Asymptotic behaviour of the weighted trace of the Schrödinger operator

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Abstract

This paper is devoted to the study of the asymptotic behaviour of the weighted trace of the Schrödinger operator with operator coefficient.

§1. Introduction

Let H be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. In the Hilbert space $H_1 = L_2\left(\mathbb{R}_n^+; H\right)$ we consider the differential operator L generated by the expression

(1.1)
$$-\Delta u + Q(x)u, \quad x \in \mathbb{R}_n^+$$

and the boundary condition

(1.2)
$$u(x_1, x_2, \ldots, x_n)\big|_{x_n=0} = 0,$$

where we denote by H_1 the Hilbert space of H valued square integrable functions with the scalar product

(1.3)
$$(f, g) = \int\limits_{\mathbb{R}_n^+} (f(x), g(x))_H \, dx, \quad f(x), g(x) \in H_1.$$

Denote by $\mathbb{R}_n^+$ the set of points $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n$, $x_n \geq 0$. Here $Q(x)$ is a self-adjoint operator acting on H. We shall assume that $Q(x)$ satisfies the following conditions:

(1^0) The domain of definition $D(Q(x)) = D$ of $Q(x)$ is independent of x and $\overline{D} = H$ and $Q^*(x) = Q(x) \geq I$, where I is the identity operator on H. It is also assumed that $Q^{-1}(x)$ is completely continuous operator on H for each $x \in \mathbb{R}_n^+$.

(2^0) $Q^{-1}(x) \in \sigma_1$, $\forall x \in \mathbb{R}_n^+$ and $\int_{\mathbb{R}_n^+} \left\|Q^{-k}(x)\right\|_1 dx < \infty$ for any $k > 0$, where σ_1 is the space of trace class operators and $\|\,.\,\|_1$ is the norm in σ_1 [5].

(3^0) $\left\| Q^{\gamma}(s) e^{-ctQ(s)} \right\|_1 \leq C \left\| Q^{\gamma}(x) e^{-f(c)tQ(x)} \right\|_1,$

$$\left\| Q^{\beta}(x) Q^{-\beta}(s) \right\| \leq C, |x - s| \leq 1,$$

where $\gamma = 0, \beta$ $f(c) > 0$ and β is a real number.

(4^0) $\left\| Q^{\gamma}(x) [Q(s) - Q(x)] Q^{-\gamma - b}(s) \right\| < C |x - s|,$

$$|x - s| \leq 1 \quad , \quad 0 < b < 3/2.$$

(5^0) $\int\limits_{\mathbb{R}_n^+} Tr \left[Q^{\gamma}(x) e^{-qtQ(x)} \right] dx = O(1) \int\limits_{\mathbb{R}_n^+} Tr \left[Q^{\gamma}(x) e^{-tQ(x)} \right] dx, \quad t \to 0,$ where

q is a positive constant.

(6^0) Let $\alpha_1(x) \leq \alpha_2(x) \leq \ldots \leq \alpha_n(x) \leq \ldots$ be eigenvalues of the operator $Q(x)$.
Denote

$$\rho_{\beta}(\lambda) = \frac{1}{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right)} \sum_j \int\limits_{\alpha_j(x) \leq \lambda} \alpha_j^{\beta}(x) [\lambda - \alpha_j(x)]^{n/2} dx,$$

and assume that $\lim\limits_{\lambda \to \infty} \rho_{\beta}(\lambda) = \infty$ and for large λ, the condition $\lambda \rho_{\beta}'(\lambda) < a \rho_{\beta}(\lambda)$is
satisfied for any positive number a. In this paper we used one and the same symbol
C in order to denote any constant which may differ from each other.

Let $\varphi_k(x)' s$ be functions that have compact support, continuous second order
derivative and satisfying $\varphi_k(x)|_{x_n = 0} = 0$, $f_k \in D$. Let us denote by D' the summation
$\sum\limits_{k=1}^{m} \varphi_k(x) f_k$. In the Hilbert space H_1 we consider the operator L' generated by
$-\Delta u + Q(x) u$with the domain of definition D'. This operator is symmetric and
positive definite. The closure L of the operator L' is self adjoint.

The spectrum of the operator L generated by the expression (1.1) and by the
boundary condition (1.2) consists of eigenvalues. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ be
the eigenvalues of this operator, and $\psi_1(x), \psi_2(x), \ldots, \psi_n(x), \ldots$ the corresponding
complete vector-valued orthonormal eigenfunctions.

