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Determination of the Optimum Parameter Tolerances for Transducers: A Sensitivity Approach

Cevat ERDAL
İstanbul Technical University, Faculty of Electrical-Electronics Engineering
80626, Maslak-İstanbul-TURKEY
e-mail: cerdal@triton.elk.itu.edu.tr
Cevdet ACAR
Yeditepe University, Faculty of Engineering & Architecture,
Electrical-Electronics Engineering Department,
Kayışdağı, İstanbul-TURKEY
e-mail: acar@yeditepe.edu.tr

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Abstract

Transducers are essential system components used in the process control industry and also in many engineering areas. In this study, a method is proposed to determine the optimum parameter tolerances by the use of the parameter sensitivities of the most general transfer function of a transducer. A sensitivity measure for transducers is also defined.

Key words: Transducers, Sensitivity, Optimum parameter tolerances.

Introduction

The most important instrument in the process control industry and in many engineering areas is the transducer. This instrument is commercially available in a wide variety of types and ranges to meet correspondingly diverse application requirements. Although removing the deviations in the output quantity of a transducer caused by environmental effects is very important in the design and use of such a component, to the authors' knowledge this problem has not been examined and solved by the use of the sensitivity concept for transducers except for accelerometers (Erdal, 1998a). Considering this gap in the literature, a general method is proposed to solve this problem for transducers by the use of the sensitivity concept. For this purpose, using the most general transfer function of a transducer, the basic definitions are given concerning the subject and the parameter sensitivities on the deviation of the output quantity of a transducer. Then an upper bound for deviation of the output quantity is determined in magnitude. Furthermore, the optimum parameter tolerances satisfying this upper bound are calculated. These tolerances keep the relative error at the output of the transducer due to the parameter variations within its tolerance region. A sensitivity measure is also defined. This measure can be used to improve the sensitivity performance of the transducer and to compare various transducers with different sets of design parameter values that realize the same transfer function. Finally, the proposed method is applied to an accelerometer as an example.

Basic Definitions

The output quantity of a transducer can be written in the s-domain as follows:

$$Q_o(s) = T(s)\, Q_i(s). \tag{1}$$

where $q_o(t)$ is the output quantity and $q_i(t)$ is the input quantity.

In Equation (1), T(s) is the transfer function of a general transducer and can be written by the application of suitable simplifying assumptions in the following form (Doebelin, 1975):

$$\frac{Q_o(s)}{Q_i(s)} = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{2}$$

where $s = \sigma + j\omega$ is the complex variable of the Laplace transform, and $a_i s$, $i = 0, \ldots, n$, and $b_j s$, $j = 0, \ldots, m$, are combinations of the transducer's physical parameters. N(s) stands for the nominator polynomial and D(s) stand for the denominator polynomial of the transfer function.

The relative deviation of the output quantity of the transducer can be expressed in terms of parameter sensitivities as follows (Huelsman, 1993; Goldstein and Kuo, 1961):

$$\frac{\Delta Q_o(s)}{Q_o(s)} = \frac{\Delta T(s)}{T(s)} = \sum_{k=1}^{n} S_{x_k}^{T}(s)(\frac{\Delta x_k}{x_k}) \tag{3}$$

where $x_k,\ k = 1, \ldots, n$, denotes the nominal value of the kth physical parameter and $\Delta x_k / x_k$ is the relative deviation in the kth parameter's nominal value due to environmental effects. $S_{x_k}^{T}(s)$ is the normalized sensitivity of the transfer function, T(s) with respect to the kth parameter x_k and it is defined as follows:

$$S_{x_k}^{T}(s) = \frac{x_k}{T}\frac{\partial T}{\partial x_k}.\ k = 1, \ldots, n \tag{4}$$

The sensitivity, $S_{x_k}^{T}(s)$, can also be written in terms of gain and phase sensitivities after substituting $s = j\omega$ as follows (Ghausi and Laker, 1981):

$$S_{x_k}^{T}(j\omega) = S_{x_k}^{|T|}(\omega) + jS_{x_k}^{\beta}(\omega) \quad , \tag{5}$$

where $S_{x_k}^{|T|}$ and $S_{x_k}^{\beta}$ are respectively the normalized sensitivities of the gain function and semi-normalized sensitivity of the phase function and can be calculated as (Acar, 1979)

$$S_{x_k}^{|T|}(\omega) = Re\left\{\sum_{j=0}^{m} \frac{b_j s^j}{N(s)}\ S_{x_k}^{b_j}(s) - \sum_{i=0}^{n} \frac{a_i s^i}{D(s)}\ S_{x_k}^{a_i}(s)\right\}\Bigg|_{s=j\omega} \tag{6a}$$

