

A Class of Nonisothermal Variable Thickness Rotating Disk Problems Solved by Hypergeometric Functions

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Dedicated to Professor M. Ruşen Geçit, Chairman of the Department of Engineering Sciences at METU, on the occasion of his 50th birthday.

Abstract

Exact solutions for nonisothermal variable thickness rotating disks represented by different thickness profiles are obtained under plane stress assumption. The solutions are based on Tresca's yield criterion, its associated flow rule and linear strain hardening material behavior. Five different plastic regimes governed by different mathematical forms of the yield criterion are considered for each thickness profile. A displacement formulation is used and the resulting differential equations for the elastic and plastic regions are solved in terms of hypergeometric functions by the introduction of appropriate transformations.

Key words: Thermoelastoplasticity, Tresca's criterion, Variable thickness, Strain hardening, Rotating disks.

Introduction

Theoretical investigation of the stresses in structures like rotating disks, annular cooling fins and shrink fits is an important topic due to a large number of applications in mechanical and structural engineering (Timoshenko and Goodier, 1970; Rees, 1990; Uğural and Fenster, 1995). For this reason, interest in this topic has never ceased. It is known that (Uğural and Fenster, 1995; Eraslan and Orcan, 2002a) a more efficient and economical design can be achieved by allowing not only variation in the thickness of the disk, but also the deformation into the plastic range.

The aim of this paper is to present the analytical solutions for nonisothermal variable thickness rotating disks having different thickness profiles of engineering interest. Elastic-plastic solutions are based on Tresca's yield criterion and its associated flow rule. Five different plastic regimes governed by different forms of the yield condition are considered, assuming linear strain hardening material behavior. These plastic regimes are commonly found in cooling fins, rotating disks and shrink fits. A state of

plane stress and infinitesimal deformations are presumed. The plane stress assumption is valid as long as the thickness of the disk is small compared to its diameter (Timoshenko and Goodier, 1970). In the solutions presented here the expressions do not contain the maximum thickness h_0 ; however, it must be sufficiently small to justify the plane stress assumption.

The analytical solutions presented in this paper can be applied to various important problems in engineering. They can be used (1) to analyze deformations of rotating solid and annular disks of variable thickness, (2) to solve some combined material and topological optimization problems for variable thickness rotating machinery, (3) to analyze the deformations of annular cooling fins of variable thickness with and without rotation, (4) to solve steady or transient shrink-fit problems including a variable thickness hub, (5) to solve similar profiles using the solution procedure presented herein.

The paper is organized as follows. First, the stress states that occur in cooling fins, rotating disks and shrink fits are discussed. Later, the stress-

displacement relations for elastic and plastic deformations are derived. These relations are necessary to complete the solutions. In the following sections, the solutions for different profiles represented by 1 or 2 geometric parameters are given. Each solution begins with the description of the profile and related literature and is followed by elastic and plastic solutions. Alternate solutions are given if one solution fails to be finite at the axis of the disk. The applications of the transformation techniques are explained in detail for some representative solutions in Appendices A, B and C.

Stress states

In a recent investigation, Eraslan and Akis (2003) proposed a heat transfer model with a variable heat transfer coefficient to explain the cooling of a centrally heated thin annular fin subjected to rotation. This model involves the solution of the differential equation

$$\frac{d^2T}{dr^2} + \frac{1}{r} \frac{dT}{dr} - \left[\frac{2(A + B\omega r)}{kh_0} \right] (T - T_0) = 0 \quad (1)$$

where $T(r)$ is the temperature in the fin, T_0 the ambient temperature, ω the angular velocity, k the thermal conductivity of the material, h_0 the thickness of the disk, and A and B are parameters. For a given angular velocity ω , the temperature distribution in the fin was obtained by the analytical solution of (1). The shapes of the temperature profiles suggest a simple fit of the form

$$T(r) = T_C \left[1 - \bar{A} \sqrt{\frac{\rho}{\sigma_0}} \omega r \right] \left[1 - \bar{B} \left(\frac{r}{b} \right) + \bar{C} \left(\frac{r}{b} \right)^2 \right] \quad (2)$$

in which T_C represents the centerline temperature, ρ the mass density, σ_0 the yield limit and b the radius of the disk, and \bar{A} , \bar{B} and \bar{C} are the fit coefficients. Using the data produced by the analytical solution of Eq. (1) the fit coefficients are determined to be $\bar{A} = 0.229$, $\bar{B} = 1.19$ and $\bar{C} = 0.591$. The dimensionless temperature profiles $\theta = T/T_C$ obtained by the use of Eq. (2) at different dimensionless angular velocities $\Omega^2 = \omega^2 b^2 \rho / \sigma_0$ are shown in Figure 1. Lower edge temperatures and sharper temperature gradients are obtained as the angular velocity is increased. On the other hand, the expressions for the elastic stresses σ_r and σ_θ , and displacement u for the nonisothermal thin disk are given by (Eraslan and Akis, 2003)

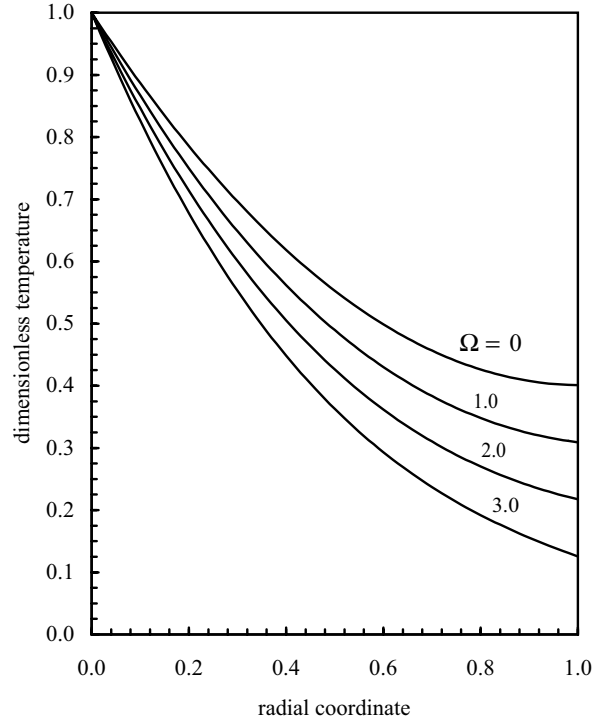


Figure 1. Temperature distributions in a centrally heated rotating disk at different rotation speeds.

$$\sigma_r = \frac{C_1}{r^2} + C_2 - \frac{1}{8}(3 + \nu)\rho\omega^2 r^2 - \frac{1}{2}E\alpha T + \frac{E\alpha}{2r^2} \int_s^r T'(\xi) \xi^2 d\xi \quad (3)$$

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 - \frac{1}{8}(1 + 3\nu)\rho\omega^2 r^2 - \frac{1}{2}E\alpha T - \frac{E\alpha}{2r^2} \int_s^r T'(\xi) \xi^2 d\xi \quad (4)$$

$$u = \frac{1}{E} \left[-\frac{(1 + \nu)C_1}{r} + (1 - \nu)C_2 r - \frac{(1 - \nu^2)\rho\omega^2 r^3}{8} \right] + \frac{1}{2}\alpha(1 + \nu)rT - \frac{\alpha(1 + \nu)}{2r} \int_s^r T'(\xi) \xi^2 d\xi \quad (5)$$

In these equations C_i is an arbitrary integration constant, ν the Poisson's ratio, E the modulus of elasticity, and α the coefficient of thermal expansion. The lower limit s in the integrals denotes the inner

boundary of the elastic region. It should be noted that

$$\int_x^y T'(r) r^2 dr = \frac{\bar{A}T_C\omega}{30b^2} \sqrt{\frac{\rho}{\sigma_0}} [10b^2(x^3 - y^3) - 15b\bar{B}(x^4 - y^4) + 18\bar{C}(x^5 - y^5)] + \frac{T_C}{b} \left[\frac{\bar{B}(x^3 - y^3)}{3} - \frac{\bar{C}(x^4 - y^4)}{2b} \right] \quad (6)$$

The purely elastic deformation behavior of the non-isothermal rotating disks can be evaluated by the use of Eqs. (3)-(6) and different boundary conditions. For a solid disk ($s = 0$) C_1 has to vanish in order to obtain finite stresses at the axis of the disk. If the edge of the disk, $r = b$, is free of any traction we have $\sigma_r(b) = 0$. This condition results in

$$C_2 = \frac{1}{8}(3 + \nu)\rho\omega^2 b^2 + \frac{1}{2}E\alpha T(b) - \frac{E\alpha}{2b^2} \int_0^b T'(r) r^2 dr \quad (7)$$

Note also that for a solid disk

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_0^r T'(\xi) \xi^2 d\xi = \lim_{r \rightarrow 0} \frac{dT}{dr} r^2 = 0 \quad (8)$$

The nondimensional stresses, $\bar{\sigma}_j = \sigma_j/\sigma_0$, and displacement, $\bar{u} = uE/\sigma_0 b$, for isothermal and non-isothermal solid disks at their corresponding elastic limit angular velocities, $\Omega = \Omega_1$, are calculated and plotted in Figure 2. In this figure, dashed lines represent the isothermal disk. Yielding commences in the isothermal disk at the axis with the yield condition $\sigma_y = \sigma_r = \sigma_\theta$. Hence, 2 adjacent plastic regions, one with a corner regime $\sigma_y = \sigma_r = \sigma_\theta$ and the other with a side regime $\sigma_y = \sigma_\theta$, develop simultaneously at the axis of the disk and propagate toward the edge with increasing angular velocities. In contrast, for the nonisothermal disk yielding begins somewhere inside the disk not at the axis with the yield condition $\sigma_y = \sigma_\theta$. The plastic region formed here propagates in both radial directions for the angular velocities $\Omega \geq \Omega_1$.

For an annular disk of inner radius a having free boundaries, the boundary conditions read $\sigma_r(a) = 0$ and $\sigma_r(b) = 0$. Accordingly, the integration constants are evaluated as

$$8(b^2 - a^2)C_1 = -a^2 b^2 (b^2 - a^2)(3 + \nu)\rho\omega^2 - 4a^2 b^2 E\alpha [T(b) - T(a)] + 4a^2 E\alpha \int_a^b T'(r) r^2 dr \quad (9)$$

$$8(b^2 - a^2)C_2 = (b^4 - a^4)(3 + \nu)\rho\omega^2 + 4E\alpha [b^2 T(b) - a^2 T(a)] - 4E\alpha \int_a^b T'(r) r^2 dr \quad (10)$$

The stresses, displacement and temperature for an annular nonisothermal rotating disk with inner radius $\bar{a} = a/b = 0.2$ at the elastic limit angular velocity Ω_1 is plotted in Figure 3. As seen in this figure, yielding commences at the inner boundary, $r = a$, for rotating speeds $\Omega \geq \Omega_1$ according to the yield condition $\sigma_y = \sigma_\theta$. Moreover, for an annular disk subjected to internal pressure (shrink fit problem) the inner boundary condition becomes $\sigma_r(a) = -P$ with P being the applied pressure. In this case, the constants C_1 and C_2 are determined to be

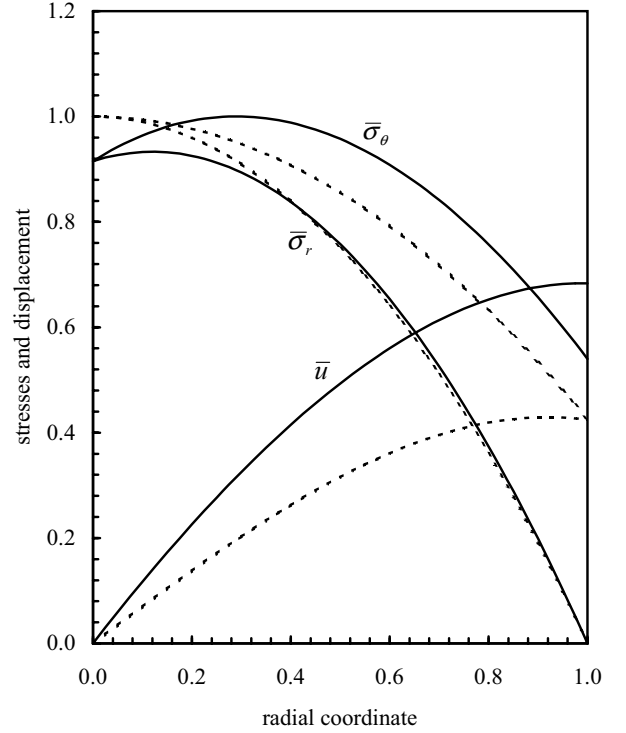


Figure 2. Stresses and displacement in a rotating isothermal (dashed lines) and nonisothermal solid disks.

$$8(b^2 - a^2)C_1 = -8a^2 b^2 P - a^2 b^2 (b^2 - a^2)(3 + \nu)\rho\omega^2 - 4a^2 b^2 E\alpha [T(b) - T(a)] + 4a^2 E\alpha \int_a^b T'(r) r^2 dr \quad (11)$$

$$8(b^2 - a^2)C_2 = 8a^2P + (b^4 - a^4)(3 + \nu)\rho\omega^2 + 4E\alpha[b^2T(b) - a^2T(a)] - 4E\alpha \int_a^b T'(r)r^2 dr \quad (12)$$

Using $\bar{a} = 0.2$ and $\bar{P} = P/\sigma_0 = 0.25$ the elastic limit angular velocity for a nonisothermal annular disk is calculated and the corresponding stresses and displacement are plotted in Figure 4. At the inner surface, the principal stress state is $\sigma_\theta > 0 > \sigma_r$. Hence, yielding will begin at this location with the yield condition $\sigma_y = \sigma_\theta - \sigma_r$.

