On the Green function of Sturm-Liouville differential equation with normal operator coefficient

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Abstract

Let H be a separable Hilbert space. In this work, we will study the Green function of the boundary value problem in $X = L_2(0,\infty;H)$ which is formed by Sturm-Liouville differential equation

$$-y'' + Q(x)y + \mu y = 0 \quad , \quad 0 \le x < \infty$$

with the boundary condition

$$y'(0) - hy(0) = 0$$

where h is a complex number, $\mu > 0$ is a real number and, for each value of x in $[0;\infty)$, $Q(x)$ is the normal operator in H.

M.S.C. 2000: 34L05, 47A10.
Key words: Hilbert space, self-adjoint, normal operator, resolvent, spectrum.

1 Introduction

Let $Q(x)$ be the normal operator which is a transformation for each value $x \in [0,\infty)$. Assume that

1. D of $Q(x)$ is independent from x and $\overline{D} = H$, where $\overline{D}$ represents closure of D in H.

2. The set of the regular points of $Q(x)$ is

$$\Lambda = \{|\arg\lambda - \pi| < \epsilon_0, \qquad 0 < \epsilon_0 < \pi, \qquad \epsilon_0 \text{ is constant}\}.$$

3. For all $x \in [0,\infty)$, let $Q^{-1}(x)$ be a completely continous operator in H.

4. Let

$$\left\| [Q(x) - Q(\xi)]Q^{-a}(x) \right\| \le \quad c|x-\xi|, \qquad c = \text{constant and} \quad 0 < a < \frac{3}{2}$$

where $|x-\xi| \le 1$.

Applied Sciences, Vol.7, 2005, pp. 161-175.

5. Let
$$\left\|Q(\xi)e^{\frac{-1}{2}|x-\xi|Q^{\frac{1}{2}}(x)}\right\| < B, \qquad B = \text{constant}$$
where $|x-\xi| > 1$.

6. Let $|\alpha_1(x)| \leq |\alpha_2(x)| \leq \cdots \leq |\alpha_n(x)| \leq \cdots$ be the eigenvalues of $Q(x)$ in H for each value of x. Let $\sum_{i=1}^{\infty} \frac{1}{|\alpha_i(x)|^{3/2}}$ be a convergent series and its sum be $F(x) \in L_1(0,\infty)$, i.e. we have
$$\int_0^\infty F(x)dx < \infty.$$

In this work, we obtain the Green function of the following boundary value problem in $L_2(0,\infty;H)$:

$$-y'' + Q(x)y + \mu y = 0, \qquad 0 \leq x < \infty, \tag{1.1}$$

$$y'(0) - hy(0) = 0 \tag{1.2}$$

where h is a fixed complex number and $\mu > 0$ is a real number.

The Green function of Sturm-Liouville In [8], we see, for the first time, the asymptotic behaviour of the eigenvalues of our problem described above. Then, the Green function of Sturm-Liouville differential equation, having unbounded self-adjoint operator coefficient have been studied in [9].

This work have been developped and generalized for many other operator equations (see [1, 3, 7, 5]).
The Green function of (1.1)-(1.2) above was examined in [4] while $Q*(x) = Q(x) \geq I$ (I is unit operator in H).
The Green function $G(x,\xi;\mu)$ of (1.1)-(1.2) is defined as a linear bounded operator function which is a transformation in H at each value x and $\xi \in [0,\infty)$. Assume that we have:

1. $G(x,\xi,\mu)$ is a continous operator function of x and ξ in $[x,\xi] = [0,\infty)$.

2. When $x \neq \xi$, $\frac{\partial G(x,\xi,\mu)}{\partial \xi}$ is a continous function of (x,ξ) and
$$\frac{\partial G(x,x+0,\mu)}{\partial \xi} - \frac{\partial G(x,x-0,\mu)}{\partial \xi} = -I$$
where I is the identity operator on H.

When $x \neq \xi$, we have:

3. $$-\frac{\partial^2 G(x,\xi;\mu)}{\partial \xi^2} + Q(x)G(x,\xi;\mu) + \mu G(x,\xi;\mu) = 0.$$

4. $$\frac{\partial G(x,0;\mu)}{\partial \xi} - hG(x,0;\mu) = 0.$$

2 An Integral Equation for the Green Function

In this section, we are looking for $G(x,\xi;\mu)$ as the solution of the following integral equation by using the parametrix method:

$$G(x,\xi;\mu) = g(x,\xi;\mu) - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]G(s,\xi;\mu)ds\,. \tag{2.1}$$

Here $g(x,\xi;\mu)$ is

$$g(x,\xi;\mu) = \frac{\chi^{-1}}{2}\left[e^{-\chi|x-\xi|} - (h-\chi)(h+\chi)^{-1}e^{-\chi(x+\xi)}\right]$$

and χ is

$$\chi = [Q(x)+\mu I]^{1/2}.$$

We want to show that (2.1) has the unique solution and this solution is the Green function of (1.1)-(1.2) above. For this, we consider the equation given in (2.1) in X_2 defined below.

