelli [4] showed that the resulting model has a dynamic equation of growth whose solution can be integrated in closed-form through the hypergeometric function ${}_2F_1$. In this paper, we extend their result to the Solow-Swan model with labor-augmenting technical progress and labor growth rate given by the generalized logistic equation.

2 The model

Let us assume a closed economy where an homogeneous good is produced according to a technology involving three inputs: physical capital, labor and technology.

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Considering a Cobb-Douglas production we arrive at $Y(t) = K(t)^\alpha[A(t)L(t)]^{1-\alpha}$, $0 < \alpha < 1$, where t stands for time, $Y(t)$ is the flow of output, $K(t)$ is the stock of physical capital, $L(t)$ is the labor force and $A(t)$ is the level of technology. We assume that $A(t)$ increases over time at the exogenous and constant rate g, that is $\dot{A}(t)/A(t) = g > 0$. Based on the feature of constant returns to scale, we can specify the economy's output in terms of effective labor as follows $y(t) = k(t)^\alpha$, $0 < \alpha < 1$, where $y(t) = Y(t)/[A(t)L(t)]$ is the output per unit of effective labor and $k(t) = K(t)/[A(t)L(t)]$ denotes the capital per unit of effective labor. Output is assumed to be used for consumption $c(t)$, or for investment $I(t)$ in physical capital. A constant fraction δ of the capital stock depreciates every period. This means that if, at the beginning of a period, the capital stock equals $K(t)$, then, at the end of it, $\delta K(t)$ will have been worn off. Therefore, the net increase in capital at any moment in time equals gross investment less capital depreciation, that is

$$\dot{K}(t) = I(t) - \delta K(t). \tag{2.1}$$

The output of the economy equals total income, whereas investments equal savings. Households save a constant fraction of their income, that is the saving rate s satisfies $0 < s < 1$. Hence, the following relations hold

$$I(t) = S(t) = sY(t). \tag{2.2}$$

Equations (2.1) and (2.2) imply that $\dot{K}(t) = sY(t) - \delta K(t)$. To keep our analysis in terms of effective labor, we divide both sides of this equation by $A(t)L(t)$, and find that

$$\frac{\dot{K}(t)}{A(t)L(t)} = \frac{sY(t)}{A(t)L(t)} - \frac{\delta K(t)}{A(t)L(t)} = sk(t)^\alpha - \delta k(t).$$

Consequently, the growth rate of the capital per effective worker writes as

$$\dot{k}(t) = \frac{d}{dt}\left(\frac{K(t)}{A(t)L(t)}\right) = \frac{\dot{K}(t)}{A(t)L(t)} - \left(g + \frac{\dot{L}(t)}{L(t)}\right)k(t).$$

If the labor growth rate $\dot{L}(t)/L(t)$ is given exogenously and equals a constant n, then the equations above yield $\dot{k}(t) = sk(t)^\alpha - (\delta + g + n)k(t)$. This is known as the fundamental differential equation of the augmented version of the Solow-Swan model. If instead we relax the assumption that the labor growth rate is constant, and assume that it has the form of the generalized logistic equation, that is $\dot{L}(t)/L(t) = a - bL(t)^\beta$, where $\beta > 0$, $a > b > 0$, $L(0) = 1$, we obtain the so-called improved labor augmented Solow-Swan model. The dynamic of per capita capital of this model is described by the following first-order nonlinear differential equation

$$\dot{k}(t) = sk(t)^\alpha - \left(\delta + g + \frac{\dot{L}(t)}{L(t)}\right)k(t). \tag{2.3}$$

Theorem 1. *Let $k(t)$ be the solution of* (2.3). *Then*

$$k(t) = \frac{e^{-(\delta+g)t}}{L(t)}\left(k(0)^{1-\alpha} + (1-\alpha)s\int_0^t e^{(1-\alpha)(\delta+g)u}L(u)^{1-\alpha}du\right)^{\frac{1}{1-\alpha}}. \tag{2.4}$$

Proof. Equation (2.3) is a Bernoulli type differential equation. Its solution is known to be found by taking the substitution $z = k^{1-\alpha}$. This yields a linear differential equation in z,

$$\dot{z} = (1-\alpha)s - (1-\alpha)\left(\delta + g + \frac{\dot{L}(t)}{L(t)}\right) z,$$

which is solved by

$$z(t) = e^{-\int_0^t (1-\alpha)\left(\delta+g+\frac{\dot{L}(u)}{L(u)}\right) du}\left(z(0) + \int_0^t (1-\alpha)s\, e^{\int_0^u (1-\alpha)\left(\delta+g+\frac{\dot{L}(v)}{L(v)}\right) dv} du\right).$$