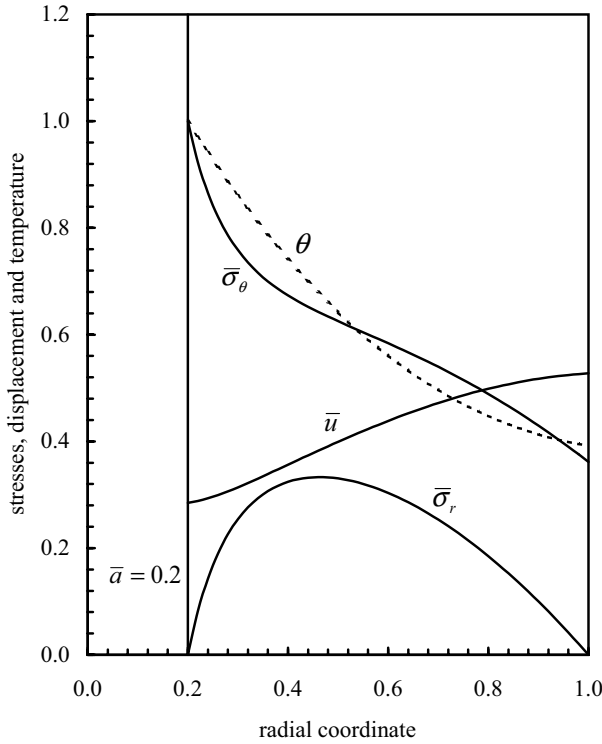


Figure 3. Stresses and displacement in a rotating non-isothermal annular disk with free boundaries.

If the annular disk is mounted on a rigid shaft, the boundary conditions become $u(a) = 0$ and $\sigma_r(b) = 0$. The constants C_1 and C_2 are thus

$$\left[\frac{a^2(1 - \nu) + b^2(1 + \nu)}{a^2b^2} \right] C_1 = -\frac{1}{8}(1 - \nu)[a^2(1 + \nu) - b^2(3 + \nu)]\rho\omega^2 + \frac{1}{2}E\alpha(1 + \nu)T(a) + \frac{1}{2}E\alpha(1 - \nu)T(b) - \frac{E\alpha(1 - \nu)}{2b^2} \int_a^b T'(r)r^2 dr \quad (13)$$

$$\left[\frac{8a^2(1 - \nu) + 8b^2(1 + \nu)}{(1 + \nu)} \right] C_2 = [a^4(1 - \nu) + b^4(3 + \nu)]\rho\omega^2 - 4a^2E\alpha T(a) + 4b^2E\alpha T(b) - 4E\alpha \int_a^b T'(r)r^2 dr \quad (14)$$

The stresses and displacement in this disk at the elastic limit heat load at a relatively low rotation speed $\Omega = 0.5$ are shown in Figure 5. The largest difference between the stresses occurs at the inner surface where $\sigma_r > 0 > \sigma_\theta$. If the heat load is further increased, yielding begins according to the yield condition $\sigma_y = \sigma_r - \sigma_\theta$. The principal stress state for an isothermal rotating disk mounted on a rigid shaft is shown in Figure 6. As seen in this figure, the disk will yield at the shaft-disk interface with an increase in Ω according to $\sigma_y = \sigma_r$.

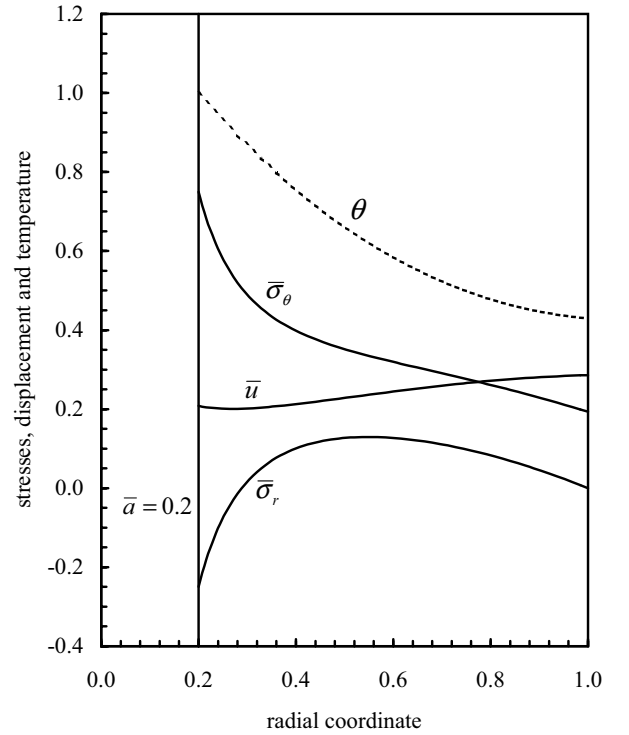


Figure 4. Stresses and displacement in a rotating non-isothermal annular disk subjected to internal pressure.

As seen in the above analysis, 5 different plastic regimes governed by different mathematical forms of Tresca's yield criterion may occur in cooling fins, rotating isothermal and nonisothermal disks and

shrink fits. Therefore to present a complete set of solutions, 5 different plastic regions with the yield criteria: 1. $\sigma_y = \sigma_r = \sigma_\theta$, 2. $\sigma_y = \sigma_\theta$, 3. $\sigma_y = \sigma_r$, 4. $\sigma_y = \sigma_r - \sigma_\theta$ and 5. $\sigma_y = \sigma_\theta - \sigma_r$ have to be taken into consideration.

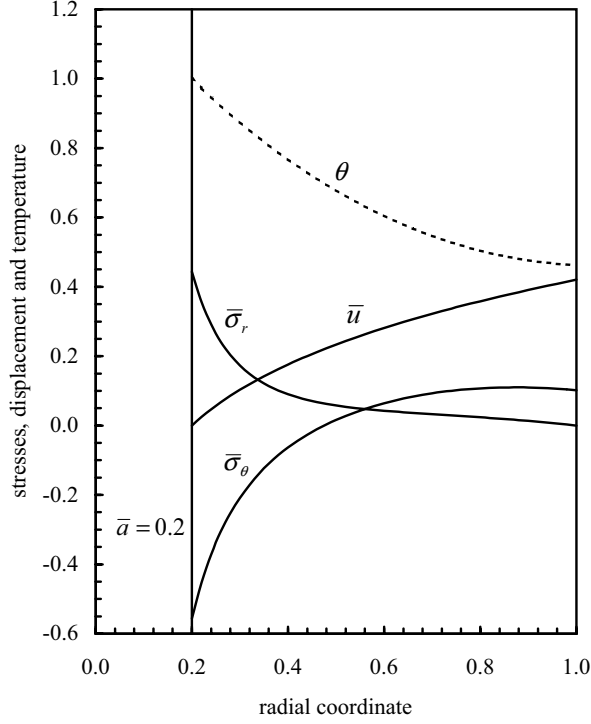


Figure 5. Stresses and displacement in a rotating non-isothermal annular disk mounted on a rigid shaft.

Stress-Displacement Relations

Elastic region

Total strains

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + \alpha T \quad (15)$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + \alpha T \quad (16)$$

are substituted in strain-displacement relations $\epsilon_r = u'$ and $\epsilon_\theta = u/r$ and solved for the stresses to yield

$$\sigma_r = \frac{E}{1-\nu^2} \left[\frac{\nu u}{r} + u' \right] - \frac{E\alpha T}{1-\nu} \quad (17)$$

$$\sigma_\theta = \frac{E}{1-\nu^2} \left[\frac{u}{r} + \nu u' \right] - \frac{E\alpha T}{1-\nu} \quad (18)$$

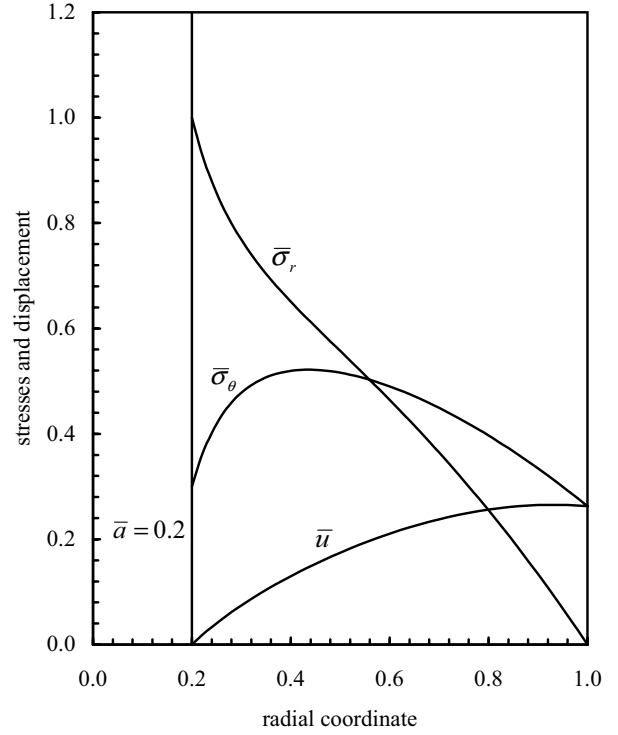


Figure 6. Stresses and displacement in a rotating annular disk mounted on a rigid shaft.

where ϵ_j is the normal strain component and T the radial temperature gradient and a prime denotes differentiation with respect to the radial coordinate r . The radial and circumferential stresses are plugged in the equation of motion

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta = -h\rho\omega^2 r^2 \quad (19)$$

to obtain the governing elastic equation in terms of the radial displacement. In the equation of motion (19), h represents the disk thickness function. The general solution of the radial displacement equation is obtained in the form

$$u(r) = C_1 P(r) + C_2 Q(r) + R(r) \quad (20)$$

in which C_i is an arbitrary integration constant, P and Q the 2 homogeneous solutions of the differential equation and R the particular integral solution. R is determined by the method of variation of parameters. It is assumed to be of the form

$$R(r) = \hat{U}_1(r)P(r) + \hat{U}_2(r)Q(r) \quad (21)$$

where

$$\hat{U}_1(r) = - \int_0^r \frac{Q(\lambda)f(\lambda)}{W_{ro}(\lambda)} d\lambda \quad (22)$$

$$\hat{U}_2(r) = \int_0^r \frac{P(\lambda)f(\lambda)}{W_{ro}(\lambda)} d\lambda \quad (23)$$

$$W_{ro}(r) = P(r)Q'(r) - Q(r)P'(r) \quad (24)$$

and $f(r)$ represents the nonhomogeneous term of the differential equation. Since $P(r)$, $Q(r)$ and $W_{ro}(r)$ are in general polynomials, the integrals (22) and (23) may be evaluated analytically by expanding the integrands into Taylor series. If these expansions are not possible because of the product $f(r)$, accurate evaluations may nevertheless be accomplished by the application of the Gaussian Quadrature rule of integration (Uğural and Fenster, 1995). With the form (20) of the radial displacement, the stresses become

$$\sigma_r = \frac{E}{1-\nu^2} \left[C_1 \left(\frac{\nu P}{r} + P' \right) + C_2 \left(\frac{\nu Q}{r} + Q' \right) + \frac{\nu R}{r} + R' \right] - \frac{E\alpha T}{1-\nu} \quad (25)$$

$$\sigma_\theta = \frac{E}{1-\nu^2} \left[C_1 \left(\frac{P}{r} + \nu P' \right) + C_2 \left(\frac{Q}{r} + \nu Q' \right) + \frac{R}{r} + \nu R' \right] - \frac{E\alpha T}{1-\nu} \quad (26)$$

