A mathematical model to reproduce the wave field generated by convergent lenses

S. Romaguera, J.V. Sánchez-Pérez, O. Valero

Abstract

Geometric optic gives a good approximation to obtain the focus generated by convergent lenses in the case that their physical characteristics do not give aberrations. However, some refractive devices present aberrations due to their inherent characteristics. Sonic lenses constructed from Sonic Band Gaps (SBG) materials provide a suitable example. In this paper we present a mathematical model, based on the theory of conformal mappings and valid for airbonesound, to predict the "focal area" generated by any kind of wave lenses with aberration problems. Furthermore, we present an abstract approach to describe the focal phenomena with the help of normed monoids.

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§1. Introduction

In the last decade, the problem of wave propagation in composite periodic media has increased interest. Now, it is well known that for these materials multiple scattered interfere to open frequency gaps (range of frequencies) where propagation is forbbiden, like happen with electrons inside crystalline materials. This behavior appears in any kind of wave: electromagnetic, elastic or acoustic. First, several authors ([11], [4]) showed that a material transparent for electromagnetic waves, can become opaque for certain range of frequencies. These materials were called Photonic Band Gaps (PBG) by analogy to the behavior of electrons in semiconductor materials.

In acoustic waves, the existence of this kind of materials was predicted theoretically at the beginning of 1990s ([2]), but the first experimental results appear at 1995 ([6]). These materials, that can stop the transmission of some range of acoustic frequencies, are called Sonic Band Gap (SBG) materials. It is interesting to remark that nearly all of experiments and physical models have been done with two-dimensional composite samples i.e., the periodicity of these composite materials appears in two dimensions ([9], [5], [8]). A typical system is made with rigid cylinders in air.

Recently ([1]) it has been obtained, theoretic an experimentally, a new property of the SBG materials: the possibility of construct refractive devices for airborne sound. The authors show the physical realization of two devices commonly used in optics:

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a Fabry-Perot interferometer and a convergent lens (Figure 1). The authors present sound level maps for each frequency showing the acoustic focus in each case for the convergent lenses.

The convergent acoustic lenses present aberrations due the physical characteristics of these composite systems. The anisotropy of the velocities inside the system and the discontinuity of the external shape are determinant to obtain a focal zone (not a focal point) using SBG materials like convergent lenses. In order to use these lenses in acoustic technology, it is necessary to carry out a mathematical model that allow us to predict the "focal area" depending on the external shape of the lense.

In this paper we give a mathematical method based on the classical theory of conformal mappings in a complex variable, to obtain this aim.

Furthermore, we show that the structure of a normed monoid provides an appropriate setting to describe focal phenomena in an abstract approach. Therefore, it is possible to employ this general method in others convergent phenomena obtained with other kind of waves.

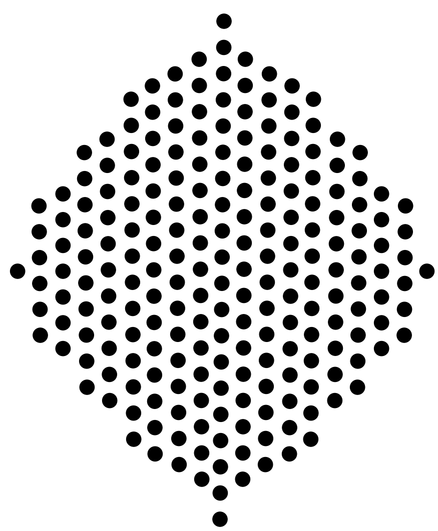

Figure 1:

§2. A geometric method for focal phenomena

Based on SBG materials, we describe in this section the "focal area" produced by acoustic waves propagation in convergent lenses for each frequency.

Our main reference for functions of a complex variable is [10].

Denote by $\mathcal{E}$ the ellipse, in the complex plane, given by

$$\mathcal{E} = \{z = u + iv \in \mathbb{C} : \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1\},$$

with $a > b > 0$.