$\mathbf{X_2}$ Space :

Let $A(x,\eta)$ be a Hilbert-Schmidt (H-S) operator function defined on H $(0 \le x,\eta < \infty)$ such that

$$\int_0^\infty dx\Big\{\int_0^\infty \|A(x,\eta)\|_2^2\, d\eta\Big\} < \infty.$$

Here, the norm is defined by

$$\|A\|_{X_2} = \left(\int_0^\infty dx \int_0^\infty \|A(x,\eta)\|_2^2\, d\eta\right)^{\frac{1}{2}}.$$

If $g(x,\xi;\mu) \in X_2$ and N is a contraction operator in X_2 which is defined by

$$NG(x,\xi;\mu) = \int_0^\infty g(x,s,\mu)[Q(s)-Q(x)]G(s,\xi;\mu)ds,$$

then the solution of the equation given in (2.1) exists and unique. Consider the spaces X_1^p, X_2^p and X_4^p $(p<0; p>0)$ whose elements are the operator functions $A(x,\xi)$ and

$$\|A(x,s)\|^2_{X_1^p} = \int_0^\infty dx\Big\{\int_0^\infty \|A(x,s)Q^p(s)\|^2 ds\Big\}$$

$$\|A(x,s)\|^2_{X_2^p} = \int_0^\infty dx\Big\{\int_0^\infty \|A(x,s)Q^p(s)\|_2^2 ds\Big\}$$

$$\|A(x,s)\|_{X_4^p} = \sup_{0\le x<\infty} \int_0^\infty \|A(x,s)Q^p(s)\| ds$$

respectively.

These spaces are defined by B.M. Levitan in [9] in which he shows that they are all Banach spaces.

By using [9], we can prove the following lemmas:

Lemma 2.1 *If* $Q(x)$ *satisfies the (1-6) conditions above then* N *is a contraction operator in* X_1 *and* X_2 *both for sufficiently large* $\mu > 0$ *.*

Lemma 2.2 *Let* $Q(x)$ *be a operator function satisfying (1) and (3) above and, in addition, we have*

$$\|Q^{-1/2}(x)Q^{1/2}(\xi)\| \leq c$$

where $|x-\xi| \leq 1$*.*

In this case, $g(x,\xi;\mu)$ *belongs to* $X_4^{(1/2)}$ *i.e. we have*

$$\sup_{0\leq x<\infty} \int_0^\infty \|g(x,\xi;\mu)Q^{1/2}(\xi)\| d\xi < \infty.$$

Lemma 2.3 *Let* $Q(x)$ *be a operator function satisfying (1) and (3) above and, in addition, we have*

$$\|Q^{1/2}(x)Q^{-1/2}(s)\| \leq c$$

where $|x-s| \leq 1$*.*

In this case, we have

$$\frac{\partial^2 g(x,\xi;\mu)}{\partial s^2} \in X_4^{(-1/2)}, \qquad x \neq s,$$

i.e. we have

$$\sup_{0\leq x<\infty} \int_0^\infty \left\| \frac{\partial^2 g}{\partial s^2} Q^{-1/2}(s) \right\| ds < \infty.$$

2.1 First Derivative of the Green Function

Consider the formal derivative of (2.1) according to ξ

$$\frac{\partial G(x,\xi;\mu)}{\partial \xi} = \frac{\partial g(x,\xi;\mu)}{\partial \xi} - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)] \frac{\partial G(s,\xi;\mu)}{\partial \xi} ds.$$

Now consider the following equation in the Banach space $X_3^{(p)}$, $(p \geq 1)$,

(2.1.1) $$K(x,\xi;\mu) = \frac{\partial g(x,\xi;\mu)}{\partial \xi} - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]K(s,\xi;\mu) ds$$

$\frac{\partial g(x,\xi;\mu)}{\partial \xi} \in X_3^{(p)}$, i.e. we have

$$\sup_{0\leq x<\infty} \int_0^\infty \left\| \frac{\partial g(x,\xi;\mu)}{\partial \xi} \right\|^{(p)} d\xi < \infty.$$

So, $K(x,\xi;\mu) \in X_3^{(p)}$ and, in particular it is an element of $X_3^{(1)} \equiv X_3$.