Plastic region I

In this region, the principal stress state corresponds to a corner regime of Tresca's hexagon with $\sigma_r = \sigma_\theta > 0$. The yield condition is given by

$$\sigma_y = \sigma_r = \sigma_\theta \quad (27)$$

For a linear strain hardening material behavior, the yield condition has the form

$$\sigma_y = \sigma_0(1 + \eta \epsilon_{EQ}) \quad (28)$$

and the inverse relation is

$$\epsilon_{EQ} = \left[\frac{\sigma_y}{\sigma_0} - 1 \right] \frac{1}{\eta} \quad (29)$$

In the above, η represents the yield limit of the material and ϵ_{EQ} the equivalent plastic strain. Using

$\sigma_r = \sigma_\theta$ the equation of motion (19) is integrated to give

$$\sigma_r(r) = \sigma_\theta(r) = \frac{EC_3}{h(r)} - \frac{\rho\omega^2}{h(r)} \int_0^r h(\xi)\xi d\xi \quad (30)$$

Consideration of the equivalence of plastic work increment, $\sigma_\theta d\epsilon_\theta^p + \sigma_r d\epsilon_r^p = \sigma_y d\epsilon_{EQ}$, together with the yield condition (27) results in $\epsilon_{EQ} = \epsilon_\theta^p + \epsilon_r^p$. The flow rule associated with the yield condition provides $\epsilon_z^p = -(\epsilon_\theta^p + \epsilon_r^p)$. Making use of the strain-displacement relations, and decomposing the total strains into their elastic and plastic parts, one obtains

$$\frac{du}{dr} + \frac{u}{r} = -\frac{1}{\eta} + \left[\frac{1}{\eta\sigma_0} + \frac{2(1-\nu)}{E} \right] \sigma_r + 2\alpha T \quad (31)$$

and therefrom the displacement

$$u(r) = \frac{C_4}{r} - \frac{r}{2\eta} + \left[\frac{1}{\eta\sigma_0} + \frac{2(1-\nu)}{E} \right] \left[\frac{1}{r} \int_0^r \sigma_r(\xi)\xi d\xi \right] + \frac{2\alpha}{r} \int_0^r T(\xi)\xi d\xi \quad (32)$$

It should be noted that

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_0^r \sigma_r(\xi)\xi d\xi = 0 \quad (33)$$

The plastic strain components are obtained by subtracting the elastic parts from the total strains as

$$\epsilon_\theta^p = \frac{u}{r} - \frac{1-\nu}{E} \sigma_r - \alpha T \quad (34)$$

$$\epsilon_r^p = \left[\frac{1}{\eta\sigma_0} + \frac{1-\nu}{E} \right] \sigma_r - \frac{u}{r} - \frac{1}{\eta} + \alpha T \quad (35)$$

Plastic region II

In this region, stresses lie in a side regime of Tresca's hexagon with $\sigma_\theta > \sigma_r > 0$. The yield condition reads

$$\sigma_y = \sigma_\theta \quad (36)$$

The increment of plastic work gives $\epsilon_{EQ} = \epsilon_\theta^p$, and according to the flow rule associated with the yield condition (36) $\epsilon_z^p = -\epsilon_\theta^p$ and $\epsilon_r^p = 0$. Since the radial strain is purely elastic and thermal and

$$\epsilon_\theta = \left[\frac{\sigma_\theta}{\sigma_0} - 1 \right] \frac{1}{\eta} + \frac{1}{E} (\sigma_\theta - \nu\sigma_r) + \alpha T \quad (37)$$

the strain-displacement relations lead to

$$\sigma_r = \frac{(1-W^2)\nu\sigma_0}{1-W^2\nu^2} + \frac{E}{1-W^2\nu^2} \left[\frac{W^2\nu u}{r} + u' - \alpha(1+W^2\nu)T \right] \quad (38)$$

$$\sigma_\theta = \frac{(1-W^2)\sigma_0}{1-W^2\nu^2} + \frac{EW^2}{1-W^2\nu^2} \left[\frac{u}{r} + \nu u' - \alpha(1+\nu)T \right] \quad (39)$$

where $W^2 = H/(1+H)$ with H being the normalized hardening parameter defined by $H = \eta\sigma_0/E$. The governing differential equation for this region is obtained by substitution of these stresses in the equation of motion (19). The general solution is obtained in the form

$$u(r) = C_5P(r) + C_6Q(r) + R(r) \quad (40)$$

The stresses become

$$\sigma_r = \frac{(1-W^2)\nu\sigma_0}{1-W^2\nu^2} + \frac{E}{1-W^2\nu^2} \left[C_5 \left(\frac{W^2\nu P}{r} + P' \right) + C_6 \left(\frac{W^2\nu Q}{r} + Q' \right) + \frac{W^2\nu R}{r} + R' - \alpha(1+W^2\nu)T \right] \quad (41)$$

$$\sigma_\theta = \frac{(1-W^2)\sigma_0}{1-W^2\nu^2} + \frac{EW^2}{1-W^2\nu^2} \left[C_5 \left(\frac{P}{r} + \nu P' \right) + C_6 \left(\frac{Q}{r} + \nu Q' \right) + \frac{R}{r} + \nu R' - \alpha(1+\nu)T \right] \quad (42)$$

The plastic strain components are determined from

$$\epsilon_\theta^p = -\epsilon_z^p = \left[\frac{\sigma_\theta}{\sigma_0} - 1 \right] \frac{1}{\eta}, \quad \text{and} \quad \epsilon_r^p = 0 \quad (43)$$

Plastic region III

In this region, stresses lie in another side regime of Tresca's hexagon with $\sigma_r > \sigma_\theta > 0$. The yield condition has the form

$$\sigma_y = \sigma_r \quad (44)$$

The increment of plastic work gives $\epsilon_{EQ} = \epsilon_r^p$, and according to the flow rule associated with the yield condition (44), $\epsilon_r^p = -\epsilon_z^p$ and $\epsilon_\theta^p = 0$. Since the circumferential strain is purely elastic and thermal and

$$\epsilon_r = \left[\frac{\sigma_r}{\sigma_0} - 1 \right] \frac{1}{\eta} + \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + \alpha T \quad (45)$$

the strain-displacement relations lead to

$$\sigma_r = \frac{(1-W^2)\sigma_0}{1-W^2\nu^2} + \frac{EW^2}{1-W^2\nu^2} \left[\frac{\nu u}{r} + u' - \alpha(1+\nu)T \right] \quad (46)$$

$$\sigma_\theta = \frac{(1-W^2)\nu\sigma_0}{1-W^2\nu^2} + \frac{E}{1-W^2\nu^2} \left[\frac{u}{r} + W^2\nu u' - \alpha(1+W^2\nu)T \right] \quad (47)$$

The governing equation for the radial displacement is obtained by the substitution of these stresses in the equation of motion (19). The general solution is obtained in the form

$$u(r) = C_7P(r) + C_8Q(r) + R(r) \quad (48)$$

Hence,

$$\sigma_r = \frac{(1-W^2)\sigma_0}{1-W^2\nu^2} + \frac{EW^2}{1-W^2\nu^2} \left[C_7 \left(\frac{\nu P}{r} + P' \right) + C_8 \left(\frac{\nu Q}{r} + Q' \right) + \frac{\nu R}{r} + R' - \alpha(1+\nu)T \right] \quad (49)$$

$$\sigma_\theta = \frac{(1-W^2)\nu\sigma_0}{1-W^2\nu^2} + \frac{E}{1-W^2\nu^2} \left[C_7 \left(\frac{P}{r} + W^2\nu P' \right) + C_8 \left(\frac{Q}{r} + W^2\nu Q' \right) + \frac{R}{r} + W^2\nu R' - \alpha(1+W^2\nu)T \right] \quad (50)$$

The plastic strain components for this region are evaluated from

$$\epsilon_r^p = -\epsilon_z^p = \left[\frac{\sigma_r}{\sigma_0} - 1 \right] \frac{1}{\eta}, \quad \text{and} \quad \epsilon_\theta^p = 0 \quad (51)$$

Plastic region IV

In this region, stresses lie in another side regime of Tresca's hexagon with $\sigma_r > 0 > \sigma_\theta$. The yield condition reads

$$\sigma_y = \sigma_r - \sigma_\theta \quad (52)$$

The increment of plastic work gives $\epsilon_{EQ} = \epsilon_r^p$, and according to the flow rule $\epsilon_r^p = -\epsilon_\theta^p$ and $\epsilon_z^p = 0$. Since

$$\epsilon_r^p = -\epsilon_\theta^p = \left[\frac{\sigma_r - \sigma_\theta}{\sigma_0} - 1 \right] \frac{1}{\eta} \quad (53)$$

the strain-displacement relations result in

$$\sigma_r = \frac{(1 - W^2)\sigma_0}{2 - W^2(1 - \nu)} + \frac{E}{(1 - \nu)[2 - W^2(1 - \nu)]} \left[\frac{[1 - W^2(1 - \nu)]u}{r} + u' \right] - \frac{E\alpha T}{1 - \nu} \quad (54)$$

$$\sigma_\theta = -\frac{(1 - W^2)\sigma_0}{2 - W^2(1 - \nu)} + \frac{E}{(1 - \nu)[2 - W^2(1 - \nu)]} \left[\frac{u}{r} + [1 - W^2(1 - \nu)]u' \right] - \frac{E\alpha T}{1 - \nu} \quad (55)$$

The governing differential equation for the radial displacement is obtained by the substitution of the stresses in the equation of motion. The general solution is obtained in the form

$$u(r) = C_9P(r) + C_{10}Q(r) + R(r) \quad (56)$$

Hence,

$$\begin{aligned} \sigma_r &= \frac{(1 - W^2)\sigma_0}{2 - W^2(1 - \nu)} + \frac{E}{(1 - \nu)[2 - W^2(1 - \nu)]} \left[C_9 \left(\frac{[1 - W^2(1 - \nu)]P}{r} + P' \right) \right. \\ &\quad \left. + C_{10} \left(\frac{[1 - W^2(1 - \nu)]Q}{r} + Q' \right) + \frac{[1 - W^2(1 - \nu)]R}{r} + R' \right] - \frac{E\alpha T}{1 - \nu} \end{aligned} \quad (57)$$

$$\begin{aligned} \sigma_\theta &= -\frac{(1 - W^2)\sigma_0}{2 - W^2(1 - \nu)} + \frac{E}{(1 - \nu)[2 - W^2(1 - \nu)]} \left[C_9 \left(\frac{P}{r} + [1 - W^2(1 - \nu)]P' \right) \right. \\ &\quad \left. + C_{10} \left(\frac{Q}{r} + [1 - W^2(1 - \nu)]Q' \right) + \frac{R}{r} + [1 - W^2(1 - \nu)]R' \right] - \frac{E\alpha T}{1 - \nu} \end{aligned} \quad (58)$$

The plastic strain components are obtained from

$$\epsilon_r^p = -\epsilon_\theta^p = \left[\frac{\sigma_r - \sigma_\theta}{\sigma_0} - 1 \right] \frac{1}{\eta}, \quad \text{and} \quad \epsilon_z^p = 0 \quad (59)$$

Plastic region V

In this region, stresses lie in another side regime of Tresca's hexagon with $\sigma_\theta > 0 > \sigma_r$. The yield condition is

$$\sigma_y = \sigma_\theta - \sigma_r \quad (60)$$

The increment of plastic work gives $\epsilon_{EQ} = \epsilon_\theta^p$, and according to the flow rule $\epsilon_\theta^p = -\epsilon_r^p$ and $\epsilon_z^p = 0$. Since

$$\epsilon_\theta^p = -\epsilon_r^p = \left[\frac{\sigma_\theta - \sigma_r}{\sigma_0} - 1 \right] \frac{1}{\eta} \quad (61)$$

the strain-displacement relations lead to

$$\sigma_r = -\frac{(1 - W^2)\sigma_0}{2 - W^2(1 - \nu)} + \frac{E}{(1 - \nu)[2 - W^2(1 - \nu)]} \left[\frac{[1 - W^2(1 - \nu)]u}{r} + u' \right] - \frac{E\alpha T}{1 - \nu} \quad (62)$$

$$\sigma_\theta = \frac{(1 - W^2)\sigma_0}{2 - W^2(1 - \nu)} + \frac{E}{(1 - \nu)[2 - W^2(1 - \nu)]} \left[\frac{u}{r} + [1 - W^2(1 - \nu)]u' \right] - \frac{E\alpha T}{1 - \nu} \quad (63)$$