$$S_{x_k}^{\beta}(\omega) = Im\left\{\sum_{j=0}^{m} \frac{b_j s^j}{N(s)}\ S_{x_k}^{b_j}(s) - \sum_{i=0}^{n} \frac{a_i s^i}{D(s)}\ S_{x_k}^{a_i}(s)\right\}\Bigg|_{s=j\omega} \tag{6b}$$

where $S_{x_k}^{b_j}$ and $S_{x_k}^{a_i}$, $i = 0, \ldots, n$, $j = 0, \ldots, m$, are normalized sensitivities of the coefficients of the nominator and the denominator polynomials with respect to the kth parameter $x_k, k = 1, \ldots, n$, respectively, and can be defined as

$$S_{x_k}^{b_j} = \frac{x_k}{b_j}\frac{\partial b_j}{\partial x_k} \tag{7a}$$

$$S_{x_k}^{a_i} = \frac{x_k}{a_i}\frac{\partial a_i}{\partial x_k}. \tag{7b}$$

The sensitivities of the transfer function can be calculated by the use of Eq.(1) and Eq.(4), and then substituting $j\omega$ instead of s, the overall relative deviation at the output of a transducer due to parameter variations can be obtained from Eq.(3), Eq.(6), Eq.(7) and Eq.(5) as follows:

$$\frac{\Delta Q_o(j\omega)}{Q_o(j\omega)} = \sum_{k=1}^{n}\left[S_{x_k}^{|T|}(\omega) + jS_{x_k}^{\beta}(\omega)\right](\frac{\Delta x_k}{x_k}). \tag{8}$$

Using the triangular inequality (Erdal, 1996, 1997) the upper bound for the overall relative deviation at the output of a transducer can be expressed as follows:

$$\begin{aligned}\left|\frac{\Delta Q_o(\omega)}{Q_o(\omega)}\right| &= \sum_{k=1}^{n}\left|S_{x_k}^{T}(s)\right|\left|\frac{\Delta x_k}{x_k}\right| \\ &\le \sum_{k=1}^{n}\left|S_{x_k}^{|T|}(\omega) + jS_{x_k}^{\beta}(\omega)\right| t_{x_k} \le t_o\end{aligned} \tag{9}$$

where t_o is the tolerance of the deviation of output, Q_o,

$$t_o = \max\left\{\sum_{k=1}^{n}\left|S_{x_k}^{T}\right| t_{x_k}\right\} \tag{10}$$

and t_{x_k} is the kth parameter tolerance defined as

$$\max\left|\frac{\Delta x_k}{x_k}\right| = t_{x_k}. \tag{11}$$

With this formula, a designer can evaluate the upper bound for the overall relative deviation at the output of a transducer, once parameter variations are known.

Calculating Optimum Parameter Tolerance

The optimum parameter tolerances are defined as the tolerances whose contributions to the upper bound of the relative error, $|\Delta Q_o(\omega)/Q_o(\omega)|$, at the output of the transducer are equal to each other. This type of definition of optimum tolerances is quite reasonable since the designer expects the contribution of each parameter variation to output deviation to be equal. In general, the designer does not know in advance how much each parameter contributes to the output error. Using the above definition of the optimum tolerances we are sure that, considering the upper bound of the error, all parameter deviations contribute equally to the output deviation. Moreover, optimum tolerances are generally not equal to each other. Formulation of these tolerances was given by Erdal *et al.* (2001). Considering this fact, we can define the optimum parameter tolerances as

$$t_{x_k} = t_o/n\left|S^T_{x_k}(\omega_k)\right|, k = 1, ..., n \tag{12}$$

where t_{x_k} is the kth parameter tolerance, t_o is the tolerance of the deviation of output of the transducer n is the parameter number, and ω_k is the angular frequency at which $\left|S^T_{x_k}(\omega)\right|$ takes its maximum value, i.e.

$$\left|S^T_{x_k}(\omega_k)\right|_{\max} = \max\left\{\left|S^T_{x_k}(\omega)\right|\right\}, \quad \omega \in [\omega_1, \omega_2] \tag{13}$$

where $\omega \in [\omega_1, \omega_2]$ describes the designer's specified frequency band. Hence $\left|S^T_{x_k}(\omega)\right| \leq \left|S^T_{x_k}(\omega_k)\right|$, $\omega \in [\omega_1,\omega_2]$. It should be noted that ω_k belongs to the interval $\omega \in [\omega_1,\omega_2]$, and $\left|S^T_{x_k}(\omega)\right|$ has its maximum value at this frequency. The designer can easily determine ω_k by plotting $\left|S^T_{x_k}(\omega)\right|$ at this interval or by using already existing mathematical programs like Matlab or Mathcad.