First, we consider a typical convergence lense: plane-convex lense whose external shape can be considered as the semiellipse $\mathcal{E}_0$, where

$$\mathcal{E}_0 = \{z \in \mathcal{E} : v \geq 0\}.$$

It is known (see for instance [10], Chapter 8) that the holomorphic function f given by

$$f(z) = \frac{1}{2}[(a-b)z + \frac{a+b}{z}]$$

is a one-to-one transformation from the unit circle $|z| = 1$ onto the ellipse $\mathcal{E}$, in such a way that the semicircle $\mathcal{S} = \{z = u + iv \in \mathbb{C} : |z| = 1, v \leq 0\}$ is transformed in the semiellipse $\mathcal{E}_0$.

Furthermore f transforms the circle $|z| = r$, where $r = [\frac{a+b}{a-b}]^{1/2}$, in the segment $[-(a^2-b^2)^{1/2}, (a^2-b^2)^{1/2}]$, as follows:

$$f(re^{it}) = \frac{1}{2}[(a-b)r + \frac{a+b}{r}]\cos(t) = (a^2-b^2)^{1/2}\cos(t)$$

for all $t \in [0, 2\pi]$.

It easy to see that f is a conformal mapping from the open set $A = \{z \in \mathbb{C} : 1 < |z| < r\}$ onto the bounded open set $B = \{z = u + iv : \frac{u^2}{a^2} + \frac{v^2}{b^2} < 1\}$, and that the unit disk is transformed onto the open set $\mathbb{C}\backslash(B \cup FrB)$, i.e. on "the outside" of the ellipse $\mathcal{E}$.

Clearly the restriction of f to the closed set

$$A_0 = \{z = u + iv : 1 \leq |z| \leq r, v \leq 0\}$$

is a one-to-one holomorphic function and its inverse g is given by

$$g(w) = \frac{w - [w^2 - (a^2 - b^2)]}{a-b}.$$

On the other hand, if we consider the family of concentric circles

$$C(r) = \{C_s : s \in]1, r[\},$$

where C_s is the circle $|z| = s$ and the family of ellipses

$$\Gamma(r) = \{\mathcal{E}_s : s \in]1, r[\},$$

where $\mathcal{E}_s$ is the ellipse with equation

$$\frac{u^2}{a_s^2} + \frac{v^2}{b_s^2} = 1,$$

where

$$a_s = \frac{1}{2}[(a-b)s + \frac{(a+b)}{s}] \quad \text{and} \quad b_s = \frac{1}{2}[(a-b)s - \frac{(a+b)}{s}].$$

Then f transforms each circle C_s on the ellipse $\mathcal{E}_s$, for all $s \in]1, r[$ (see Figure 2).

In the sequel, we will say that two ellipses $\mathcal{E}_i, \mathcal{E}_j$, with $i \neq j$, are *concentric* if there is $r > 1$ such that $\mathcal{E}_i, \mathcal{E}_j \in \Gamma(r)$.

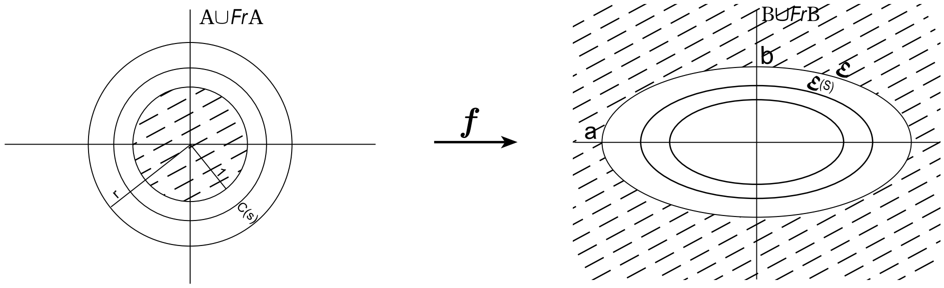

Figure 2:

This method can be applied to the more general case that the circle $|z| = \varepsilon$, where ε is an arbitrarily small positive real number, is conformally transformed onto the ellipse $\mathcal{E}$. In this situation, the holomorphic function f_ε defined by

$$f_\varepsilon(z) = \frac{1}{2}[\frac{(a-b)}{\varepsilon}z + \frac{(a+b)\varepsilon}{z}]$$

carries out the transformation in the same way as described above. Moreover, the holomorphic function g_ε defined by $g_\varepsilon(w) = \varepsilon \cdot g(w)$ is the inverse of f_ε.