The governing differential equation for the radial displacement is obtained by the substitution of the stresses in the equation of motion. The general solution is obtained in the form

$$u(r) = C_{11}P(r) + C_{12}Q(r) + R(r) \quad (64)$$

Hence,

$$\begin{aligned} \sigma_r = & -\frac{(1-W^2)\sigma_0}{2-W^2(1-\nu)} + \frac{E}{(1-\nu)[2-W^2(1-\nu)]} \left[C_{11} \left(\frac{[1-W^2(1-\nu)]P}{r} + P' \right) \right. \\ & \left. + C_{12} \left(\frac{[1-W^2(1-\nu)]Q}{r} + Q' \right) + \frac{[1-W^2(1-\nu)]R}{r} + R' \right] - \frac{E\alpha T}{1-\nu} \end{aligned} \quad (65)$$

$$\begin{aligned} \sigma_\theta = & \frac{(1-W^2)\sigma_0}{2-W^2(1-\nu)} + \frac{E}{(1-\nu)[2-W^2(1-\nu)]} \left[C_{11} \left(\frac{P}{r} + [1-W^2(1-\nu)]P' \right) \right. \\ & \left. + C_{12} \left(\frac{Q}{r} + [1-W^2(1-\nu)]Q' \right) + \frac{R}{r} + [1-W^2(1-\nu)]R' \right] - \frac{E\alpha T}{1-\nu} \end{aligned} \quad (66)$$

The plastic strain components are evaluated from

$$\epsilon_\theta^p = -\epsilon_r^p = \left[\frac{\sigma_\theta - \sigma_r}{\sigma_0} - 1 \right] \frac{1}{\eta}, \text{ and } \epsilon_z^p = 0 \quad (67)$$

Disk Profiles with One Geometric Parameter

Elliptic profile

The elliptic disk profile is described by

$$h(r) = h_0 \sqrt{1 - n \left(\frac{r}{b} \right)^2} \quad (68)$$

where n is geometrical parameter ($0 \leq n < 1$), b is the radius of the disk and h_0 is the thickness at the axis of the disk. With this form of the disk profile function, a uniform thickness disk is obtained by setting $n = 0$. For $n > 0$ the thickness profile is always convex.

This thickness profile was introduced by the author (Eraslan, 2005). Isothermal solutions for the elastic region and plastic regions I, II and III were obtained to analyze elastic-plastic behavior of rotating elliptic solid and annular disks in comparison with the solution obtained by the von Mises yield condition. Closed form solutions of the plastic regions IV and V have not appeared in the literature. Nonisothermal solutions for the elastic and 5 plastic regions are presented next. It should be noted that without having the form of $T(r)$ it is not possible to obtain the general solution of the governing differential equation. However, as indicated before a series solution for $R(r)$ may generally be obtained.

Elastic Solutions: Governing differential equation:

$$\begin{aligned} r^2(b^2 - nr^2) \frac{d^2u}{dr^2} + r(b^2 - 2nr^2) \frac{du}{dr} \\ - [b^2 - (1-\nu)nr^2] u = - \frac{(b^2 - nr^2)(1-\nu^2)\rho\omega^2 r^3}{E} \\ - n\alpha(1+\nu)r^3T + \alpha(b^2 - nr^2)(1+\nu)r^2 \frac{dT}{dr} \end{aligned} \quad (69)$$

The homogeneous solution is obtained by introducing a new variable $z = b^2 - nr^2$ and using the transformation $u(r) = ry(z)$. The result is

$$P(r) = rF \left(\alpha, \beta, \delta; 1 - n \left(\frac{r}{b} \right)^2 \right) \quad (70)$$

$$\begin{aligned} Q(r) = r\sqrt{b^2 - nr^2} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \right. \\ \left. 1 - n \left(\frac{r}{b} \right)^2 \right) \end{aligned} \quad (71)$$

In these equations $F(\alpha, \beta, \delta; z)$ is the hypergeometric function defined by (Abramowitz and Stegun, 1966; Zhang and Jin, 1996)

$$\begin{aligned} F(\alpha, \beta, \delta; z) = 1 + \frac{\alpha\beta}{\delta 1!} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\delta(\delta+1)2!} z^2 \\ + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\delta(\delta+1)(\delta+2)3!} z^3 + \dots \end{aligned} \quad (72)$$

The arguments α , β and δ in Eqs.(70)-(71) have the following meanings:

$$\alpha = \frac{3}{4} - \frac{1}{4}\sqrt{5-4\nu} \quad (73)$$

$$\beta = \frac{3}{4} + \frac{1}{4}\sqrt{5-4\nu} \quad (74)$$

$$\delta = \frac{1}{2} \quad (75)$$

This solution is not finite at the axis of the disk.

Alternate solution: The solution given below is finite at the axis of the disk.

$$P(r) = r F\left(\alpha, \beta, \delta; n\left(\frac{r}{b}\right)^2\right) \quad (76)$$

$$Q(r) = P(r) \int \frac{dr}{r^3 \sqrt{b^2 - nr^2} [F(\alpha, \beta, \delta; n(\frac{r}{b})^2)]^2} \quad (77)$$

where α and β are given by Eqs.(73)-(74) and $\delta = 2$. The details of these solutions are presented in Appendices A and B, respectively.

Plastic region I: The stresses:

$$\sigma_r(r) = \sigma_\theta(r) = \frac{EC_3}{\sqrt{b^2 - nr^2}} + \frac{(b^2 - nr^2)\rho\omega^2}{3n} \quad (78)$$

The displacement:

$$u(r) = \frac{C_4}{r} - \frac{r}{2\eta} + \left[\frac{1}{\eta\sigma_0} + \frac{2(1-\nu)}{E} \right] \left[\frac{(2b^2 - nr^2)\rho\omega^2 r}{12n} - \frac{EC_3\sqrt{b^2 - nr^2}}{nr} \right] + \frac{2\alpha}{r} \int_0^r T(\xi)\xi d\xi \quad (79)$$

This equation is valid for $r > 0$ and in view of Eq. (33) $u(0) = 0$ since C_4 has to vanish if this plastic region begins at $r = 0$.

Plastic region II: Governing differential equation:

$$\begin{aligned} r^2(b^2 - nr^2)\frac{d^2u}{dr^2} + r(b^2 - 2nr^2)\frac{du}{dr} - W^2[b^2 \\ -(1-\nu)nr^2]u = -\frac{(b^2 - nr^2)(1 - W^2\nu^2)\rho\omega^2 r^3}{E} \\ + \frac{(1 - W^2)[b^2(1-\nu) - (1-2\nu)nr^2]\sigma_0 r}{E} \\ -\alpha \{ [2 - W^2(1-\nu)]nr^2 - b^2(1 - W^2) \} rT \\ + \alpha(b^2 - nr^2)(1 + W^2\nu)r^2\frac{dT}{dr} \end{aligned} \quad (80)$$

Homogeneous solution:

$$P(r) = r^{-W} F\left(\alpha, \beta, \delta; n\left(\frac{r}{b}\right)^2\right) \quad (81)$$

$$Q(r) = r^W F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n\left(\frac{r}{b}\right)^2\right) \quad (82)$$

where

$$\alpha = \frac{1}{4} - \frac{W}{2} - \frac{1}{4}\sqrt{1 + 4W^2(1-\nu)} \quad (83)$$

$$\beta = \frac{1}{4} - \frac{W}{2} + \frac{1}{4}\sqrt{1 + 4W^2(1-\nu)} \quad (84)$$

$$\delta = 1 - W \quad (85)$$

Plastic region III: Governing differential equation:

$$\begin{aligned} r^2W^2(b^2 - nr^2)\frac{d^2u}{dr^2} + rW^2(b^2 - 2nr^2)\frac{du}{dr} - [b^2 \\ -(1 - W^2\nu)nr^2]u = -\frac{(b^2 - nr^2)(1 - W^2\nu^2)\rho\omega^2 r^3}{E} \\ - \frac{(1 - W^2)[b^2(1-\nu) - (2-\nu)nr^2]\sigma_0 r}{E} \\ -\alpha \{ b^2(1 - W^2) - nr^2[1 - W^2(2+\nu)] \} rT \\ + \alpha W^2(b^2 - nr^2)(1 + \nu)r^2\frac{dT}{dr} \end{aligned} \quad (86)$$

Homogeneous solution:

$$P(r) = r^{-1/W} F\left(\alpha, \beta, \delta; n\left(\frac{r}{b}\right)^2\right) \quad (87)$$

$$Q(r) = r^{1/W} F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n\left(\frac{r}{b}\right)^2\right) \quad (88)$$

where

$$\alpha = \frac{1}{4} - \frac{1}{2W} - \frac{1}{4W}\sqrt{4 + W^2(1-4\nu)} \quad (89)$$

$$\beta = \frac{1}{4} - \frac{1}{2W} + \frac{1}{4W}\sqrt{4 + W^2(1-4\nu)} \quad (90)$$

$$\delta = 1 - \frac{1}{W} \quad (91)$$

Plastic region IV: Governing differential equation:

$$\begin{aligned} & r^2(b^2 - nr^2)\frac{d^2u}{dr^2} + r(b^2 - 2nr^2)\frac{du}{dr} - [b^2 \\ & \quad - W^2nr^2(1 - \nu)]u \\ = & -\frac{(b^2 - nr^2)(1 - \nu)[2 - W^2(1 - \nu)]\rho\omega^2r^3}{E} \\ & -\frac{(2b^2 - 3nr^2)(1 - \nu)(1 - W^2)\sigma_0r}{E} - n\alpha[2 \\ & - W^2(1 - \nu)]r^3T + \alpha(b^2 - nr^2)[2 - W^2(1 - \nu)]r^2\frac{dT}{dr} \end{aligned} \quad (92)$$

Homogeneous solution:

$$P(r) = rF\left(\alpha, \beta, \delta; 1 - n\left(\frac{r}{b}\right)^2\right) \quad (93)$$

$$\begin{aligned} Q(r) = r\sqrt{b^2 - nr^2}F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \right. \\ \left. 1 - n\left(\frac{r}{b}\right)^2\right) \end{aligned} \quad (94)$$

where

$$\alpha = \frac{3}{4} - \frac{1}{4}\sqrt{1 + 4W^2(1 - \nu)} \quad (95)$$

$$\beta = \frac{3}{4} + \frac{1}{4}\sqrt{1 + 4W^2(1 - \nu)} \quad (96)$$

$$\delta = \frac{1}{2} \quad (97)$$

Alternate solution:

$$P(r) = rF\left(\alpha, \beta, \delta; n\left(\frac{r}{b}\right)^2\right) \quad (98)$$

$$Q(r) = P(r) \int \frac{dr}{r^3\sqrt{b^2 - nr^2} [F(\alpha, \beta, \delta; n(\frac{r}{b})^2)]^2} \quad (99)$$

where α and β are given by Eqs. (95)-(96) and $\delta = 2$.

Plastic region V: Governing differential equation:

$$\begin{aligned} & r^2(b^2 - nr^2)\frac{d^2u}{dr^2} + r(b^2 - 2nr^2)\frac{du}{dr} \\ & \quad - [b^2 - W^2nr^2(1 - \nu)]u \\ = & -\frac{(b^2 - nr^2)(1 - \nu)[2 - W^2(1 - \nu)]\rho\omega^2r^3}{E} \\ & \quad + \frac{(2b^2 - 3nr^2)(1 - W^2)(1 - \nu)\sigma_0r}{E} \\ & - n\alpha[2 - W^2(1 - \nu)]r^3T + \alpha(b^2 - nr^2)[2 \\ & \quad - W^2(1 - \nu)]r^2\frac{dT}{dr} \end{aligned} \quad (100)$$

Homogeneous solution: same as in plastic region IV.

Hyperbolic profile

The hyperbolic disk profile is given by

$$h(r) = h_0 \left(\frac{b+r}{b}\right)^{-k} \quad (101)$$

where k is a geometrical parameter ($k > 0$). With this form of the disk profile function, a uniform thickness disk is obtained by setting $k = 0$. For $k > 0$ the profile is always concave. The form of the hyperbolic profile $h(r) = h_0(r/b)^{-k}$ commonly used by researchers (see for example Güven, 1998a) is not as convenient since h is not finite as $r \rightarrow 0$.

This thickness profile was proposed by the author (Eraslan, 2004). Isothermal solutions were obtained for the elastic region and plastic regions I, II and III. These solutions were used to study elastic-plastic deformations of rotating solid disks in the absence of a radial temperature gradient. Solutions of the plastic regions IV and V have not appeared in the literature. Homogeneous solutions and nonhomogeneous terms of the elastic and 5 plastic regions are given below.

