The inner-product of Iso-Taxicab Geometry

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Abstract

The Iso-Taxicab Geometry is a non-Euclidean Geometry. It is defined in 1989 by K.O. Sowell. The aim of this paper is, firstly, to present basics of Iso-Taxicab Geometry, discuss the similarities to Taxicab Geometry (another non-Euclidean Geometry) and finally to define the inner-product of Iso-Taxicab Geometry.

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1 Introduction

E. F. Krause has defined a new geometry, the *taxicab geometry* in 1975, by using the metric $d_T(A,B) = |a_1 - b_1| + |a_2 - b_2|$ for $A = (a_1, a_2)$, $B = (b_1, b_2)$ in the Euclidean plane. Using the similar argument, K.O.Sowell has introduced a new type of taxicab geometry, *Iso-Taxicab geometry* in 1989. As it is mentioned by Sowell in [3], in iso-taxicab geometry three distance functions arise, depending upon the relative positions of the points A and B.

At the origin three axes occur: the x-axis, the y-axis and the y'-axis. This latter axis forms an angle of 60^0 with both the x-axis and the y-axis.

The three axes separate the plane into six regions, called *hextants*. These hextants will be numbered $I - VI$ in a counterclockwise direction beginning with the hextant where the coordinates of the points are both positive (Figure 1).

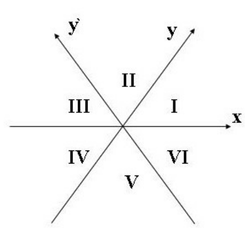

Figure 1:

Points will still be denoted by ordered pairs $A = (a_1, a_2)$ with respect to $x - axis$ and $y - axis$ (Figure 2).

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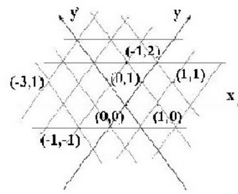

Figure 2:

At any point in the plane three lines may be drawn parallel to the axes, which in turn separate the plane into six regions. Two points could have a $I-IV$ orientation, a $II-V$ orientation or a $III-VI$ orientation to one another. The three distance functions for iso-taxicab geometry are:

(i) if the two points have $I-IV$ orientation (Figure 3), then we use

$$d_I(A,B) = |a_1 - b_1| + |a_2 - b_2|\,;$$

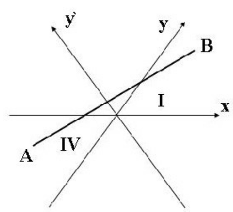

Figure 3: $I-IV$ orientation

(ii) if the two points have $II-V$ orientation (Figure 4), then we use

$$d_I(A,B) = |a_2 - b_2|\,;$$

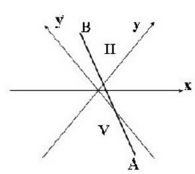

Figure 4: $II-V$ orientation

(iii) if the two points have $III-VI$ orientation (Figure 5), then we use

$$d_I(A,B) = |a_1 - b_1|\,;$$

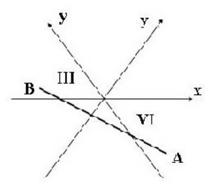

Figure 5: $III - VI$ orientation

If the two points lie on a line parallel to the x-axis, then formula (iii) is used; if the two points lie on a line parallel to the y-axis or to the y'-axis, then formula (ii) is used.

The similarities and differences of taxicab and iso-taxicab geometries are also given in [3]; e.g., we have:

1) Since the angle measure is not dependent upon the distance function, angles may be measured as in both Euclidean and Taxicab Geometry.

2) Only Euclidean plane geometry has a rotation invariant metric; therefore one should expect properties involving angle measurement to be altered in taxicab geometries. Specifically, the side-angle-side axiom may be assumed in neither iso-taxicab geometry nor in taxicab geometry. Obviously, any theorem which relies on the side-angle-side axiom is also invalid in each of these taxicab geometries. Among these are the angle-side-angle theorem and the side-side-side theorem.