Here,

$$(2.1.2)\qquad \frac{\partial g(x,\xi;\mu)}{\partial \xi}=\begin{cases}\frac{-1}{2}e^{-\chi(\xi-x)}+\frac{1}{2}(h+\chi)^{-1}(h-\chi)e^{-\chi(x+\xi)}, & x<\xi\\ \frac{1}{2}e^{-\chi(x-\xi)}+\frac{1}{2}(h+\chi)^{-1}(h-\chi)e^{-\chi(x+\xi)}, & x>\xi.\end{cases}$$

Assume that $\xi < x$ (note that we can do the same process for $\xi > x$). The integration of (2.1.1) gives us

$$(2.1.3)\quad \int\limits_{\infty}^{\xi} K(x,\xi;\mu)d\xi = g(x,\xi;\mu) - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\int\limits_{\infty}^{\xi} K(s,\xi;\mu)dsd\xi.$$

This is same as the equation (2.1) and we have

$$\int\limits_{\infty}^{\xi} K(s,\xi;\mu)d\xi = G(x,\xi;\mu)$$

since (2.1) has the unique solution.

Now, we will show that the operator function $K(x,\xi;\mu)$ is continous with respect to $\xi \neq x$,

$$\|K(x,\xi+h;\mu)-K(x,\xi;\mu)\| \longrightarrow 0$$

when $h \longrightarrow 0$.

For this reason, we write the equation (2.1.1) as below:

$$(2.1.4)\qquad K(x,\xi;\mu)-\frac{\partial g(x,\xi;\mu)}{\partial \xi} = -\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi}ds-$$

$$-\int\limits_0^\infty \{g(x,s;\mu)[Q(s)-Q(x)]\}\{K(s,\xi;\mu)-\frac{\partial g(s,\xi;\mu)}{\partial \xi}\}ds.$$

Let

$$L(x,\xi;\mu)=K(x,\xi;\mu)-\frac{\partial g(x,\xi;\mu)}{\partial \xi}$$

and

$$l(x,\xi;\mu)=\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi}ds.$$

Hence (2.1.4) becomes

$$(2.1.5)\qquad L(x,\xi;\mu)=l(x,\xi;\mu)-\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]L(s,\xi;\mu)ds.$$

If we denote

$$\Delta L(x,\xi;\mu)=L(x,\xi+h;\mu)-L(x,\xi;\mu)$$

and

$$\Delta l(x,\xi;\mu)=l(x,\xi+h;\mu)-l(x,\xi;\mu)$$

then, by (2.1.5), we obtain

$$\Delta L = \Delta l - N(\Delta L). \tag{2.1.6}$$

We study the equation (2.1.6) in the space X_5. We know from [9] that X_5 is Banach space of operator functions $A(x,\xi)$, $(0 \le x; \xi < \infty)$, defined on H with

$$\|A(x,\xi)\|_{X_5} = \sup_{0\le x<\infty}\sup_{0\le\xi<\infty}\|A(x,\xi)\|_H$$

As in Lemma 2.1, it can be seen that N is a contraction operator for sufficiently large $\mu > 0$ in X_5.

According to this, $(I+N)^{-1}$ exist and is bounded in X_5. Let $\|(I+N)^{-1}\|_{X_5} = A$. In this case, we have

$$\|\Delta L\|_{X_5} \le A\|\Delta l\|_{X_5}. \tag{2.1.7}$$

Lemma 2.4 *For arbitrary $\epsilon > 0$, when $|h| < t$ there exists $\delta > 0$ such that*

$$\|L(x,\xi+h;\mu) - L(x,\xi;\mu)\|_{X_5} < \epsilon \tag{2.1.8}$$

when $|h| < t$.

So, the operator function $K(x,\xi;\mu) - \frac{\partial g(x,\xi;\mu)}{\partial\xi}$ is continous with respect to ξ. Moreover,
$K(x,\xi;\mu) = \frac{\partial G(x,\xi;\mu)}{\partial\xi}$ is continous for $\xi \ne x$. Let us see that $\frac{\partial g}{\partial\xi}$ is continous with respect to ξ when $\xi \ne x$.

$$\frac{\partial g(x,\xi;\mu)}{\partial\xi} = \begin{cases} \frac{-1}{2}e^{-\chi(\xi-x)} + \frac{1}{2}(h+\chi)^{-1}(h-\chi)e^{-\chi(x+\xi)}, & x<\xi \\ \frac{1}{2}e^{-\chi(x-\xi)} + \frac{1}{2}(h+\chi)^{-1}(h-\chi)e^{-\chi(x+\xi)}, & x>\xi \end{cases}$$

Let $\xi < x$. In this case, we have

$$\frac{\partial g}{\partial\xi} = \frac{1}{2}e^{-\chi(x-\xi)} + \frac{1}{2}(h+\chi)^{-1}(h-\chi)e^{-\chi(x+\xi)}.$$