Definition of Sensitivity Measure

According to Blostein's definition (Blostein, 1963; Erdal, 1998b, 1998c) a sensitivity measure, M_o, can be calculated as

$$M_o(\omega) = \sum_{k=1}^{n} \left|S^T_{x_k}\right| \quad . \tag{14}$$

M_o can be used not only to improve the sensitivity performance of a transducer but also allows comparing various transducers with the same input-output relation at a certain frequency.

Example

In the following, according to the proposed formula, the optimum parameter tolerances will be calculated for an accelerometer whose schematic diagram is shown in Figure (Carvalho, 1993; Doebelin, 1975; Erdal, 1998a)

The transfer function between the output voltage of the accelerometer and acceleration to be measured can be obtained in the s-domain as (Carvalho, 1993; Doebelin, 1975)

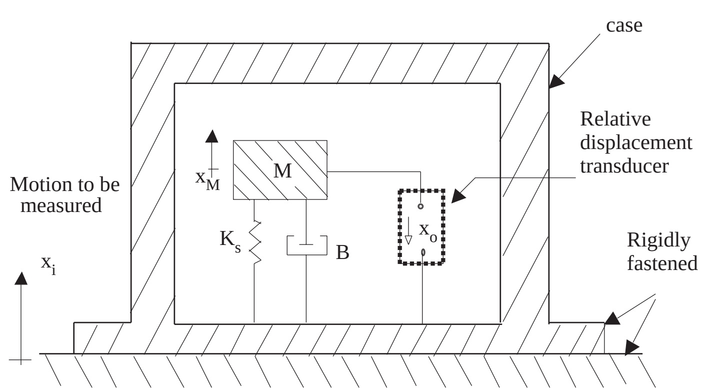

Figure. An accelerometer, where $E_0 = K_e X_0$.

$$\frac{E_o(s)}{A(s)} = T(s) = \frac{MK_e}{Ms^2 + Bs + K_s} \tag{15}$$

where

$e_o(t)$	output voltage $=K_e\, x_0$
$x_0(t)$	relative displacement
K_e	relative displacement transducer constant
a(t)	dynamic acceleration to be measured
M	seismic mass
B	viscous-friction coefficient
K_s	spring constant

Considering

$$x_k \in \{M, B, K_s, K_e\} \tag{16a}$$

$$\frac{\Delta x_k}{x_k} \in \left\{ \frac{\Delta M}{M}, \frac{\Delta B}{B}, \frac{\Delta K_s}{K_s} \frac{\Delta K_e}{K_e} \right\} \quad , \tag{16b}$$

the parameter sensitivities can be calculated from Eq.(15) according to the proposed formula and be substituted into Eq.(9) to obtain the upper bound for the overall relative deviation in the output voltage of the accelerometer as follows:

$$\begin{aligned} \left|\tfrac{\Delta E_o(\omega)}{E_o(\omega)}\right| &\le \left[(-MK_s\omega^2 + B^2\omega^2 + K_s^2)^2 + B^2M^2\omega^6\right]^{1/2} t_M/N(\omega) \\ &+ \left[(-B^2\omega^2)^2 + (BM\omega^3 - BK_s\omega)^2\right]^{1/2} t_B/N(\omega) \\ &+ \left[(MK_s\omega^2 - K_s^2)^2 + B^2K_s\omega^2\right]^{1/2} t_{K_s}/N(\omega) + t_{K_e} \le t_o \end{aligned} \tag{17}$$

where $N(\omega)$ is the denominator polynomial

$$N(\omega) = (-M\omega^2 + K_s)^2 + B^2\omega^2. \tag{18}$$

Let us give the acceleration specifications as follows

$$\begin{aligned} &M = 0.05kg; K_s = 3 \times 10^3 N/m; \\ &B = 17.15 N/m/s; \\ &K_e = 3V/mm; \zeta = 0.7 \end{aligned} \tag{19}$$

where ζ is the damping ratio, calculated from $\zeta = B/2\sqrt{K_s M}$ for this example.

Assuming that the designer wants $|\Delta E_o/E_o|$ to be less than or equal to 0.01, then the optimum parameter tolerances are obtained from Eq.(12) as follows:

$$t_M = 0.2\%, t_B = t_{K_e} = t_{K_s} = 0.25\% \tag{20}$$

Choosing the parameter tolerances as given above, the designer can guarantee that the maximum deviation in the output voltage of the accelerometer due to parameter variations caused by environmental effects will be less than or equal to 0.01.