The aim of this paper is to find asymptotic behaviour of $N_{\beta}(\lambda)$ as $\lambda \to \infty$. We
define $N_{\beta}(\lambda)$ as follows

$$(1.4) \qquad N_{\beta}(\lambda) = \sum_{\lambda_k < \lambda} \int\limits_{\mathbb{R}_n^+} \left(Q^{\beta}(x) \psi_k(x), \psi_k(x) \right) dx,$$

and it is called weighted-trace of the operator L, where β is a real constant.

§2. Asymptotic behaviour of Green function

Let us consider the following boundary value problem:

$$(2.1) \qquad \frac{\partial u}{\partial t} = \Delta u - Q(x) u, \quad x \in \mathbb{R}_n^+, \quad t > 0,$$

$$(2.2) \qquad u(x, 0) = \Psi(x), \quad x \in \mathbb{R}_n^+, \quad \Psi(x) \in H_1$$

(2.3) $$u\left(t,x\right)\big|_{x_n=0}=0.$$

Let $G(t,x,s)$ be the Green function of this problem. The Green function for the problem (2.1)-(2.3) corresponding $Q(x)=0$ is

(2.4) $$G_0\left(t,x,s\right)=\left(2\sqrt{\pi t}\right)^{-n}\left[e^{-\frac{|x-s|^2}{4t}}-e^{-\frac{|x-s'|^2}{4t}}\right]I,$$

where $s=(s_1,s_2,s_3)$ and $s'=(s_1,s_2,-s_3)$, $(s_3>0)$. We shall seek, using the method of E.E.Levi [4], the Green function of (1.1)-(1.3) in the form

(2.5) $$G\left(t,x,s\right)=e^{-tQ(s)}G_0\left(t,x,s\right)+$$

$$+\int_0^t d\tau\int_{\mathbb{R}_n^+}G_0\left(t-\tau,x,\xi\right)e^{-(t-\tau)Q(\xi)}\varphi\left(\tau,\xi,s\right)d\xi,$$

where $\varphi\left(t,x,s\right)$ is a operator valued function to be determined. In order that G(t,x,s) shall be a solution of (2.1) it is necassery that $\varphi\left(t,x,s\right)$ shall be a solution of the integral equation

(2.6) $$\varphi\left(t,x,s\right)=K\left(t,x,s\right)+\int_0^t d\tau\int_{\mathbb{R}_n^+}K\left(t-\tau,x,\xi\right)\varphi\left(\tau,\xi,s\right)d\xi,$$

where

(2.7) $$K\left(t,x,s\right)=\left[Q\left(s\right)-Q\left(x\right)\right]e^{-tQ(s)}G_0\left(t,x,s\right)$$

If we solve the equations by the method of successive approximation with the initial approximation $K_1\left(t,x,s\right)=K\left(t,x,s\right)$, the we obtain the iteration formula:

(2.8) $$K_{n+1}\left(t,x,s\right)=\int_0^t d\tau\int_{\mathbb{R}_n^+}K\left(t-\tau,x,\xi\right)K_n\left(\tau,\xi,s\right)d\xi,\quad n=1,2,\dots$$

If the conditions (1^0)-(4^0) are satisfied then it can be shown that the series $\sum_{k=1}^{\infty}\left\|K_n\left(t,x,s\right)\right\|_1$ is uniformly convergent, $t>0$, $x,s\in\mathbb{R}_n^+$. Therefore $\varphi\left(t,x,s\right)=\sum_{n=1}^{\infty}K_n\left(t,x,s\right)$ is a solution of the integral equation (2.6). Finding an estimate for $K_n\left(t,x,s\right)$ and $Q^\gamma\left(x\right)\varphi\left(t,x,s\right)$ we conclude our results with the following theorem.

Theorem 1. *If $Q(x)$ satisfies the conditions (1^0)-(4^0), then the following asymptotic formula holds*

$$Q^\gamma\left(x\right)G\left(t,x,s\right)=Q^\gamma\left(x\right)e^{-tQ(s)}G_0\left(t,x,s\right)+$$

$$+O\left(1\right)t^{-\frac{n}{2}+\delta}e^{-\frac{|x-s|^2}{t}}\left[\left\|Q^{-k}\left(s\right)\right\|_1+\left\|Q^\gamma\left(s\right)e^{-ctQ(s)}\right\|_1\right],$$

where δ is a positive number.