Hence we may propose the function g_ε, where ε is a parameter that depends from frequency of acustic wave, as a suitable tool to describe the "focal area", because it transforms the compact region

$$B_0 = \{z = u + iv \in B \cup FrB : v \geq 0\}$$

onto the compact region

$$A_\varepsilon = \{z = u + iv : \varepsilon \leq |z| \leq 1, v \leq 0\},$$

which reproduces the sound level map for acoustic frequency associated to the parameter ε (Figure 3). Finally, the "focal zone" is the compact region

$$\{z = u + iv : |z| \leq \varepsilon, v \leq 0\}.$$

Consequently, the corresponding "focal area" of convergent phenomena is $\pi\varepsilon^2/2$. This method is valid for any convergent lense. In particular, for a biconvex lense (see Figure 1 above).

In Figure 4 we present the tendency of the parameter ε with respect to the acoustic frequency, which has been obtained from experimental data.

§3. An abstract approach to focal phenomena

In this section we show that a certain structure of Topological Algebra, namely normed monoids, provides an appropriate setting to describe the general convergent phenomena.

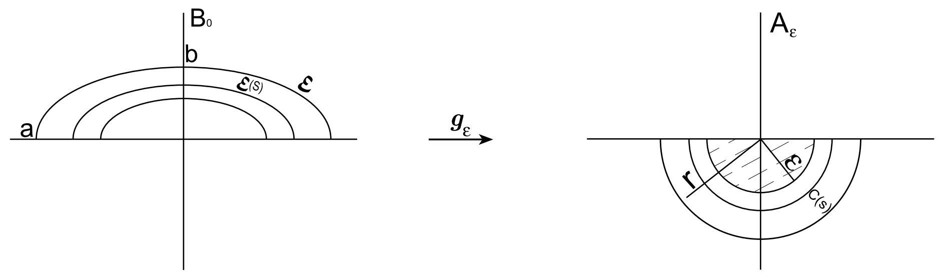

Figure 3:

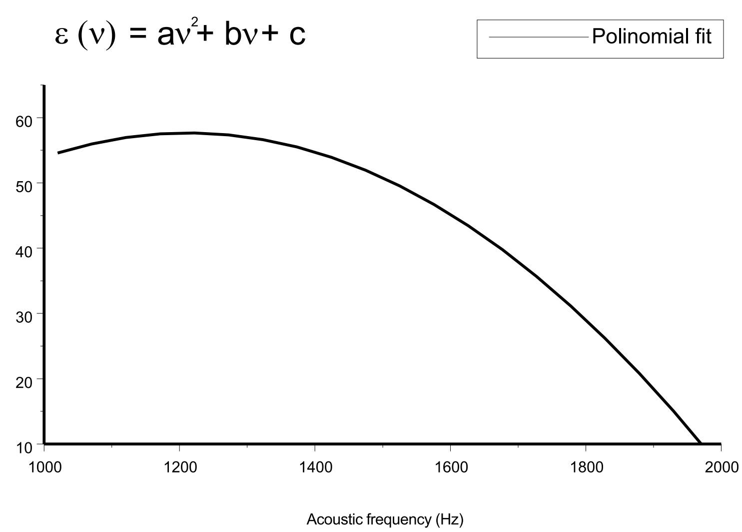

Figure 4:

The letter $\mathbb{R}^+$ will denote the set of nonnegative real numbers. In order to fix the terminology let us recall a few concepts.

A *monoid is* a semigroup $(X,+)$ with neutral element e. An *homomorphism* from a monoid $(X,+)$ to a monoid $(Y,\otimes)$ is a function $f : X \to Y$ such that $f(x+y) = f(x) \otimes f(y)$.