Put $x - \xi = t$. Assume that h is a smaller number. We can show that

$$\|\frac{\partial g(x,\xi+h;\mu)}{\partial\xi} - \frac{\partial g(x,\xi;\mu)}{\partial\xi}\| \longrightarrow 0$$

when $h \longrightarrow 0$. With respect to the spectral expansion formula, we can write

$$\begin{aligned} \|e^{-\chi(t-h)} - e^{-\chi t}\|_H &= \sup_{\|f\|=1}\left|\int_\sigma (e^{-\sqrt{\lambda+\mu(t-h)}} - e^{-\sqrt{\lambda+\mu t}})d(E_\lambda f,f)\right| \le \\ &\le \sup_{\|f\|=1}\int_\sigma \left|(e^{-\sqrt{\lambda+\mu(t-h)}} - e^{-\sqrt{\lambda+\mu t}})\right|d(E_\lambda f,f) \\ &= \sup_{\|f\|=1}\left(\int_{\sigma\cap K_R}\left|(e^{-\sqrt{\lambda+\mu(t-h)}} - e^{-\sqrt{\lambda+\mu t}})\right|\right)d(E_\lambda f,f) + \\ &+ \int_{\sigma\cap K'_R}\left|(e^{-\sqrt{\lambda+\mu(t-h)}} - e^{-\sqrt{\lambda+\mu t}})\right|d(E_\lambda f,f) \\ &= I_1 + I_2 \end{aligned}$$

Here, K_R is a circle with center at origin and radius R, K'_R is the exterior of the circle K_R and $E_\lambda = E_\lambda(x)$ is a resolution of the identity corresponding to $Q(x)$ [12].
Let $\epsilon > 0$ be an arbitrary positive number. Assume that $|h| < t$. We can choose a sufficiently large R for $|\lambda+\mu| > R$. We have

$$\left|e^{-\sqrt{\lambda+\mu}(t-h)} - e^{-\sqrt{\lambda+\mu}t}\right| < \frac{\epsilon}{2}$$

In this case, we have

$$I_2 < \frac{\epsilon}{2}.$$

Now, let R be in I_2. We can choose a sufficiently small $\delta > 0$ such that $|h| < \delta$ for $|\lambda+\mu| < R$

$$\left|e^{-\sqrt{\lambda+\mu(t-h)}} - e^{-\sqrt{\lambda+\mu t}}\right| = \left|e^{-\sqrt{\lambda+\mu t}}(e^{-\sqrt{\lambda+\mu h}} - 1)\right| < \frac{\epsilon}{2}$$

Similarly, we have

$$I_1 < \frac{\epsilon}{2}.$$

Therefore, there are $\delta > 0$ such that

$$\|e^{-\chi(t-h)} - e^{-\chi t}\|_H < \epsilon$$

when $|h| < \delta$. Recall that we can repeat the same process for the case $\xi > x$.

We can show that $\frac{\partial g}{\partial \xi}$ has a jump at the $x = \xi$:

Since, in (2.1.2), the second terms are continous with respect to ξ we need to see the first terms. Assume that $h > 0$. We have

$$\left.\frac{\partial g}{\partial \xi}\right|_{\xi=x+h} = -\frac{1}{2}e^{-h\chi} \quad , \quad \left.\frac{\partial g}{\partial \xi}\right|_{\xi=x-h} = \frac{1}{2}e^{-h\chi}$$

$$\left[\left.\frac{\partial g}{\partial \xi}\right|_{\xi=x+h} + \frac{1}{2}I\right]\chi^{-2} = \frac{1}{2}(e^{-h\chi} - I)\chi^{-2} = \alpha(x,\xi,h)$$

$$\begin{aligned}
\|\alpha(x,\xi,h)\| &= \sup_{\|f\|=1}\left|\int_\sigma -\frac{1}{2}\frac{1}{\lambda+\mu}[e^{-h\sqrt{\lambda+\mu}} - 1]d(E_\lambda f,f)\right| \\
&\le \sup_{\|f\|=1}\int_\sigma\left|\frac{1}{\lambda+\mu}[e^{-h\sqrt{\lambda+\mu}} - 1]\right|d(E_\lambda f,f) \\
&= \sup_{\|f\|=1}\int_{\sigma\cap K_R}\left|\frac{1}{\lambda+\mu}[e^{-h\sqrt{\lambda+\mu}} - 1]\right|d(E_\lambda f,f) + \\
&+ \sup_{\|f\|=1}\int_{\sigma\cap K'_R}\left|\frac{1}{\lambda+\mu}[e^{-h\sqrt{\lambda+\mu}} - 1]\right|d(E_\lambda f,f) \\
&= I_1 + I_2
\end{aligned}$$

For $\mathbf{I_2}$: Let $\epsilon > 0$ be an arbitrary number. We can choose R such that we have $\lambda \in \sigma \cap K'_R$ with $\left|\frac{1}{\lambda+\mu}\right| < \frac{\epsilon}{4}$. In this case, $I_2 < \frac{\epsilon}{2}$.