If the designer wants $|\Delta E_o/E_o|$ to be less than or equal to 0.1, then the parameter tolerances chosen must be ten times larger than the ones in Eq.(20) and so forth.

For example, the values of the parameter sensitivities can be found from plottings in Mathcad mathematical program for $\omega_k = 100$ rad/s as follows:

$$\left|S_M^T\right| = 1.14, \left|S_B^T\right| = 0.566, \left|S_{K_e}^T\right| = 1, \left|S_{K_s}^T\right| = 0.99 \tag{21}$$

Using Eq. (21), the sensitivity measure M_o can be calculated for this accelerometer with the given set of parameters from Eq.(14) as follows:

$$M_o(\omega_k = 100\, rad/s) = 3.696 \tag{22}$$

This sensitivity measure can be used to improve the sensitivity performance of the transducer and to compare various transducers with different sets of design parameter values that realize the same transfer function.

Conclusion

Using the parameter sensitivities of the transfer function of a transducer, a general method is proposed to determine the optimum parameter tolerances for any kind of transducer by an appropriate approach. If the parameter tolerances chosen are less than or equal to the optimum parameter tolerances, the relative error in the the output quantity of a transducer due to the parameter variations always stays within the prescribed tolerance region denoted by t_o. Furthermore, a sensitivity measure is defined and calculated. This sensitivity measure, M_o, can be used not only to improve the sensitivity performance of a transducer but also allows the comparison of various transducers with the same input-output relation at a certain frequency.

Nomenclature

T(s)	transfer function
a_i	ith coefficient of the denominator of the transfer function
b_j	jth coefficient of the nominator of the transfer function
x_k	kth parameter
$S^T_{x_k}(s)$	sensitivity of the transfer function with respect to the kth parameter
t_{x_k}	kth parameter tolerance
t_o	tolerance of the deviation of the output of the transducer
e(t)	voltage, V
$x_0(t)$	relative displacement, m
K_e	relative displacement transducer constant, V/m
a(t)	dynamic acceleration to be measured, m/s^2
M	seismic mass, kg
B	viscous-friction coefficient, N/m/s
K_s	spring constant, N/m
ζ	damping ratio

References

Acar, C., Sensitivity and Tolerance Analysis, İ.T.Ü. Press, İstanbul, 1979.

Blostein, M.L., "Some Bounds on Sensitivity in RLC Networks", Proc. 1st Allerton Conf. Circuit System Theory, 488-501, 1963.

Doebelin, E.O., Measurement Systems, McGraw-Hill Book Company, USA, 1975.

Erdal C., "A Sensitivity Measure for Armature-Controlled DC Servomotors and Calculating Optimum Parameter Tolerances" Proceedings of IEEE 8th Mediterranean Electrotechnical Conference, MELECON'96, Bari, Italy, v.1, 342-345, 1996.

Erdal, C., "On the Determination of the Optimum Parameter Tolerances for a Positional Servo System: A Sensitivity Approach", Proceedings of the 23^{rd} Annual International Conference on Industrial Electronics, Control, and Instrumentation, IECON'97, New Orleans, Louisiana, USA, v.1, 1-3, 1997.

Erdal, C., "A Sensitivity Measure for an Accelerometer and a Method for Calculation Optimum Parameter Tolerances", Proceedings of 7th IASTED International European Conference on Circuit Theory and Design, MIC'98, Grindelwald, Switzerland, 207-210, 1998.

Erdal, C., "Determination of the Optimum Parameter Tolerances for a Permanent Magnet-step Motor: A Sensitivity Approach", Proceedings of the Mediterranean Electrotechnical Conference, MELECON'98, Tel-Aviv, Israel, 2, 1161-1164, 1998.

Erdal C., "A Sensitivity Measure for an Electronic, Proportional-Integral (PI) Controller and a Method for Calculating Optimum Parameter Tolerances", UKACC International Conference on CONTROL'98, IEE Conference Publication, Swansea, UK, 1, 275-277, 1998.

Erdal, C., Toker, A. and Acar, C., "Ota-C Based Proportional-Integral-Derivative (PID) Controller and Calculating Optimum Parameter Tolerances", Turk. J. Elec. Engin., 9, 2, 189-198, 2001.

Ghausi, M.S., and Laker, K.R., Modern Filter Design, Prentice-Hall Inc., NJ, 1981.

Goldstein, A.J. and Kuo, F.F., "Multiparameter Sensitivity", IRE Transactions on Circuit Theory, CT-18, 177-178, 1961.

Huelsman, L.P., Active and Passive Analog Filter Design, McGraw-Hill Inc., USA, 1993.

Martins De Carvalho, J.L., Dynamical Systems and Automatic Control, Prentice-Hall Inc. New York, 1993.