3) The taxicab circle is a square and $\pi_T = 4$. The iso-taxicab circle is hexagon. Since the circumference of iso-taxicab circle is six times the radius, $\pi_I = 3$.

2 The Iso-taxicab inner product

Within the new geometry we can define an inner-productas in [1],

Definition 1. *Let* $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ *be two vectors in* R^2*. Then the taxicab inner-product is given by*

$$\langle \alpha, \beta \rangle_T = \varepsilon_1 \left| a_1 b_1 \right| + \varepsilon_2 \left| a_2 b_2 \right|,$$

where $\varepsilon_i = \begin{cases} 1, & a_i b_i \rangle 0 \\ -1, & a_i b_i \langle 0 \end{cases} \quad , \quad i = 1, 2$

We can define as well the inner-product of Iso-Taxicab Geometry - which is the aim of this paper.

First we note that since we choose 2 vectors out of 6, we have 15 choices for the vectors in different hextants. Also we have 6 choice for the vectors in the same hextants. Hence, in total we have 21 cases.

We note as well that all vectors are assumed to be standard vectors (passing through the origin) and that the orientation is counterclockwise direction. Thus the vectors on coordinate axes will be taken as in the next hextant.

Definition 2. *Let $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2) \in R^2$. We define the iso-taxicab inner-product by*

	α	β	$\langle\alpha,\beta\rangle_I$
1	I	I	$\|a_1b_1\| + \|a_2b_2\|$
2	II	II	$\|a_2b_2\|$
3	III	III	$\|a_1b_1\|$
4	IV	IV	$\|a_1b_1\| + \|a_2b_2\|$
5	V	V	$\|a_2b_2\|$
6	VI	VI	$\|a_1b_1\|$
7	II	I	$-\|a_1b_1\| + \|a_2b_2\|$
8	III	I	$-\|a_1b_1\|$
9	III	II	$\|a_1b_1\| + \|a_2b_2\|$
10	IV	I	$-\|a_1b_1\| - \|a_2b_2\|$
11	IV	II	$-\|a_2b_2\|$
12	IV	III	$\|a_1b_1\| - \|a_2b_2\|$
13	V	I	$\|a_1b_1\| - \|a_2b_2\|$
14	V	II	$-\|a_2b_2\|$
15	V	III	$-\|a_1b_1\| - \|a_2b_2\|$
16	V	IV	$-\|a_1b_1\| + \|a_2b_2\|$
17	VI	I	$\|a_1b_1\|$
18	VI	II	$-\|a_1b_1\| - \|a_2b_2\|$
19	VI	III	$-\|a_1b_1\|$
20	VI	IV	$-\|a_1b_1\|$
21	VI	V	$\|a_1b_1\| + \|a_2b_2\|$

Remark. Examining the sign of a_1, a_2, b_1, b_2, leads to

$$\langle\alpha,\beta\rangle_I = \begin{cases} a_1b_1 + a_2b_2 & , \quad 1,4,7,8,9,10,11,12,13,15,16,17,18,20,21 \\ a_2b_2 & , \quad 2,5,14 \\ a_1b_1 & , \quad 3,6,19 \end{cases}$$

Regarding the properties of this inner-product, we have the following

Theorem 3. *The iso-taxicab inner-product is positive definite, symmetric and bilinear.*

Proof. We need to show that, for any $\alpha, \beta, \gamma \in R^2$ and $r \in R$,

(1) $\langle\alpha,\alpha\rangle_I \geq 0$, and $\alpha = 0 \Leftrightarrow \langle\alpha,\alpha\rangle_I = 0$
(2) $\langle\alpha,\beta\rangle_I = \langle\beta,\alpha\rangle_I$
(3) $\langle r_1\alpha + r_2\beta, \gamma\rangle_I = r_1\langle\alpha,\gamma\rangle_I + r_2\langle\beta,\gamma\rangle_I$

Note that we have three cases:

Case 1. The vectors are of the form $1, 4, 7, 8, 9, 10, 11, 12, 13, 15$, $16, 17, 18, 20$, or 21.

Case 2. The vectors are of the form $2, 5, 14$.