For $\mathbf{I_1}$: Let R be as in I_2. We can choose $\delta > 0$ and $h < \delta$ such that $\lambda \in \sigma \cap K_R$ and $\left|\frac{1}{\lambda+\mu}(e^{-h\sqrt{\lambda+\mu}} - 1)\right| < \frac{\epsilon}{2}$. In this case, $I_1 < \frac{\epsilon}{2}$.

So, for $\epsilon > 0$, there exists $\delta > 0$ such that $h < \delta$ such that $\|\alpha(x,\xi;\mu)\|_H < \epsilon$. Thus, we obtain

$$\left\| \left[\left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x+h} - \frac{1}{2}I \right] \chi^{-2} \right\|_H \xrightarrow[h \to 0]{} 0$$

By the same way, we can see that

$$\left\| \left[\left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x-h} + \frac{1}{2}I \right] \chi^{-2} \right\|_H \xrightarrow[h \to 0]{} 0$$

Hence, we have

$$\begin{aligned}
&\left\| \left[\left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x+h} - \left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x-h} + I \right] \chi^{-2} \right\|_H = \\
= &\left\| \left[\left.\frac{-\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x+h} - \frac{1}{2}I \right] \chi^{-2} + \left[\left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x-h} + \frac{1}{2}I \right] \chi^{-2} \right\|_H \leq \\
\leq &\left\| \left[\left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x+h} - \frac{1}{2}I \right] \chi^{-2} \right\|_H \\
+ &\left\| \left[\left.\frac{-\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x-h} + \frac{1}{2}I \right] \chi^{-2} \right\|_H \xrightarrow[h \to 0]{} 0
\end{aligned}$$

If we consider the remaining terms in $\frac{\partial g}{\partial \xi}$ which are continous with respect to ξ then we have

$$\left\| \left[\left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x+h} - \left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x-h} + I \right] \chi^{-2} \right\|_H \xrightarrow[h \to 0]{} 0$$

Since, $\frac{\partial G}{\partial \xi}$ has the same jump as $\frac{\partial g}{\partial \xi}$ for $\xi = x$ the operator function $\frac{\partial G}{\partial \xi} - \frac{\partial g}{\partial \xi}$ is continous for $\xi = x$.

Thus, we obtain

$$\left\| \left[\left.\frac{\partial G(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x+h} - \left.\frac{\partial G(x,\xi;\mu)}{\partial \xi}\right|_{\xi=x-h} + I \right] \chi^{-2} \right\|_H \xrightarrow[h \to 0]{} 0 \tag{2.1.9}$$

Let $f \in D = D[Q(x)]$. Obviously, f belongs to the domain of definition of the operator $[Q(x)+\mu I]$. This gives $[Q(x)+\mu I]f = g$. Then we obtain

$$\begin{aligned} & \left\| \left[\left.\frac{\partial G}{\partial \xi}\right|_{\xi=x+h} - \left.\frac{\partial G}{\partial \xi}\right|_{\xi=x-h} + I \right] f \right\| \\ \leq & \left\| \left[\left.\frac{\partial G}{\partial \xi}\right|_{\xi=x+h} - \left.\frac{\partial G}{\partial \xi}\right|_{\xi=x-h} + I \right] \chi^{-2} \right\|_H \|g(x)\|_H < \\ < & \epsilon \|g(x)\|_H \end{aligned}$$

for small values of h.
Furthermore, we have

$$[G'_\xi(x, x+0, \mu) - G'_\xi(x, x-0, \mu)]f = -f$$

for $\xi = x$.

Thus, we obtain that $G(x,\xi;\mu)$ has a jump at $\xi = x$ and this jump equals to $-I$, that is,

$$[G'_\xi(x, x+0;\mu) - G'_\xi(x, x-0;\mu)] = -I.$$

2.2 Second Derivative of the Green Function

Consider

$$\frac{\partial G(x,\xi;\mu)}{\partial \xi} = \frac{\partial g(x,\xi;\mu)}{\partial \xi} - \int_0^\infty g(x,s,\mu)[Q(s)-Q(x)]\frac{\partial G(s,\xi;\mu)}{\partial \xi} ds. \tag{2.2.1}$$