Elastic solutions: Governing differential equation:

$$\begin{aligned} & r^2\frac{d^2u}{dr^2} + \left(\frac{r}{b+r}\right)[b+r(1-k)]\frac{du}{dr} - \frac{b+r(1+k\nu)}{b+r}u \\ = & -\frac{(1-\nu^2)\rho\omega^2r^3}{E} - \frac{k\alpha(1+\nu)r^2T}{b+r} + \alpha(1+\nu)r^2\frac{dT}{dr} \end{aligned} \quad (102)$$

The homogeneous solution is obtained by introducing a new variable $z = [b/(r+b)]$ and using the transformation $u(r) = ry(z)$. The result is

$$P(r) = r\left(\frac{b}{r+b}\right)^\alpha F\left(\alpha, \beta, \delta; \left(\frac{b}{r+b}\right)\right) \quad (103)$$

$$\begin{aligned} Q(r) = r\left(\frac{b}{r+b}\right)^{\alpha-\delta+1} \\ \times F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \left(\frac{b}{r+b}\right)\right) \end{aligned} \quad (104)$$

where

$$\delta = 1 - \sqrt{k^2 + 4(1 + k\nu)} \quad (107)$$

$$\alpha = 1 - \frac{k}{2} - \frac{1}{2}\sqrt{k^2 + 4(1 + k\nu)} \quad (105)$$

This solution is not finite at the axis of the disk.

$$\beta = 2 + \frac{k}{2} - \frac{1}{2}\sqrt{k^2 + 4(1 + k\nu)} \quad (106)$$

Alternate solution: The solution given below is finite at the axis of the disk.

$$P(r) = r F(\alpha, \beta, \delta; -\frac{r}{b}) = r \left(1 + \frac{r}{b}\right)^{-\alpha} F\left(\alpha, \delta - \beta, \delta; \frac{r}{b+r}\right) \quad (108)$$

$$Q(r) = P(r) \int \frac{(b+r)^k dr}{r^3 [F(\alpha, \beta, \delta; -\frac{r}{b})]^2} \quad (109)$$

where

$$\alpha = 1 - \frac{k}{2} - \frac{1}{2}\sqrt{k^2 + 4(1 + k\nu)} \quad (110)$$

$$\beta = 1 - \frac{k}{2} + \frac{1}{2}\sqrt{k^2 + 4(1 + k\nu)} \quad (111)$$

$$\delta = 3 \quad (112)$$

Plastic region I: The stresses:

$$\sigma_r(r) = \sigma_\theta(r) = EC_3(b+r)^k + \frac{(b+r)[b - (1-k)r]\rho\omega^2}{(1-k)(2-k)} \quad (113)$$

The displacement:

$$u(r) = \frac{C_4}{r} - \frac{r}{2\eta} + \left[\frac{1}{\eta\sigma_0} + \frac{2(1-\nu)}{E} \right] \left[\frac{[6b^2 + 4bkr - 3(1-k)r^2]\rho\omega^2 r}{12(1-k)(2-k)} \right. \\ \left. + \frac{EC_3}{(1+k)(2+k)r} \{ (b+r)^{1+k} [(1+k)r - b] + b^{2+k} \} \right] + \frac{2\alpha}{r} \int_0^r T(\xi)\xi d\xi \quad (114)$$

Plastic region II: Governing differential equation:

$$r^2 \frac{d^2 u}{dr^2} + \left(\frac{r}{b+r} \right) [b+r(1-k)] \frac{du}{dr} - \frac{W^2[b+r(1+k\nu)]}{b+r} u = \\ - \frac{(1-W^2\nu^2)\rho\omega^2 r^3}{E} + \frac{(1-W^2)\{b(1-\nu) + [1-(1-k)\nu]r\}\sigma_0 r}{E(b+r)} \\ + \frac{\alpha\{b(1-W^2) + r[1-k-W^2(1+k\nu)]\}rT}{b+r} + \alpha(1+W^2\nu)r^2 \frac{dT}{dr} \quad (115)$$

Homogeneous solution:

$$P(r) = r^{-W} F\left(\alpha, \beta, \delta; -\frac{r}{b}\right) \quad (116)$$

$$Q(r) = r^W F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; -\frac{r}{b}\right) \quad (117)$$

$$\alpha = -\frac{k}{2} - W - \frac{1}{2}\sqrt{k^2 + 4W^2(1 + k\nu)} \quad (118)$$

$$\beta = -\frac{k}{2} - W + \frac{1}{2}\sqrt{k^2 + 4W^2(1 + k\nu)} \quad (119)$$

$$\delta = 1 - 2W \quad (120)$$

Plastic region III: Governing differential equation:

$$\begin{aligned} W^2 r^2 \frac{d^2 u}{dr^2} + \left(\frac{W^2 r}{b+r}\right) [b+r(1-k)] \frac{du}{dr} - \frac{b+r(1+kW^2\nu)}{b+r} u = \\ - \frac{(1-W^2\nu^2)\rho\omega^2 r^3}{E} - \frac{(1-W^2)[b(1-\nu) + (1-k-\nu)r]\sigma_0 r}{E(b+r)} \\ - \frac{\alpha \{b(1-W^2) + r[1-W^2(1-k-k\nu)]\} r T}{b+r} + W^2 \alpha (1+\nu) r^2 \frac{dT}{dr} \end{aligned} \quad (121)$$

Homogeneous solution:

$$P(r) = r^{-1/W} F\left(\alpha, \beta, \delta; -\frac{r}{b}\right) \quad (122)$$

$$Q(r) = r^{1/W} F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; -\frac{r}{b}\right) \quad (123)$$

where

$$\alpha = -\frac{k}{2} - \frac{1}{W} - \frac{1}{2W}\sqrt{k^2 W^2 + 4(1 + kW^2\nu)} \quad (124)$$

$$\beta = -\frac{k}{2} - \frac{1}{W} + \frac{1}{2W}\sqrt{k^2 W^2 + 4(1 + kW^2\nu)} \quad (125)$$

$$\delta = 1 - \frac{2}{W} \quad (126)$$

Plastic region IV: Governing differential equation:

$$\begin{aligned} r^2 \frac{d^2 u}{dr^2} + \left(\frac{r}{b+r}\right) [b+r(1-k)] \frac{du}{dr} - \frac{b+r\{1+k[1-W^2(1-\nu)]\}}{b+r} u = \\ - \frac{[2-W^2(1-\nu)](1-\nu)\rho\omega^2 r^3}{E} - \frac{[2b+r(2-k)](1-W^2)(1-\nu)\sigma_0 r}{E(b+r)} \\ - \frac{k\alpha[2-W^2(1-\nu)]r^2 T}{b+r} + \alpha[2-W^2(1-\nu)]r^2 \frac{dT}{dr} \end{aligned} \quad (127)$$

Homogeneous solution:

$$P(r) = r \left(\frac{b}{r+b}\right)^\alpha F\left(\alpha, \beta, \delta; \left(\frac{b}{r+b}\right)\right) \quad (128)$$

$$Q(r) = r \left(\frac{b}{r+b} \right)^{\alpha-\delta+1} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \left(\frac{b}{r+b} \right) \right) \quad (129)$$

where

$$\alpha = 1 - \frac{k}{2} - \frac{1}{2} \sqrt{4 + k^2 + 4k[1 - W^2(1 - \nu)]} \quad (130)$$

$$\beta = 2 + \frac{k}{2} - \frac{1}{2} \sqrt{4 + k^2 + 4k[1 - W^2(1 - \nu)]} \quad (131)$$

$$\delta = 1 - \sqrt{4 + k^2 + 4k[1 - W^2(1 - \nu)]} \quad (132)$$

Alternate solution:

$$P(r) = r F(\alpha, \beta, \delta; -\frac{r}{b}) \quad (133)$$

$$Q(r) = P(r) \int \frac{(b+r)^k dr}{r^3 [F(\alpha, \beta, \delta; -\frac{r}{b})]^2} \quad (134)$$

where

$$\alpha = 1 - \frac{k}{2} - \frac{1}{2} \sqrt{4 + k^2 + 4k[1 - W^2(1 - \nu)]} \quad (135)$$

$$\beta = 1 - \frac{k}{2} + \frac{1}{2} \sqrt{4 + k^2 + 4k[1 - W^2(1 - \nu)]} \quad (136)$$

$$\delta = 3 \quad (137)$$

Plastic region V: Governing differential equation:

$$\begin{aligned} r^2 \frac{d^2 u}{dr^2} + \left(\frac{r}{b+r} \right) [b+r(1-k)] \frac{du}{dr} - \frac{b+r \{1+k[1-W^2(1-\nu)]\}}{b+r} u = \\ - \frac{[2-W^2(1-\nu)](1-\nu)\rho\omega^2 r^3}{E} + \frac{[2b+r(2-k)](1-W^2)(1-\nu)\sigma_0 r}{E(b+r)} \\ - \frac{k\alpha[2-W^2(1-\nu)]r^2 T}{b+r} + \alpha[2-W^2(1-\nu)]r^2 \frac{dT}{dr} \end{aligned} \quad (138)$$

Homogeneous solution: same as in plastic region IV.

Disk Profiles with Two Geometric Parameters

Exponential profile

The exponential disk profile is described by the thickness function

$$h(r) = h_0 e^{-n(\frac{r}{b})^k} \quad (139)$$

where n and k are geometrical parameters. The parameter n determines the thickness at the edge of the disk relative to h_0 while k determines the shape of the profile.

This thickness profile was first introduced by Güven (1995a) and has also been used by other researchers. Isothermal solutions for the elastic region and plastic regions I and II were obtained by Eraslan

and Orcan (2002a). A solution of plastic region III was obtained and used for the stress analysis of rotating concave exponential solid disks (Eraslan and Orcan, 2002b). The solutions of plastic regions I and II were used by the author (Eraslan, 2002a) to compare the predictions of Tresca and von Mises yield criteria in estimating the stress distribution for rotating exponential solid disks in the fully plastic state. In a later work, the author (Eraslan 2002b) used this profile to analyze the deformation behavior of non-

linearly hardening rotating annular disks mounted on rigid shafts. This numerical analysis was based on von the Mises yield condition, deformation theory of plasticity and Swift's hardening law. Nonisothermal solutions for the elastic and plastic regions I, II and III have not appeared in the literature. Furthermore, the solutions for the plastic regions IV and V are original. These solutions are presented next.

Elastic solution: Governing differential equation:

$$r^2 \frac{d^2 u}{dr^2} + r \left[1 - k n \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - \left[1 + kn\nu \left(\frac{r}{b} \right)^k \right] u = - \frac{(1 - \nu^2) \rho \omega^2 r^3}{E} - kn\alpha \left(\frac{r}{b} \right)^k (1 + \nu) r T + \alpha(1 + \nu) r^2 \frac{dT}{dr} \quad (140)$$

The homogeneous solution is obtained by introducing a new variable $z = n(r/b)^k$ and using the transformation $u(r) = ry(z)$. The result is

$$P(r) = r F_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) \quad (141)$$

$$Q(r) = \frac{1}{r} F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \quad (142)$$

where $F_C(\alpha, \beta; z)$ is the confluent hypergeometric function given by (Abramowitz and Stegun, 1966)

$$F_C(\alpha, \beta; z) = 1 + \frac{\alpha}{\beta 1!} z + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1) 2!} z^2 + \frac{\alpha(\alpha + 1)(\alpha + 2)}{\beta(\beta + 1)(\beta + 2) 3!} z^3 + \dots \quad (143)$$

and the arguments are defined by

$$\alpha = \frac{1}{k} + \frac{\nu}{k} \quad (144)$$

$$\beta = 1 + \frac{2}{k} \quad (145)$$

The details of this solution are presented in Appendix C.