Since

$$e^{-(1-\alpha)\int_0^t \left(\delta+g+\frac{\dot{L}(u)}{L(u)}\right) du} = e^{-(1-\alpha)[(\delta+g)u + lnL(u)]_0^t} = e^{-(1-\alpha)(\delta+g)t} L(t)^{-(1-\alpha)},$$

we obtain

$$z(t) = e^{-(1-\alpha)(\delta+g)t} L(t)^{-(1-\alpha)}\left(z(0) + (1-\alpha)s\int_0^t e^{(1-\alpha)(\delta+g)u} L(u)^{1-\alpha} du\right).$$

The statement follows by rewriting this equation in terms of k. □

3 The model solution expressed via the hypergeometric function ${}_2F_1$

The aim of this section is to write the model solution $k(t)$ stated in Theorem 1 in terms of the hypergeometric function ${}_2F_1$. For this purpose, first we recall that ${}_2F_1$ has the integral representation (see Watson and Whittaker [8])

$$(3.5) \qquad {}_2F_1(c_1, c_2, c_3; z) = \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_3 - c_1)} \int_0^1 t^{c_1-1}(1-t)^{c_3-c_1-1}(1-zt)^{-c_2} dt,$$

where $z \in \mathbb{C}$, $c_1, c_2, c_3 \in \mathbb{C}$ are such that $Re(c_1) > 0$, $Re(c_3 - c_1) > 0$, and Γ is the Euler gamma function. Next, we remind that the generalized logistic law $\dot{L}(t) = aL(t) - bL(t)^{1+\beta}$ is a Bernoulli type differential equation, and so its solution can be found to be given by

$$(3.6) \qquad L(t) = e^{at}\left(1 - \frac{b}{a} + \frac{b}{a}\, e^{\beta a t}\right)^{-\frac{1}{\beta}}.$$

This growth law is a generalization of the logistic equation as it can be seen by taking $\beta = 1$ in (3.6) (see Tsoularis [7] for an analysis of logistic growth models).

Theorem 2. *Let* $\gamma_1 = (1-\alpha)(\delta+g+a) > 0$, $\gamma_2 = \beta a > 0$, $\gamma_3 = (1-\alpha)/\beta > 0$, *and* $B = b/(b-a) < 0$. *Let* ${}_2F_1$ *be the hypergeometric function. The solution* $k(t)$ *of* (2.3) *writes in closed-form as*

$$k(t) = e^{-(\delta+g)t} L(t)^{-1} \left\{ k(0)^{1-\alpha} + (1-\alpha)s\left(1-\frac{b}{a}\right)^{-\frac{1-\alpha}{\beta}} \frac{1}{\gamma_1} \cdot \right.$$

$$\left. \cdot \left[e^{\gamma_1 t} {}_2F_1\left(\frac{\gamma_1}{\gamma_2}, \gamma_3, 1+\frac{\gamma_1}{\gamma_2}; Be^{\gamma_2 t}\right) - {}_2F_1\left(\frac{\gamma_1}{\gamma_2}, \gamma_3, 1+\frac{\gamma_1}{\gamma_2}; B\right)\right]\right\}^{\frac{1}{1-\alpha}}.$$

Proof. The statement will follow by writing the integral in the statement of Theorem 1 in terms of the function ${}_2F_1$. Let start replacing $L(t)$ with its expression written in (3.6). This yields

$$\int_0^t e^{(1-\alpha)(\delta+g)u} L(u)^{1-\alpha} du$$

$$= \int_0^t e^{(1-\alpha)(\delta+g)u} e^{(1-\alpha)au} \left(1-\frac{b}{a}+\frac{b}{a}\, e^{\beta a u}\right)^{-\frac{1-\alpha}{\beta}} du,$$

$$= \left(1-\frac{b}{a}\right)^{-\frac{1-\alpha}{\beta}} \int_0^t e^{(1-\alpha)(\delta+g+a)u} \left(1-\frac{b}{b-a}\, e^{\beta a u}\right)^{-\frac{1-\alpha}{\beta}} du.$$