Let us write (2.2.1) as

$$\begin{aligned} \frac{\partial G}{\partial \xi} - \frac{\partial g}{\partial \xi} = & -\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi} ds - \\ & -\int_0^\infty \{g(x,s;\mu)[Q(s)-Q(x)]\} \left[\frac{\partial G}{\partial \xi} - \frac{\partial g}{\partial \xi}\right] ds. \end{aligned} \tag{2.2.2}$$

Let

$$L(x,\xi;\mu) = \frac{\partial G}{\partial \xi} - \frac{\partial g}{\partial \xi}$$

and

$$l(x,\xi;\mu) = -\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi} ds.$$

In this case, (2.2.2) becomes

$$L(x,\xi;\mu) = l(x,\xi;\mu) - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]L(s,\xi;\mu) ds.$$

Differentiating formally each side of this equation with respect to ξ , we obtain

$$\frac{\partial L(x,\xi;\mu)}{\partial \xi} = \frac{\partial l(x,\xi;\mu)}{\partial \xi} - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial L}{\partial \xi} ds$$

By using the fact that

$$\frac{\partial g(x,x+0;\mu)}{\partial \xi} - \frac{\partial g(x,x-0;\mu)}{\partial \xi} = -I$$

and

$$\begin{aligned} l(x,s;\mu) &= \int_0^{\xi-0} g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi} ds - \\ &- \int_{\xi+0}^{\infty} g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi} ds, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\partial l}{\partial \xi} = &- \int_0^{\infty} g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial^2 g(s,\xi;\mu)}{\partial \xi^2} ds + \\ &+ g(x,\xi;\mu)[Q(\xi)-Q(x)]\left\{\frac{\partial g(\xi+0,\xi;\mu)}{\partial \xi} - \frac{\partial g(\xi-0,\xi;\mu)}{\partial \xi}\right\}. \end{aligned}$$

$$\frac{\partial l}{\partial \xi} = -g(x,\xi;\mu)[Q(\xi)-Q(x)] - \int_0^{\infty} g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial^2 g(s,\xi;\mu)}{\partial \xi^2} ds$$

$$\frac{\partial l}{\partial \xi} = l_1(x,\xi;\mu).$$

Lemma 2.5 *If the operator function $Q(x)$ satisfies the conditions of Lemma 2.1 then N is a contraction operator in $X_1^{(s)}$, $X_2^{(s)}$ and $X_4^{(s)}$ for large values of $\mu > 0$.*

Since there is a unique solution of

$$M(x,\xi;\mu) = l_1(x,\xi;\mu) - \int_0^{\infty} g(x,s;\mu)[Q(s)-Q(x)]M(s,\xi;\mu)ds \tag{2.2.3}$$

and this solution belongs to $X_4^{(-1/2)}$, l_1 belongs to $X_4^{(-1/2)}$ according to Lemma 4.1. If $f \in D\{Q(x)\}$ is a solution of (2.2.3) we can show that f satisfies

$$\frac{\partial L}{\partial \xi}(f) = M(f) \qquad (s \neq \xi).$$

Let $f \in D\{Q(x)\} = D$. Then, (2.2.3) becomes

$$M(f) = l_1(f) - \int_0^{\infty} g[Q(s)-Q(x)]M(s,\xi;\mu)(f)ds.$$

The integration from ξ_0 to ξ

$$\int_{\xi_0}^{\xi} M(f)d\xi = \int_{\xi_0}^{\xi} l_1(f)d\xi - \int_0^{\infty} g[Q(s)-Q(x)]\left(\int_{\xi_0}^{\xi} M(s,\xi;\mu)(f)d\xi\right)ds$$

gives

$$[L(x,\xi;\mu) - L(x,\xi_0;\mu)] = [l(x,\xi;\mu) - l(x,\xi_0;\mu)]f - \tag{2.2.4}$$
$$- \int_0^\infty g[Q(s) - Q(x)][L(s,\xi;\mu) - L(s,\xi_0;\mu)]f\, ds.$$

If we show

$$\int\limits_{\xi_0}^{\xi} l_1(x,\xi;\mu)(f)d\xi = [l(x,\xi;\mu) - l(x,\xi_0;\mu)]f$$

we can obtain the following equation from the uniqueness of the solution of (2.2.4):

$$\int\limits_{\xi_0}^{\xi} M(x,\xi;\mu)(f)d\xi = [L(x,\xi;\mu) - L(x,\xi_0;\mu)](f)$$

i.e. we have

$$M(x,\xi;\mu)(f) = \frac{\partial L}{\partial \xi}(f) \tag{2.2.5}$$

since

$$L(x,\xi;\mu) = \frac{\partial G(x,\xi;\mu)}{\partial \xi} - \frac{\partial g(x,\xi;\mu)}{\partial \xi}.$$