Plastic region I: The stresses:

$$\sigma_r(r) = \sigma_\theta(r) = EC_3 e^{n(\frac{r}{b})^k} - \rho \omega^2 e^{n(\frac{r}{b})^k} \int_0^r e^{-n(\frac{\xi}{b})^k} \xi d\xi \quad (146)$$

The displacement:

$$u(r) = \frac{C_4}{r} - \frac{r}{2\eta} + \left[\frac{1}{\eta\sigma_0} + \frac{2(1 - \nu)}{E} \right] \left[\frac{EC_3}{r} \int_0^r e^{n(\frac{\xi}{b})^k} \xi d\xi - \frac{\rho \omega^2}{r} \int_0^r \int_0^\varphi e^{-n(\frac{\xi}{b})^k} e^{n(\frac{\varphi}{b})^k} \xi d\xi d\varphi \right] + \frac{2\alpha}{r} \int_0^r T(\xi) \xi d\xi \quad (147)$$

Note that

$$\lim_{r \rightarrow 0} \frac{\int_0^r e^{n(\frac{\xi}{b})^k} \xi d\xi}{r} = 0 \quad (148)$$

$$\lim_{r \rightarrow 0} \frac{\int_0^r \int_0^\varphi e^{-n(\frac{\xi}{b})^k} e^{n(\frac{\varphi}{b})^k} \xi d\xi d\varphi}{r} = 0 \quad (149)$$

The integrals in Eqs. (146) and (147) may be evaluated analytically if n , b and k are assigned numerical values. For example, if $n = 1/2$, $b = 1$ and $k = 2$ one finds by expanding the integrand into Taylor series

$$\int_0^r e^{n(\frac{\xi}{b})^k} \xi d\xi = \frac{r^2}{2} \left[1 + \frac{r^2}{4} + \frac{r^4}{24} + \frac{r^6}{192} + \frac{r^8}{1920} + \frac{r^{10}}{23040} + \dots \right] \quad (150)$$

Plastic region II: Governing differential equation:

$$\begin{aligned} r^2 \frac{d^2 u}{dr^2} + r \left[1 - kn \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - W^2 \left[1 + kn\nu \left(\frac{r}{b} \right)^k \right] u = - \frac{(1 - W^2 \nu^2) \rho \omega^2 r^3}{E} \\ + \frac{(1 - W^2) \left\{ 1 - \nu \left[1 - kn \left(\frac{r}{b} \right)^k \right] \right\} \sigma_0 r}{E} + \alpha \left[1 - W^2 - (1 + W^2 \nu) kn \left(\frac{r}{b} \right)^k \right] rT + \alpha (1 + W^2 \nu) r^2 \frac{dT}{dr} \end{aligned} \quad (151)$$

Homogeneous solution:

$$P(r) = r^{-W} F_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) \quad (152)$$

$$Q(r) = r^W F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \quad (153)$$

where

$$\alpha = -\frac{W}{k} + \frac{W^2 \nu}{k} \quad (154)$$

$$\beta = 1 - \frac{2W}{k} \quad (155)$$

Plastic region III: Governing differential equation:

$$\begin{aligned} W^2 r^2 \frac{d^2 u}{dr^2} + W^2 r \left[1 - kn \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - \left[1 + kn\nu W^2 \left(\frac{r}{b} \right)^k \right] u = - \frac{(1 - W^2 \nu^2) \rho \omega^2 r^3}{E} \\ - \frac{(1 - W^2) \left[1 - \nu - kn \left(\frac{r}{b} \right)^k \right] \sigma_0 r}{E} - \alpha \left[1 - W^2 + kn W^2 \left(\frac{r}{b} \right)^k (1 + \nu) \right] rT + \alpha W^2 (1 + \nu) r^2 \frac{dT}{dr} \end{aligned} \quad (156)$$

Homogeneous solution:

$$P(r) = r^{-1/W} F_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) \quad (157)$$

$$Q(r) = r^{1/W} F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \quad (158)$$

where

$$\alpha = -\frac{1}{kW} + \frac{\nu}{k} \quad (159)$$

$$\beta = 1 - \frac{2}{kW} \quad (160)$$

Plastic region IV: Governing differential equation:

$$\begin{aligned} r^2 \frac{d^2 u}{dr^2} + r \left[1 - kn \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - \left\{ 1 + kn[1 - W^2(1 - \nu)] \left(\frac{r}{b} \right)^k \right\} u = \\ - \frac{[2 - W^2(1 - \nu)](1 - \nu)\rho\omega^2 r^3}{E} - \frac{[2 - kn \left(\frac{r}{b} \right)^k] (1 - W^2)(1 - \nu)\sigma_0 r}{E} \\ - \alpha kn[2 - W^2(1 - \nu)] \left(\frac{r}{b} \right)^k rT + \alpha[2 - W^2(1 - \nu)]r^2 \frac{dT}{dr} \end{aligned} \quad (161)$$

Homogeneous solution:

$$P(r) = rF_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) \quad (162)$$

$$Q(r) = \frac{1}{r} F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \quad (163)$$

where

$$\alpha = \frac{2}{k} - \frac{W^2(1 - \nu)}{k} \quad (164)$$

$$\beta = 1 + \frac{2}{k} \quad (165)$$

Plastic region V: Governing differential equation:

$$\begin{aligned} r^2 \frac{d^2 u}{dr^2} + r \left[1 - kn \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - \left\{ 1 + kn[1 - W^2(1 - \nu)] \left(\frac{r}{b} \right)^k \right\} u = \\ - \frac{[2 - W^2(1 - \nu)](1 - \nu)\rho\omega^2 r^3}{E} + \frac{[2 - kn \left(\frac{r}{b} \right)^k] (1 - W^2)(1 - \nu)\sigma_0 r}{E} \\ - \alpha[2 - W^2(1 - \nu)]kn \left(\frac{r}{b} \right)^k rT + \alpha[2 - W^2(1 - \nu)]r^2 \frac{dT}{dr} \end{aligned} \quad (166)$$

Homogeneous solution: same as in plastic region IV.

Disk profile in a power function form

The disk profile is described by

$$h(r) = h_0 \left[1 - n \left(\frac{r}{b} \right)^k \right]^k \quad (167)$$

where n and k are geometrical parameters $0 \leq n < 1$. The parameter n determines the thickness at the edge of the disk while k determines the shape of the profile. Based on the values of the parameter k the profile may be convex, linear or concave.

The power function profile in the form $h(r) = h_0 [1 - nr]^k$ was proposed by Güven (1995b). Orcan and Eraslan (2002) used Güven's profile and obtained isothermal solutions for elastic and plastic regions I and II. An isothermal homogeneous elastic solution was used by Güven (1998b) to obtain the stress distribution in a stationary annular disk subjected to external pressure. A computational analysis using Güven's profile was carried out by Eraslan and Argeso (2002) to calculate elastic and plastic limit angular velocities in solid and annular disks. The von Mises yield condition was used to deter-

mine plastic limit velocities. Another computational study considering this profile was carried out by the author (Eraslan, 2002b), in which the deformation behavior of a rotating annular disk having a power function form thickness profile was investigated using radially constrained and free boundary conditions. However, the form (167) of the profile considered in this work is slightly different than the one proposed by Güven (1995b). In this respect, all the solutions given below may be considered original.

Elastic solutions: Governing differential equation:

$$r^2(b - nr) \frac{d^2u}{dr^2} + r[b - nr(1 + k)] \frac{du}{dr} - [b - nr(1 - k\nu)] u = -\frac{(b - nr)(1 - \nu^2)\rho\omega^2 r^3}{E} - \alpha kn(1 + \nu)r^2 T + \alpha(b - nr)(1 + \nu)r^2 \frac{dT}{dr} \quad (168)$$

The homogeneous solution is obtained by introducing a new variable $z = b - nr$ and using the transformation $u(r) = ry(z)$. The result is

$$P(r) = r F\left(\alpha, \beta, \delta; 1 - \frac{nr}{b}\right) \quad (169)$$

$$Q(r) = r(b - nr)^{1-k} F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; 1 - \frac{nr}{b}\right) \quad (170)$$

where

$$\alpha = 1 + \frac{k}{2} - \frac{1}{2}\sqrt{k^2 + 4(1 - k\nu)} \quad (171)$$

$$\beta = 1 + \frac{k}{2} + \frac{1}{2}\sqrt{k^2 + 4(1 - k\nu)} \quad (172)$$

$$\delta = k \quad (173)$$

This solution is not finite at the axis of the disk.

Alternate solution: The solution given below is finite at the axis of the disk.

$$P(r) = r F\left(\alpha, \beta, \delta; \frac{nr}{b}\right) \quad (174)$$

$$Q(r) = P(r) \int \frac{dr}{r^3(b - nr)^k [F(\alpha, \beta, \delta; \frac{nr}{b})]^2} \quad (175)$$

where the parameters α and β are given by Eqs. (171)-(172) and $\delta = 3$.

Plastic region I: The stresses:

$$\sigma_r(r) = \sigma_\theta(r) = \frac{EC_3}{[1 - n(\frac{r}{b})]^k} + \frac{(b - nr)[b + nr(1 + k)]\rho\omega^2}{n^2(1 + k)(2 + k)} \quad (176)$$

The displacement:

$$u(r) = \frac{C_4}{r} - \frac{r}{2\eta} + \left[\frac{1}{\eta\sigma_0} + \frac{2(1-\nu)}{E} \right] \left\{ \frac{[6b^2 + 3bknr - 2n^2r^2(1+k)]\rho\omega^2}{6n^2(1+k)(2+k)} - \frac{EC_3(b-nr)}{n(1-k) \left[1 - n\left(\frac{r}{b}\right)^k\right] r} \right\} + \frac{2\alpha}{r} \int_0^r T(\xi)\xi d\xi \quad (177)$$

Plastic region II: Governing differential equation:

$$\begin{aligned} r^2(b-nr)\frac{d^2u}{dr^2} + r[b-nr(1+k)]\frac{du}{dr} - W^2[b-nr(1-k\nu)]u = \\ -\frac{(b-nr)(1-W^2\nu^2)\rho\omega^2r^3}{E} + \frac{(1-W^2)\{b(1-\nu) - nr[1-\nu(1+k)]\}\sigma_0r}{E} \\ +\alpha\{b(1-W^2) - nr[1+k - W^2(1-k\nu)]\}rT + \alpha(b-nr)(1+W^2\nu)r^2\frac{dT}{dr} \end{aligned} \quad (178)$$

Homogeneous solution:

$$P(r) = r^{-W} F\left(\alpha, \beta, \delta; \frac{nr}{b}\right) \quad (179)$$

$$Q(r) = r^W F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \frac{nr}{b}\right) \quad (180)$$

where

$$\alpha = \frac{k}{2} - W - \frac{1}{2}\sqrt{k^2 + 4W^2(1-k\nu)} \quad (181)$$

$$\beta = \frac{k}{2} - W + \frac{1}{2}\sqrt{k^2 + 4W^2(1-k\nu)} \quad (182)$$

$$\delta = 1 - 2W \quad (183)$$

Plastic region III: Governing differential equation:

$$\begin{aligned} r^2W^2(b-nr)\frac{d^2u}{dr^2} + rW^2[b-nr(1+k)]\frac{du}{dr} - [b-nr(1-kW^2\nu)]u = \\ -\frac{(b-nr)(1-W^2\nu^2)\rho\omega^2r^3}{E} - \frac{(1-W^2)[b(1-\nu) - nr(1+k-\nu)]\sigma_0r}{E} \\ -\alpha\{b(1-W^2) - nr[1-W^2(1+k(1+\nu))]\}rT + \alpha W^2(b-nr)(1+\nu)r^2\frac{dT}{dr} \end{aligned} \quad (184)$$

Homogeneous solution:

$$P(r) = r^{-1/W} F\left(\alpha, \beta, \delta; \frac{nr}{b}\right) \quad (185)$$

$$Q(r) = r^{1/W} F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \frac{nr}{b}\right) \quad (186)$$

where

$$\alpha = \frac{k}{2} - \frac{1}{W} - \frac{1}{2W}\sqrt{k^2W^2 + 4(1-kW^2\nu)} \quad (187)$$

$$\beta = \frac{k}{2} - \frac{1}{W} + \frac{1}{2W} \sqrt{k^2 W^2 + 4(1 - kW^2\nu)} \quad (188)$$

$$\delta = 1 - \frac{2}{W} \quad (189)$$

Plastic region IV: Governing differential equation:

$$\begin{aligned} r^2(b - nr) \frac{d^2u}{dr^2} + r[b - nr(1 + k)] \frac{du}{dr} - \{b - nr[1 - k(1 - W^2(1 - \nu))]\} u = \\ - \frac{(b - nr)[2 - W^2(1 - \nu)](1 - \nu)\rho\omega^2 r^3}{E} - \frac{(1 - W^2)[2b - nr(2 + k)](1 - \nu)\sigma_0 r}{E} \\ - \alpha kn[2 - W^2(1 - \nu)]r^2 T + \alpha(b - nr)[2 - W^2(1 - \nu)]r^2 \frac{dT}{dr} \end{aligned} \quad (190)$$

Homogeneous solution:

$$P(r) = rF\left(\alpha, \beta, \delta; 1 - \frac{nr}{b}\right) \quad (191)$$

$$Q(r) = r(b - nr)^{1-k} F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; 1 - \frac{nr}{b}\right) \quad (192)$$

where

$$\alpha = 1 + \frac{k}{2} - \frac{1}{2} \sqrt{4 + k^2 - 4k[1 - W^2(1 - \nu)]} \quad (193)$$

$$\beta = 1 + \frac{k}{2} + \frac{1}{2} \sqrt{4 + k^2 - 4k[1 - W^2(1 - \nu)]} \quad (194)$$

$$\delta = k \quad (195)$$

Alternate Solution:

$$P(r) = r F\left(\alpha, \beta, \delta; \frac{nr}{b}\right) \quad (196)$$

$$Q(r) = P(r) \int \frac{dr}{r^3(b - nr)^k [F(\alpha, \beta, \delta; \frac{nr}{b})]^2} \quad (197)$$

where the parameters α and β are given by Eqs. (193)-(194) and $\delta = 3$.