Case 3. The vectors are of the form $3, 6, 19$.

Case 1 is, simply, an Euclidean inner-product. Case 2 and Case 3 are similar. So, we will only prove the form 2 of Case 2.

Proof of 2 of Case 2.

Let the vectors α and β be of the form 2. Namely, $\alpha = (a_1, a_2) \in II$, $\beta = (b_1, b_2) \in II$. Then,

(1) $\langle\alpha,\alpha\rangle_I = a_2^2 \geq 0$ and $\langle\alpha,\alpha\rangle_I = 0 \Leftrightarrow \alpha = 0$.

Note. Although for the vectors of the form $\alpha = (a_1, 0)$ where $a_1 \neq 0$, we have $\langle\alpha,\alpha\rangle_I = 0$ for $\alpha \neq 0$, notice that α is not a vector in the form 2.

(2) $\langle\alpha,\beta\rangle_I = |a_2b_2| = |b_2a_2| = \langle\beta,\alpha\rangle_I$

(3) Let $\alpha = (a_1, a_2), \beta = (b_1, b_2), \gamma = (c_1, c_2) \in R^2$ and $r_1, r_2 \in R$. Then,

$$\langle r_1\alpha + r_2\beta, \gamma\rangle_I \overset{?}{=} r_1 \langle \alpha, \gamma\rangle_I + r_2 \langle \beta, \gamma\rangle_I$$

In this case, we have three subcases. Denote $\varphi = r_1\alpha + r_2\beta$. Then, $\langle \varphi, \gamma\rangle_I$ may also be one of the form of Case 1, Case 2, or Case 3.

Subcase 1. Let $\langle \varphi, \gamma\rangle_I$ be one of the form of Case 1. Then,

$$\begin{aligned}
\langle r_1\alpha + r_2\beta, \gamma\rangle_I &= \langle (r_1a_1 + r_2b_1, r_1a_2 + r_2b_2), (c_1, c_2)\rangle_I \\
&= (r_1a_1 + r_2b_1) \cdot c_1 + (r_1a_2 + r_2b_2) \cdot c_2 \\
&= r_1a_1 \cdot c_1 + r_2b_1 \cdot c_1 + r_1a_2 \cdot c_2 + r_2b_2 \cdot c_2 \\
&= r_1(a_1c_1 + a_2c_2) + r_2(b_1c_1 + b_2c_2) \\
&= r_1 \langle \alpha, \gamma\rangle_I + r_2 \langle \beta, \gamma\rangle_I
\end{aligned}$$

as desired.

Subcase 2. Let $\langle \varphi, \gamma\rangle_I$ be one of the form of Case 2. Then,

$$\begin{aligned}
\langle r_1\alpha + r_2\beta, \gamma\rangle_I &= \langle (r_1a_1 + r_2b_1, r_1a_2 + r_2b_2), (c_1, c_2)\rangle_I \\
&= (r_1a_2 + r_2b_2) \cdot c_2 \\
&= r_1a_2 \cdot c_2 + r_2b_2 \cdot c_2 \\
&= r_1 \langle \alpha, \gamma\rangle_I + r_2 \langle \beta, \gamma\rangle_I
\end{aligned}$$

as desired.

Subcase 3. Finally, let $\langle \varphi, \gamma\rangle_I$ be one of the form of Case 3. Then,

$$\begin{aligned}
\langle r_1\alpha + r_2\beta, \gamma\rangle_I &= \langle (r_1a_1 + r_2b_1, r_1a_2 + r_2b_2), (c_1, c_2)\rangle_I \\
&= (r_1a_1 + r_2b_1) \cdot c_1 \\
&= r_1a_1 \cdot c_1 + r_2b_1 \cdot c_1 \\
&= r_1 \langle \alpha, \gamma\rangle_I + r_2 \langle \beta, \gamma\rangle_I
\end{aligned}$$

as desired. This proves the property (3) - namely, the linearity; the bi-linearity can be similarly proved. This proves Case 2; finally, the proof of Case 3 is analogous to the one used in Case 2. □

References

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