Hence, $\frac{\partial^2 G(s,\xi;\mu)}{\partial \xi^2}$ exists and belongs to $X_4^{(-1/2)}$ at $s \neq \xi$. So we need to prove (2.2.5). For this, consider

$$l(x,\xi;\mu) = -\int_0^\infty g(x,s;\mu)[Q(s) - Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi} ds.$$

Let us choose arbitrary $\overline{\xi}$ in (ξ_0, ξ). Then $Q^{-1}(\xi)Q(\overline{\xi})$ is bounded operator in H. Indeed, let $\xi > \overline{\xi}$. In this case, we can write $\xi = \overline{\xi} + k + r$ (k integer $0 < r < 1$). Assume that $\|Q(x)Q^{-1}(\xi)\| < c$ when $|x - \xi| \leq 1$. Obviously,

$$Q^{-1}(\xi).Q(\overline{\xi}) = Q^{-1}(\xi)Q(\xi - 1)Q^{-1}(\xi - 1)\cdots Q^{-1}(\overline{\xi} + r)Q(\overline{\xi}).$$

Consequently,

$$\begin{aligned} &\|Q^{-1}(\xi).Q(\overline{\xi})\| \leq \\ \leq\ &\|Q^{-1}(\xi)Q(\xi-1)\|\|Q^{-1}(\xi-1)Q(\xi-2)\|\cdots\|Q^{-1}(\overline{\xi}+r)Q(\xi_n)\| \leq c. \end{aligned}$$

This says that $Q^{-1}(\xi).Q(\overline{\xi})$ is a bounded operator.

Applying l to f

$$\begin{aligned} l(x,\xi;\mu)(f) &= \int_0^\infty g(x,s;\mu)[Q(s) - Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi}(f)ds \\ &= \int_{|x-s|\leq 1} g(x,s;\mu)[Q(s) - Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi}(f)ds + \\ &+ \int_{|x-s|>1} g(x,s;\mu)[Q(s) - Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi}(f)ds \\ &= a_1 + a_2 \end{aligned}$$

Let us examine the $a_1(x,\xi;\mu)$ in two cases [9]:
1) When $|x-\xi|>2$,

$$|s-\xi| \geq |x-\xi| - |x-s| > 1$$

is the function under the integral and its derivative with respect to ξ is a bounded operator.
2) When $|x-\xi|<2$: Let $f = Q^{-1}(\xi)h,\ h \in H$. In this case, we have the integral which is itself under integral and its derivative with respect to ξ are bounded operator.

Let us examine $a_2(x,\xi;\mu)$:

$$a_2(x,\xi;\mu) = \int_{|s-\xi|>1} g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial g(s,\xi;\mu)}{\partial \xi}(f)ds$$

Here, the function, under the integral, is bounded operator function because of the multiplier $g(x,\xi;\mu)$ and the condition 5). In addition, $a_2(x,\xi;\mu)$ can be differentiated under the integral sign similiar with bounded operator. This integral is bounded operator when it is derived with respect to ξ.
Indeed, it is clear when $|s-\xi| < \gamma > 0$. We take the element f as
$f = Q^{-1}(s)[Q(s)Q^{-1}(\bar{\xi})]Q(\bar{\xi})f$ for the value of $s-\xi$ which is near zero.
Therefore the operator function, under the integral, can be differentiated with respect to ξ as in bounded operator. We obtain equation (2.2.3) that is the formal derivation written for $\frac{\partial l}{\partial \xi}$.

2.3 The Case that Green Function Satisfies the Third Condition

The derivation of the following equation with respect to ξ

$$\frac{\partial G}{\partial \xi} = \frac{\partial g(x,\xi;\mu)}{\partial \xi} - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial G(s,\xi;\mu)}{\partial \xi}ds$$

gives

$$\frac{\partial^2 G}{\partial \xi^2} = G(x,\xi;\mu)[Q(\xi)+\mu I]$$

We have

$$\frac{\partial^2 G}{\partial \xi^2} = g[Q(\xi)+\mu I] - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial^2 G(s,\xi;\mu)}{\partial \xi^2}ds \tag{2.3.1}$$

For $f \in D$ (D is the definition set of $Q(x)$) we have

$$\frac{\partial^2 G}{\partial \xi^2}(f) = g[Q(\xi)+\mu I](f) - \tag{2.3.2}$$

$$-\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial^2 G(s,\xi;\mu)}{\partial \xi^2}(f)ds.$$