Plastic region V: Governing differential equation:

$$\begin{aligned} r^2(b - nr) \frac{d^2u}{dr^2} + r[b - nr(1 + k)] \frac{du}{dr} - \{b - nr[1 - k(1 - W^2(1 - \nu))]\} u = \\ - \frac{(b - nr)[2 - W^2(1 - \nu)](1 - \nu)\rho\omega^2 r^3}{E} + \frac{(1 - W^2)[2b - nr(2 + k)](1 - \nu)\sigma_0 r}{E} \\ - \alpha kn[2 - W^2(1 - \nu)]r^2 T + \alpha(b - nr)[2 - W^2(1 - \nu)]r^2 \frac{dT}{dr} \end{aligned} \quad (198)$$

Homogeneous solution: same as in plastic region IV.

Parabolic disk type I

This parabolic disk profile is described by the thickness function

$$h(r) = h_0 \left[1 - n \left(\frac{r}{b} \right)^k \right] \quad (199)$$

where n and k are geometrical parameters ($0 \leq n < 1$, $k > 0$). With this form of the disk profile function, a uniform thickness disk is obtained by setting $n = 0$ and a linearly decreasing thickness is obtained

by setting $k = 1$. Furthermore, if $k < 1$ the profile is concave and if $k > 1$ it is convex.

This profile was suggested by the author Eraslan (2003). A similar profile in the form $h(r) = h_0 [1 - nr^k]$ was used by Güven (1998b). Isothermal elastic and plastic solutions for regions I, II and III were presented in (Eraslan, 2003). Homogeneous solutions and nonhomogeneous terms for these regions together with the solutions for plastic regions IV and V are given next.

Elastic solution: Governing differential equation:

$$\begin{aligned} r^2 \left[1 - n \left(\frac{r}{b} \right)^k \right] \frac{d^2 u}{dr^2} + r \left[1 - n(1+k) \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - \left[1 - n(1-k\nu) \left(\frac{r}{b} \right)^k \right] u = \\ - \frac{\left[1 - n \left(\frac{r}{b} \right)^k \right] (1-\nu^2) \rho \omega^2 r^3}{E} - \alpha k n \left(\frac{r}{b} \right)^k (1+\nu) r T + \alpha \left[1 - n \left(\frac{r}{b} \right)^k \right] (1+\nu) r^2 \frac{dT}{dr} \end{aligned} \quad (200)$$

The homogeneous solution is obtained by introducing a new variable $z = n(r/b)^k$ and using the transformation $u(r) = ry(z)$. The result is

$$P(r) = rF \left(\alpha, \beta, \delta; n \left(\frac{r}{b} \right)^k \right) \quad (201)$$

$$Q(r) = \frac{1}{r} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n \left(\frac{r}{b} \right)^k \right) \quad (202)$$

where

$$\alpha = \frac{1}{2} + \frac{1}{k} - \frac{1}{2k} \sqrt{k^2 + 4(1-k\nu)} \quad (203)$$

$$\beta = \frac{1}{2} + \frac{1}{k} + \frac{1}{2k} \sqrt{k^2 + 4(1-k\nu)} \quad (204)$$

$$\delta = 1 + \frac{2}{k} \quad (205)$$

Plastic region I: The stresses:

$$\sigma_r(r) = \sigma_\theta(r) = \frac{EC_3}{1 - n \left(\frac{r}{b} \right)^k} - \frac{\left[2 + k - 2n \left(\frac{r}{b} \right)^k \right] \rho \omega^2 r^2}{2(2+k) \left[1 - n \left(\frac{r}{b} \right)^k \right]} \quad (206)$$

The displacement:

$$\begin{aligned} u(r) = \frac{C_4}{r} - \frac{r}{2\eta} + \left[\frac{1}{\eta\sigma_0} + \frac{2(1-\nu)}{E} \right] \left[\frac{EC_3 r}{2} F \left(\frac{2}{k}, 1, 1 + \frac{2}{k}; n \left(\frac{r}{b} \right)^k \right) \right. \\ \left. - \frac{\left[2 + k F \left(\frac{4}{k}, 1, 1 + \frac{4}{k}; n \left(\frac{r}{b} \right)^k \right) \right] \rho \omega^2 r^3}{8(2+k)} \right] + \frac{2\alpha}{r} \int_0^r T(\xi) \xi d\xi \end{aligned} \quad (207)$$

in which $F(\alpha, \beta, \delta; z)$ is again a hypergeometric function defined by (72).

Plastic region II: Governing differential equation:

$$\begin{aligned}
 & r^2 \left[1 - n \left(\frac{r}{b} \right)^k \right] \frac{d^2 u}{dr^2} + r \left[1 - n(1+k) \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} \\
 & - W^2 \left[1 - n(1 - k\nu) \left(\frac{r}{b} \right)^k \right] u = - \frac{\left[1 - n \left(\frac{r}{b} \right)^k \right] (1 - W^2 \nu^2) \rho \omega^2 r^3}{E} \\
 & \quad + \frac{(1 - W^2) \left[1 - \nu - n(1 - \nu - k\nu) \left(\frac{r}{b} \right)^k \right] \sigma_0 r}{E} \\
 & + \alpha \left\{ 1 - W^2 - n[1 + k - W^2(1 - k\nu)] \left(\frac{r}{b} \right)^k \right\} rT + \alpha \left[1 - n \left(\frac{r}{b} \right)^k \right] (1 + W^2 \nu) r^2 \frac{dT}{dr}
 \end{aligned} \tag{208}$$

Homogeneous solution:

$$P(r) = r^{-W} F \left(\alpha, \beta, \delta; n \left(\frac{r}{b} \right)^k \right) \tag{209}$$

$$Q(r) = r^W F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n \left(\frac{r}{b} \right)^k \right) \tag{210}$$

where

$$\alpha = \frac{1}{2} - \frac{W}{k} - \frac{1}{2k} \sqrt{k^2 + 4W^2(1 - k\nu)} \tag{211}$$

$$\beta = \frac{1}{2} - \frac{W}{k} + \frac{1}{2k} \sqrt{k^2 + 4W^2(1 - k\nu)} \tag{212}$$

$$\delta = 1 - \frac{2W}{k} \tag{213}$$

Plastic region III: Governing differential equation:

$$\begin{aligned}
 & W^2 r^2 \left[1 - n \left(\frac{r}{b} \right)^k \right] \frac{d^2 u}{dr^2} + W^2 r \left[1 - n(1+k) \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} \\
 & - \left[1 - n(1 - kW^2 \nu) \left(\frac{r}{b} \right)^k \right] u = - \frac{\left[1 - n \left(\frac{r}{b} \right)^k \right] (1 - W^2 \nu^2) \rho \omega^2 r^3}{E} \\
 & \quad - \frac{(1 - W^2) \left[1 - \nu - n(1 + k - \nu) \left(\frac{r}{b} \right)^k \right] \sigma_0 r}{E} \\
 & - \alpha \left\{ 1 - W^2 - n[1 - W^2(1 + k(1 + \nu))] \left(\frac{r}{b} \right)^k \right\} rT + \alpha W^2 \left[1 - n \left(\frac{r}{b} \right)^k \right] (1 + \nu) r^2 \frac{dT}{dr}
 \end{aligned} \tag{214}$$

Homogeneous solution:

$$P(r) = r^{-1/W} F \left(\alpha, \beta, \delta; n \left(\frac{r}{b} \right)^k \right) \tag{215}$$

$$Q(r) = r^{1/W} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n \left(\frac{r}{b} \right)^k \right) \tag{216}$$

where

$$\alpha = \frac{1}{2} - \frac{1}{kW} - \frac{1}{2kW} \sqrt{k^2 W^2 + 4(1 - kW^2\nu)} \quad (217)$$

$$\beta = \frac{1}{2} - \frac{1}{kW} + \frac{1}{2kW} \sqrt{k^2 W^2 + 4(1 - kW^2\nu)} \quad (218)$$

$$\delta = 1 - \frac{2}{kW} \quad (219)$$

Plastic region IV: Governing differential equation:

$$\begin{aligned} r^2 \left[1 - n \left(\frac{r}{b} \right)^k \right] \frac{d^2 u}{dr^2} + r \left[1 - n(1+k) \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - \left\{ 1 - n[1 - k(1 - W^2(1 - \nu))] \left(\frac{r}{b} \right)^k \right\} u = \\ - \frac{\left[1 - n \left(\frac{r}{b} \right)^k \right] [2 - W^2(1 - \nu)](1 - \nu) \rho \omega^2 r^3}{E} - \frac{\left[2 - n(2+k) \left(\frac{r}{b} \right)^k \right] (1 - W^2)(1 - \nu) \sigma_0 r}{E} \\ - \alpha kn[2 - W^2(1 - \nu)] \left(\frac{r}{b} \right)^k rT + \alpha \left[1 - n \left(\frac{r}{b} \right)^k \right] [2 - W^2(1 - \nu)] r^2 \frac{dT}{dr} \end{aligned} \quad (220)$$

Homogeneous solution:

$$P(r) = rF \left(\alpha, \beta, \delta; n \left(\frac{r}{b} \right)^k \right) \quad (221)$$

$$Q(r) = \frac{1}{r} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n \left(\frac{r}{b} \right)^k \right) \quad (222)$$

where

$$\alpha = \frac{1}{2} + \frac{1}{k} - \frac{1}{2k} \sqrt{4 + k^2 - 4k[1 - W^2(1 - \nu)]} \quad (223)$$

$$\beta = \frac{1}{2} + \frac{1}{k} + \frac{1}{2k} \sqrt{4 + k^2 - 4k[1 - W^2(1 - \nu)]} \quad (224)$$

$$\delta = 1 + \frac{2}{k} \quad (225)$$

Plastic region V: Governing differential equation:

$$\begin{aligned} r^2 \left[1 - n \left(\frac{r}{b} \right)^k \right] \frac{d^2 u}{dr^2} + r \left[1 - n(1+k) \left(\frac{r}{b} \right)^k \right] \frac{du}{dr} - \left\{ 1 - n[1 - k(1 - W^2(1 - \nu))] \left(\frac{r}{b} \right)^k \right\} u = \\ - \frac{\left[1 - n \left(\frac{r}{b} \right)^k \right] [2 - W^2(1 - \nu)](1 - \nu) \rho \omega^2 r^3}{E} + \frac{\left[2 - n(2+k) \left(\frac{r}{b} \right)^k \right] (1 - W^2)(1 - \nu) \sigma_0 r}{E} \\ - \alpha kn[2 - W^2(1 - \nu)] \left(\frac{r}{b} \right)^k rT + \alpha \left[1 - n \left(\frac{r}{b} \right)^k \right] [2 - W^2(1 - \nu)] r^2 \frac{dT}{dr} \end{aligned} \quad (226)$$

Homogeneous solution: same as in plastic region IV.

Parabolic profile type II

This parabolic disk profile is defined as

$$h(r) = h_0 \left[1 - \left(\frac{r}{b+n} \right)^k \right] \quad (227)$$

where n and k are geometrical parameters ($n > 0$, $k > 0$). A uniform thickness disk is obtained by setting $n \rightarrow \infty$ and a linearly decreasing disk thickness is obtained by the use of $k = 1$. If $k < 1$ the profile

is concave and if $k > 1$ it is convex. Furthermore, the shape of the profile is smoothed as n increases.

This profile was proposed by the author Eraslan (2003). Isothermal elastic and plastic solutions for regions I, II and III may be found in (Eraslan, 2003). Homogeneous solutions and nonhomogeneous terms for these regions together with the solutions for plastic regions IV and V are given below.