Let us recall ([7]) that a *norm* on a monoid $(X,+)$ is a function $p : X \to \mathbb{R}^+$ such that (i) $p(x) = e \Leftrightarrow x = e$, and (ii) $p(x+y) \leq p(x) + p(y)$.

A *normed monoid* is a pair (X,p) such that X is a monoid and p is a norm on X.

Let (X,p) and (Y,q) be two normed monoids. An *isometry* from (X,p) to (Y,q) is an homomorphism f such that $q(f(x)) = p(x)$ for all $x \in X$. (X,p) and (Y,q) are called *isometric* if there is a bijective isometry f from (X,p) onto (Y,q).

Although the application of our abstract approach to the method given in the above section can be made in the realm of metric spaces, we will construct this theory in the more general setting of quasi-metric spaces, because it provides a suitable context to explain some processes that appear in some fields of Theoretical Computer Science; they will be not discussed here.

Our main reference to quasi-metric spaces is [3].

A *quasi-metric* on a set X is a nonnegative real-valued function d defined on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x,y) = 0$ if and only if $x = y$.

(ii) $d(x,y) \leq d(x,z) + d(y,z)$.

A *quasi-metric space* is a pair (X,d) such that X is a (nonempty) set and d is a quasi-metric on X.

Let $(X,\ d)$ be a quasi-metric space such that d satisfies the following condition:

(i) Given $r \in \mathbb{R}^+$ and $x \in X$, exists $y \in X$ such that $d(x,y) = r$.

Let a, b two elements of $\mathbb{R}^+$ such that $a > b > 0$ and let $r = [\frac{a+b}{a-b}]^{1/2}$.

We define the function $a : [0,r] \to [(a^2-b^2)^{1/2}, +\infty]$ by

$$a(s) = a_s := \frac{1}{2}[(a-b)s + \frac{(a+b)}{s}].$$

where we assume that $a(0) = +\infty$.

Consider $F_1 \in X$ and $(a^2-b^2)^{1/2} \in \mathbb{R}^+$. By condition (i), there exists $F_2 \in X$ such that

$$d(F_1, F_2) = (a^2-b^2)^{1/2}.$$

On the other hand, let

$$C_s = \{x \in X : d(0,x) = s\}$$

and

$$\xi_s = \{x \in X : d(F_1,x) + d(F_2,x) = 2a_s\},$$

where s, $a_s \in \mathbb{R}^+$. If $s > 0$ the set ξ_s will be called the *ellipse* with focus F_1 and F_2 and C_s will be called the *circle* with center 0 and radius s.

Set $\mathcal{C}(r) = \{C_s : s \in [0,r]\}$. For each C_s, C_t in $\mathcal{C}(r)$, put $C_s + C_t = C_{s\wedge t}$. It is easy to see that the addition operation $+$ is well-defined on $\mathcal{C}(r)$. Now set $\Gamma(r) = \{\xi_s : s \in [0,r]\}$. For each ξ_s, ξ_t in $\Gamma(r)$, put $\xi_s + \xi_t = \xi_{s\wedge t}$. It easy to see that the addition operation $+$ is well-defined on $\Gamma(r)$.

Clearly $(\mathcal{C}(r), +)$ and $(\Gamma(r), +)$ are Abelian monoid with neutral element C_r and ξ_r, respectively.

Now we define a function $p : \mathcal{C}(r) \to \mathbb{R}^+$ by $p(C_s) = r - s$.

Proposition 1. *p is a norm on $\mathcal{C}(r)$.*

Proof. Suppose that $p(C_s) = 0$, then $r - s = 0$. Thus $r = s$. Therefore $p(C_s) = 0$ if only if $C_s = C_r$.

On the other hand,

$$p(C_s + C_t) = p(C_{s\wedge t}) = r - (s \wedge t) \leq (r - s) + (r - t) = p(C_s) + p(C_t)$$

for all $t, s \in [0,r]$. □

Let us recall that a *quasi-metric Abelian monoid* ([7]) is a pair (X, d) such that X is an Abelian monoid and d is a subinvariant quasi-metric on X, i.e. $d(x+z, y+z) \leq d(x,y)$ for all $x, y, z \in X$.