Let $[Q(\xi)+\mu I](f) = \alpha$. According to this, (2.3.2) can be rewritten as

$$\frac{\partial^2 G}{\partial \xi^2}[Q(\xi)+\mu I]^{-1}\alpha = g(x,\xi;\mu)\alpha - \tag{2.3.3}$$

$$-\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\frac{\partial^2 G(s,\xi;\mu)}{\partial \xi^2}[Q(\xi)+\mu I]^{-1}\alpha ds.$$

Let us compare (2.3.3) and (2.1). If (2.1) has a unique solution, we obtain

$$\frac{\partial^2 G}{\partial \xi^2}[Q(\xi)+\mu I]^{-1}\alpha = G(x,\xi;\mu)\alpha. \tag{2.3.4}$$

If $[Q(\xi)+\mu I]^{-1}\alpha = f$ and $\alpha = [Q(\xi)+\mu I]f$ then (2.3.4) can be rewritten as

$$\frac{\partial^2 G}{\partial \xi^2} f = G(x,\xi;\mu)[Q(\xi)+\mu I]f.$$

Since the set of the elements f is dense in H for every $\xi \geq 0$, we have (consider $\overline{D} = H$)

$$-\frac{\partial^2 G}{\partial \xi^2} + G(x,\xi;\mu)[Q(\xi)+\mu I] = 0, \qquad (\xi \neq x).$$

2.4 To Satisfy the Boundary Condition

We will show that $G(x,\xi;\mu)$ satisfies the boundary condition

$$\left.\frac{\partial G(x,\xi;\mu)}{\partial \xi}\right|_{\xi=0} - hG(x,\xi;\mu)\Big|_{\xi=0} = 0. \tag{2.4.1}$$

The derivative of the equation

$$G(x,\xi;\mu) = g(x,\xi;\mu) - \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]G(s,\xi;\mu)ds$$

at $\xi = 0$ is

$$\left.\frac{\partial G(x,\xi;\mu)}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial g(x,\xi;\mu)}{\partial \xi}\right|_{\xi=0} - \tag{2.4.2}$$

$$-\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\left.\frac{\partial G(x,\xi;\mu)}{\partial \xi}\right|_{\xi=0} ds$$

$$hG(x,\xi;\mu)\Big|_{\xi=0} = g(x,\xi;\mu)h- \tag{2.4.3}$$

$$-\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]G(s,\xi;\mu)\Big|_{\xi=0} hds.$$

Replace (2.4.2) and (2.4.3) in (2.4.1)

$$\begin{aligned}\left.\frac{\partial G(x,\xi;\mu)}{\partial\xi}\right|_{\xi=0} &- \left.hG(x,\xi;\mu)\right|_{\xi=0} = \left.\frac{\partial g(x,\xi;\mu)}{\partial\xi}\right|_{\xi=0} - \\ &- \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\left.\frac{\partial G(s,\xi;\mu)}{\partial\xi}\right|_{\xi=0} ds - \\ &- g(s,\xi;\mu)h + \\ &+ \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\left.hG(s,\xi;\mu)\right|_{\xi=0} ds.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\left.\frac{\partial G}{\partial\xi}\right|_{\xi=0} - \left.hG\right|_{\xi=0} = &- \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\left.\frac{\partial G(s,\xi;\mu)}{\partial\xi}\right|_{\xi=0} ds + \\ &+ \int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\left.hG(s,\xi;\mu)\right|_{\xi=0} ds\end{aligned}$$

since $\left.\frac{\partial g}{\partial\xi}\right|_{\xi=0} - \left.hg\right|_{\xi=0} = 0$. Thus, we obtain

$$\begin{aligned}&\left[\frac{\partial G}{\partial\xi} - hG\right]_{\xi=0} = \\ &= -\int_0^\infty g(x,s;\mu)[Q(s)-Q(x)]\left[\frac{\partial G(s,\xi;\mu)}{\partial\xi} - hG(s,\xi;\mu)\right]_{\xi=0} ds.\end{aligned}$$

Since N is a contraction operator the homogenous equation obtained above has only the zero solution. This gives

$$\left.\frac{\partial G}{\partial\xi} - hG\right|_{\xi=0} = 0.$$

If we construct the integral operator in H_1

$$A_\mu f = \int_0^\infty G(x,\xi;\mu)f(\xi)d\xi, \qquad \mu > 0$$

by using the obtained Green function, then A_μ is an operator of Hilbert-Schmidt type in H_1 since

$$\int_0^\infty\int_0^\infty \|G(x,\xi;\mu)\|_2^2 dx d\xi < \infty.$$

By using the Green function, one can examine the boundary value problem of the non-homegeneous equation

$$\begin{gathered}-y'' + Q(x)y + \mu y = 0, \qquad f(x) \in L_2(0,\infty;H), \\ y'(0) - hy(0) = 0.\end{gathered}$$

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