Elastic solution: Governing differential equation:

$$\begin{aligned} & r^2 \left[1 - \left(\frac{r}{b+n} \right)^k \right] \frac{d^2u}{dr^2} + r \left[1 - (1+k) \left(\frac{r}{b+n} \right)^k \right] \frac{du}{dr} \\ & - \left[1 - (1-k\nu) \left(\frac{r}{b+n} \right)^k \right] u = - \frac{\left[1 - \left(\frac{r}{b+n} \right)^k \right] (1-\nu^2) \rho \omega^2 r^3}{E} \\ & - \alpha k \left(\frac{r}{b+n} \right)^k (1+\nu) r T + \alpha \left[1 - \left(\frac{r}{b+n} \right)^k \right] (1+\nu) r^2 \frac{dT}{dr} \end{aligned} \quad (228)$$

The homogeneous solution is obtained by introducing a new variable $z = [r/(b+n)]^k$ and using the transformation $u(r) = ry(z)$. The result is

$$P(r) = rF \left(\alpha, \beta, \delta; \left(\frac{r}{b+n} \right)^k \right) \quad (229)$$

$$Q(r) = \frac{1}{r} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \left(\frac{r}{b+n} \right)^k \right) \quad (230)$$

where arguments α , β and δ are given by Eqs. (203)-(205).

Plastic region I: The stresses:

$$\sigma_r(r) = \sigma_\theta(r) = \frac{EC_3}{\left[1 - \left(\frac{r}{n+b} \right)^k \right]} - \left\{ \frac{\rho \omega^2 r^2}{2+k} + \frac{k \rho \omega^2 r^2}{2(2+k) \left[1 - \left(\frac{r}{n+b} \right)^k \right]} \right\} \quad (231)$$

The displacement:

$$\begin{aligned} u(r) = & \frac{C_4}{r} - \frac{r}{2\eta} + \left[\frac{1}{\eta \sigma_0} + \frac{2(1-\nu)}{E} \right] \left[\frac{EC_3 r}{4(2+k)} F \left(\frac{2}{k}, 1, 1 + \frac{2}{k}; \left(\frac{r}{n+b} \right)^k \right) \right. \\ & \left. - \frac{\left[2 + k F \left(\frac{4}{k}, 1, 1 + \frac{4}{k}; \left(\frac{r}{n+b} \right)^k \right) \right] \rho \omega^2 r^3}{8(2+k)} \right] + \frac{2\alpha}{r} \int_0^r T(\xi) \xi d\xi \end{aligned} \quad (232)$$

Plastic region II: Governing differential equation:

$$\begin{aligned}
& r^2 \left[1 - \left(\frac{r}{b+n} \right)^k \right] \frac{d^2 u}{dr^2} + r \left[1 - (1+k) \left(\frac{r}{b+n} \right)^k \right] \frac{du}{dr} \\
& - W^2 \left[1 - (1-k\nu) \left(\frac{r}{b+n} \right)^k \right] u = - \frac{\left[1 - \left(\frac{r}{b+n} \right)^k \right] (1 - W^2 \nu^2) \rho \omega^2 r^3}{E} \\
& + \frac{(1 - W^2) \left[1 - \nu - \left(\frac{r}{b+n} \right)^k (1 - \nu - k\nu) \right] \sigma_0 r}{E} \\
& + \alpha \left\{ 1 - W^2 - [1 + k - W^2(1 - k\nu)] \left(\frac{r}{b+n} \right)^k \right\} rT \\
& + \alpha \left[1 - \left(\frac{r}{b+n} \right)^k \right] (1 + W^2 \nu) r^2 \frac{dT}{dr}
\end{aligned} \tag{233}$$

Homogeneous solution:

$$P(r) = r^{-W} F \left(\alpha, \beta, \delta; \left(\frac{r}{b+n} \right)^k \right) \tag{234}$$

$$Q(r) = r^W F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \left(\frac{r}{b+n} \right)^k \right) \tag{235}$$

The arguments α , β and δ are given by Eqs. (211)-(213).

Plastic region III: Governing differential equation:

$$\begin{aligned}
& r^2 W^2 \left[1 - \left(\frac{r}{b+n} \right)^k \right] \frac{d^2 u}{dr^2} + r W^2 \left[1 - (1+k) \left(\frac{r}{b+n} \right)^k \right] \frac{du}{dr} \\
& - \left[1 - (1 - kW^2 \nu) \left(\frac{r}{b+n} \right)^k \right] u = - \frac{\left[1 - \left(\frac{r}{b+n} \right)^k \right] (1 - W^2 \nu^2) \rho \omega^2 r^3}{E} \\
& - \frac{(1 - W^2) \left[1 - \nu - (1 + k - \nu) \left(\frac{r}{b+n} \right)^k \right] \sigma_0 r}{E} \\
& - \alpha \left\{ 1 - W^2 - [1 - W^2(1 + k(1 + \nu))] \left(\frac{r}{b+n} \right)^k \right\} rT \\
& + \alpha W^2 \left[1 - \left(\frac{r}{b+n} \right)^k \right] (1 + \nu) r^2 \frac{dT}{dr}
\end{aligned} \tag{236}$$

Homogeneous solution:

$$P(r) = r^{-1/W} F \left(\alpha, \beta, \delta; \left(\frac{r}{b+n} \right)^k \right) \tag{237}$$

$$Q(r) = r^{1/W} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \left(\frac{r}{b+n} \right)^k \right) \quad (238)$$

The arguments α , β and δ are given by Eqs. (217)-(219).

Plastic region IV: Governing differential equation:

$$\begin{aligned} r^2 \left[1 - \left(\frac{r}{b+n} \right)^k \right] \frac{d^2u}{dr^2} + r \left[1 - (1+k) \left(\frac{r}{b+n} \right)^k \right] \frac{du}{dr} - \left\{ 1 - [1 - k(1 - W^2(1 - \nu))] \left(\frac{r}{b+n} \right)^k \right\} u = \\ - \frac{\left[1 - \left(\frac{r}{b+n} \right)^k \right] [2 - W^2(1 - \nu)](1 - \nu)\rho\omega^2 r^3}{E} - \frac{\left[2 - (2+k) \left(\frac{r}{b+n} \right)^k \right] (1 - W^2)(1 - \nu)\sigma_0 r}{E} \\ - \alpha k [2 - W^2(1 - \nu)] \left(\frac{r}{b+n} \right)^k rT + \alpha \left[1 - \left(\frac{r}{b+n} \right)^k \right] [2 - W^2(1 - \nu)] r^2 \frac{dT}{dr} \end{aligned} \quad (239)$$

Homogeneous solution:

$$P(r) = rF \left(\alpha, \beta, \delta; \left(\frac{r}{b+n} \right)^k \right) \quad (240)$$

$$Q(r) = \frac{1}{r} F \left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \left(\frac{r}{b+n} \right)^k \right) \quad (241)$$

The arguments α , β and δ are given by Eqs. (223)-(225).

Plastic region V: Governing differential equation:

$$\begin{aligned} r^2 \left[1 - \left(\frac{r}{b+n} \right)^k \right] \frac{d^2u}{dr^2} + r \left[1 - (1+k) \left(\frac{r}{b+n} \right)^k \right] \frac{du}{dr} - \left\{ 1 - [1 - k(1 - W^2(1 - \nu))] \left(\frac{r}{b+n} \right)^k \right\} u = \\ - \frac{\left[1 - \left(\frac{r}{b+n} \right)^k \right] [2 - W^2(1 - \nu)](1 - \nu)\rho\omega^2 r^3}{E} + \frac{\left[2 - (2+k) \left(\frac{r}{b+n} \right)^k \right] (1 - W^2)(1 - \nu)\sigma_0 r}{E} \\ - \alpha k [2 - W^2(1 - \nu)] \left(\frac{r}{b+n} \right)^k rT + \alpha \left[1 - \left(\frac{r}{b+n} \right)^k \right] [2 - W^2(1 - \nu)] r^2 \frac{dT}{dr} \end{aligned} \quad (242)$$

Homogeneous solution: same as in plastic region IV.

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Appendix A
Solution of Eq. (69)

Introducing a new variable $z = b^2 - nr^2$ and using the transformation $u(r) = ry(z)$ the homogeneous equation is transformed into

$$z(b^2 - z)\frac{d^2y}{dz^2} + \frac{1}{2}(b^2 - 5z)\frac{dy}{dz} - \frac{(1 + \nu)}{4}y = 0 \quad (\text{A1})$$

This is the standard form of the hypergeometric differential equation with the solution (Zhang and Jin, 1996)

$$y(z) = C_1F\left(\alpha, \beta, \delta; \frac{z}{b^2}\right) + C_2\sqrt{z}F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; \frac{z}{b^2}\right) \quad (\text{A2})$$

in which the arguments have been defined by Eqs. (73)-(75). Back transformation $ry(z) = u(r)$ gives

$$u(r) = C_1rF\left(\alpha, \beta, \delta; 1 - n\left(\frac{r}{b}\right)^2\right) + C_2r\sqrt{b^2 - nr^2}F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; 1 - n\left(\frac{r}{b}\right)^2\right) \quad (\text{A3})$$

Hence,

$$P(r) = rF\left(\alpha, \beta, \delta; 1 - n\left(\frac{r}{b}\right)^2\right) \quad (\text{A4})$$

$$Q(r) = r\sqrt{b^2 - nr^2}F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; 1 - n\left(\frac{r}{b}\right)^2\right) \quad (\text{A5})$$

Appendix B
 Alternate Solution of Eq. (69)

One solution to the homogeneous equation is obtained as

$$P(r) = rF\left(\alpha, \beta, \delta; n\left(\frac{r}{b}\right)^2\right) \quad (\text{B1})$$

in which α and β are given by Eqs. (73)-(74) and $\delta = 2$. The homogeneous solution is assumed to be of the form $u(r) = P(r) \cdot V(r)$. Substituting in Eq. (69) we get a differential equation for $V(r)$:

$$r^2(b^2 - nr^2)P\frac{d^2V}{dr^2} + \left[r(b^2 - 2nr^2)P + 2r^2(b^2 - nr^2)\frac{dP}{dr}\right]\frac{dV}{dr} = 0 \quad (\text{B2})$$

with the solution

$$\begin{aligned} V(r) &= C_1 + C_2 \int \frac{dr}{r\sqrt{b^2 - nr^2} [P(r)]^2} \\ &= C_1 + C_2 \int \frac{dr}{r^3\sqrt{b^2 - nr^2} \left[F\left(\alpha, \beta, \delta; n\left(\frac{r}{b}\right)^2\right)\right]^2} \end{aligned} \quad (\text{B3})$$

In view of Eq. (20) we see that

$$Q(r) = P(r) \int \frac{dr}{r^3 \sqrt{b^2 - nr^2} [F(\alpha, \beta, \delta; n(\frac{r}{b})^2)]^2} \quad (B4)$$

The integration in Eq. (B4) may be carried out analytically if the Poisson's ratio is assigned a numerical value. For $\nu = 3/10$ the result is

$$\begin{aligned} \int \frac{dr}{r^3 \sqrt{b^2 - nr^2} [F(\alpha, \beta, \delta; n(\frac{r}{b})^2)]^2} = & -\frac{1}{2br^2} + \frac{7n \ln r}{40b^3} + \frac{2663n^2r^2}{38400b^5} \\ & + \frac{542597n^3r^4}{18432000b^7} + \frac{383330149n^4r^6}{22118400000b^9} + \frac{208132660421n^5r^8}{17694720000000b^{11}} \\ & + \frac{183819510478993n^6r^{10}}{21233664000000000b^{13}} + \frac{27380392583073829n^7r^{12}}{4076863488000000000b^{15}} + \dots \end{aligned} \quad (B5)$$

Appendix C Solution of Eq. (140)

Introducing a new variable $z = n(r/b)^k$ and using the transformation $u(r) = ry(z)$ the homogeneous equation is transformed into

$$z \frac{d^2y}{dz^2} + \frac{1}{k} [2 + k(1 - z)] \frac{dy}{dz} - \frac{1}{k} (1 + \nu)y = 0 \quad (C1)$$

This is the standard form of the confluent hypergeometric differential equation with the solution (Abramowitz and Stegun, 1966)

$$y(z) = C_1 F_C(\alpha, \beta; z) + \hat{C}_2 z^{-\frac{2}{k}} F_C(\alpha - \beta + 1, 2 - \beta; z) \quad (C2)$$

in which the arguments have been defined by Eqs. (144)-(145). Back transforming using $ry(z) = u(r)$ we obtain

$$\begin{aligned} u(r) &= C_1 r F_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) + \hat{C}_2 r \left(n \left(\frac{r}{b} \right)^k \right)^{-2/k} F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \\ &= C_1 r F_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) + \frac{\hat{C}_2 b^2 n^{-2/k}}{r} F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \\ &= C_1 r F_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) + C_2 \frac{1}{r} F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \end{aligned} \quad (C3)$$

Hence,

$$P(r) = r F_C \left(\alpha, \beta; n \left(\frac{r}{b} \right)^k \right) \quad (C4)$$

$$Q(r) = \frac{1}{r} F_C \left(\alpha - \beta + 1, 2 - \beta; n \left(\frac{r}{b} \right)^k \right) \quad (C4)$$

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