§3. Asymptotic behaviour of $N_\beta(\lambda)$

Since the Green function of (2.1)-(2.2) is $G(t, x, s)$, then

$$(3.1) \qquad u(t, x) = \int_{\mathbb{R}_n^+} G(t, x, s) \, \Psi(s) \, ds.$$

On the other hand, since $u(t, x) = e^{-tL} \psi(x)$, then we may write

$$(3.2) \qquad e^{-tL} \Psi(x) = \int_{\mathbb{R}_n^+} G(t, x, s) \, \Psi(s) \, ds.$$

Therefore e^{-tL} is an integral operator with the kernel $G(t, x, s)$. Using the conditions (5^0) and from the Theorem 1 we have

$$(3.3) \qquad \int_{\mathbb{R}_n^+} Tr\left[Q^\gamma G(t, x, x)\right] dx \sim \left(2\sqrt{\pi t}\right)^{-n} \int_{\mathbb{R}_n^+} Tr\left[Q^\gamma e^{-tQ(x)}\right] dx.$$

Considering (3.3) we obtain

$$\int_0^\infty TrG(t, x, x) \, dx < \infty.$$

Since e^{-tL} is a positive operator we may write [1]

$$(3.4) \qquad Tre^{-tL} = \int_{\mathbb{R}_n^+} TrG(t, x, x) \, dx.$$

This implies that e^{-tL} is the trace operator in $H_1 = L_2\left(\mathbb{R}_n^+; H\right)$ and therefore the spectrum of the operator L is discrete: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$, $\lim_{n\to\infty} \lambda_n = \infty$. It can be shown that $Q^\beta e^{-tL}$ is a trace operator in H_1 and that

$$(3.5) \qquad Tr\left[Q^\beta e^{-tL}\right] = \int_{\mathbb{R}_n^+} Tr\left[Q^\beta G(t, x, x)\right] dx = \int_0^\infty e^{-\lambda t} N_\beta(\lambda)$$

From (3.3) and (3.5) we have

$$(3.6) \qquad \int_0^\infty e^{-\lambda t} N_\beta(\lambda) \sim \left(2\sqrt{\pi t}\right)^{-n} \int_{\mathbb{R}_n^+} Tr\left[Q^\beta e^{-tQ(x)}\right] dx.$$

As in [2] we write

$$(3.7) \qquad \left(2\sqrt{\pi t}\right)^{-n} \int\limits_{\mathbb{R}_n^+} \alpha_j^\beta\left(x\right) e^{-t\alpha_j(x)} dx = \int\limits_0^\infty e^{-\lambda t} d\rho_j\left(\lambda\right),$$

where $\rho_j\left(\lambda\right) = \frac{1}{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right)} \sum\limits_j \int\limits_{\alpha_j(x)\leq\lambda} \alpha_j^\beta\left(x\right) \left[\lambda - \alpha_j\left(x\right)\right]^{n/2} dx.$

Using the inequality above we get

$$(3.8) \qquad \left(2\sqrt{\pi t}\right)^{-n} \int\limits_{\mathbb{R}_n^+} Tr\left[Q^\beta e^{-tQ(x)}\right] dx = \int\limits_0^\infty e^{-\lambda t} d\rho_\beta\left(\lambda\right).$$

Using (3.6) and (3.8) we obtain

$$(3.9) \qquad \int\limits_0^\infty e^{-\lambda t} N_\beta\left(\lambda\right) = \int\limits_0^\infty e^{-\lambda t} d\rho_\beta\left(\lambda\right) \text{ as } t \to 0.$$

Using the condition (6^0) and application of a Tauberian theorem of Korenblum, B.I., (see [3]) to (3.9) we obtain that

$$(3.10) \qquad N_\beta\left(\lambda\right) \sim \frac{1}{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right)} \sum\limits_j \int\limits_{\alpha_j(x)\leq\lambda} \alpha_j^\beta\left(x\right) \left[\lambda - \alpha_j\left(x\right)\right]^{n/2} dx$$

as $\lambda \to \infty$.

Theorem2. *If $Q(x)$ satisfies the conditions (1^0)-(6^0), then the asymptotic formula (3.10) holds for the weighted trace $N_\beta\left(\lambda\right)$ of the operator L.*

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