Define d_p on $\mathcal{C}(r) \times \mathcal{C}(r)$ as follows:

$$d_p(C_s, C_t) = \begin{cases} \min\{p(C_h) : C_t = C_s + C_h\} \wedge r & \text{if } C_s \in \mathcal{C}(r) \text{ and } C_t \in C_s + \mathcal{C}(r) \\ r & \text{if } C_s \in \mathcal{C}(r) \text{ and } C_t \notin C_s + \mathcal{C}(r) \end{cases}.$$

Thus

$$d_p(C_s, C_t) = \begin{cases} p(C_t) & \text{if } t < s \\ r & \text{if } t > s \\ 0 & \text{if } t = s \end{cases}.$$

From the above definition and Proposition 1 in [7], we deduce that $(\mathcal{C}(r), d_p)$ is a quasi-metric Abelian monoid.

Next we define a function $q : \Gamma(r) \to \mathbb{R}^+$ by $q(\xi_s) = r - s$.

Proposition 2. *q is a norm on $\Gamma(r)$.*

Proof. Suppose that $q(\xi_s) = 0$, then $r - s = 0$. Thus $r = s$. Therefore $q(\xi_s) = 0$ if only if $\xi_s = \xi_r$.

On the other hand,

$$q(\xi_s + \xi_t) = q(\xi_{s\wedge t}) = r - (s \wedge t) \leq (r - s) + (r - t) = q(\xi_s) + q(\xi_t)$$

for all $t, s \in [0,r]$. □

Since, by Proposition 2 and Proposition 1 in [7], q induces a quasi-metric d_q on $\Gamma(r)$, which is defined as follows:

$$d_q(\xi_s, \xi_t) = \begin{cases} \min\{q(\xi_h) : \xi_t = \xi_s + \xi_h\} \wedge r & \text{if } \xi_s \in \Gamma(r) \text{ and } \xi_t \in \xi_s + \Gamma(r) \\ r & \text{if } \xi_s \in \Gamma(r) \text{ and } \xi_t \notin \xi_s + \Gamma(r) \end{cases}.$$

Then

$$d_q(\xi_s, \xi_t) = \begin{cases} q(\xi_t) & \text{if } t < s \\ r & \text{if } t > s \\ 0 & \text{if } t = s. \end{cases}$$

Now we define $\Phi : (\mathcal{C}(r), p) \to (\Gamma(r), q)$ by $\Phi(C_s) = \xi_s$ for all $s \in [0, r]$.

The following result shows that $(C(r), p)$ and $(\Gamma(r), q)$ are isometric normed monoid.

Proposition 3. *Φ is a bijective isometry.*

Proof. Let $C_s \in \mathcal{C}(r)$, then

$$q(\Phi(C_s)) = q(\xi_s) = r - s = p(C_s).$$

Hence $q(\Phi(C_s)) = p(C_s)$ for all $s \in [0, r]$.

Let C_s, C_t be two elements of $\mathcal{C}(r)$, then

$$\Phi(C_s + C_t) = \Phi(C_{s\wedge t}) = \xi_{s\wedge t} = \xi_s + \xi_t = \Phi(C_s) + \Phi(C_t).$$

So Φ is a homomorphism on $\mathcal{C}(r)$.

Next we show that Φ is bijective on $\mathcal{C}(r)$. Indeed, let $C_s, C_t \in \mathcal{C}(r)$ such that $\Phi(C_s) = \Phi(C_t)$, then $\xi_s = \xi_t$. Consequently $s = t$ and $C_s = C_t$.

Clearly, Φ is a surjective function on $\mathcal{C}(r)$. □

Remark. Note that if X is the Euclidean plane $\mathbb{R}^2$, then for a fixed $r > 1$ and each $s \in [1, r]$, C_s and ξ_s are the circle $|z| = s$ and the ellipse $\frac{u^2}{a_s^2} + \frac{v^2}{b_s^2} = 1$, respectively, where a_s and b_s are the semiaxis defined in Section 2. Therefore the family of circles and ellipses of the preceding section constitute a particular case of the construction made here.

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