Operating the change of variable $x = e^{\gamma_2 u}$, and using the definition of γ_1, γ_2, γ_3 and B, the above integral writes

$$\int_0^t e^{(1-\alpha)(\delta+g+a)u} \left(1-\frac{b}{b-a}\, e^{\beta a u}\right)^{-\frac{1-\alpha}{\beta}} du = \int_0^t e^{\gamma_1 u} \left(1-Be^{\gamma_2 u}\right)^{-\gamma_3} du,$$

$$= \frac{1}{\gamma_2} \int_1^{e^{\gamma_2 t}} x^{\frac{\gamma_1}{\gamma_2}-1} (1-Bx)^{-\gamma_3}\, dx,$$

$$= \frac{1}{\gamma_2} \left(\int_0^{e^{\gamma_2 t}} x^{\frac{\gamma_1}{\gamma_2}-1} (1-Bx)^{-\gamma_3}\, dx - \int_0^1 x^{\frac{\gamma_1}{\gamma_2}-1} (1-Bx)^{-\gamma_3}\, dx \right). \tag{3.7}$$

As $\gamma_1/\gamma_2 > 0$, the integrals in (3.7) are convergent since

$$\int_0^v x^{\frac{\gamma_1}{\gamma_2}-1} (1-Bx)^{-\gamma_3}\, dx \sim \int_0^v x^{\frac{\gamma_1}{\gamma_2}-1} dx, \quad \text{for } v \geq 1.$$

Setting $r = e^{-\gamma_2 t}x$, by (3.5) the first integral in (3.7) writes

$$\int_0^{e^{\gamma_2 t}} x^{\frac{\gamma_1}{\gamma_2}-1}(1-Bx)^{-\gamma_3}\,dx = e^{\gamma_1 t}\int_0^1 r^{\frac{\gamma_1}{\gamma_2}-1}\left(1-Be^{\gamma_2 t}r\right)^{-\gamma_3}(1-r)^0 dr$$

$$= e^{\gamma_1 t}\,\frac{\Gamma\left(\dfrac{\gamma_1}{\gamma_2}\right)\Gamma(1)}{\Gamma\left(1+\dfrac{\gamma_1}{\gamma_2}\right)}\;{}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,1+\frac{\gamma_1}{\gamma_2};Be^{\gamma_2 t}\right).$$

Since $\Gamma(1) = 1$ and $\Gamma(v+1) = v\,\Gamma(v)$, for any $v > 0$, it follows

$$\int_0^{e^{\gamma_2 t}} x^{\frac{\gamma_1}{\gamma_2}-1}(1-Bx)^{-\gamma_3}\,dx = e^{\gamma_1 t}\,\frac{\gamma_2}{\gamma_1}\;{}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,1+\frac{\gamma_1}{\gamma_2};Be^{\gamma_2 t}\right).$$

Using again (3.5), the second integral in (3.7) becomes

$$\int_0^1 x^{\frac{\gamma_1}{\gamma_2}-1}(1-Bx)^{-\gamma_3}\,dx = \int_0^1 x^{\frac{\gamma_1}{\gamma_2}-1}(1-Bx)^{-\gamma_3}(1-x)^0\,dx,$$

$$= \frac{\Gamma\left(\dfrac{\gamma_1}{\gamma_2}\right)\Gamma(1)}{\Gamma\left(1+\dfrac{\gamma_1}{\gamma_2}\right)}\;{}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,1+\frac{\gamma_1}{\gamma_2};B\right),$$

Hence

$$\int_0^1 x^{\frac{\gamma_1}{\gamma_2}-1}(1-Bx)^{-\gamma_3}\,dx = \frac{\gamma_2}{\gamma_1}\;{}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,1+\frac{\gamma_1}{\gamma_2};B\right).$$

In conclusion, we have found that

$$\int_0^t e^{(1-\alpha)(\delta+g)u}L(u)^{1-\alpha}du = \left(1-\frac{b}{a}\right)^{-\frac{1-\alpha}{\gamma}}\;\frac{1}{\gamma_1}\;\cdot$$

$$\cdot\;\left[e^{\gamma_1 t}{}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,1+\frac{\gamma_1}{\gamma_2};Be^{\gamma_2 t}\right) - {}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,1+\frac{\gamma_1}{\gamma_2};B\right)\right].$$

The statement now follows immediately from Theorem 1. □

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