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The Buckling of Laminated Cylindrical Thin Shells under Torsion Varying as a Linear Function of Time

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Abstract

The buckling of laminated orthotropic cylindrical thin shells under torsion, which is a linear function of time, has been investigated. First, fundamental relations and the modified Donnell type stability equations of the laminated cylindrical thin shells are derived. Applying Galerkin's method, a differential equation having a variable coefficient depending on time is obtained and by applying the Ritz-type variational method to these equations, general formulas for static and dynamic critical loads, corresponding wave numbers and the dynamic factor are obtained.

Finally after performing the computations, the effects of the variations of the numbers and ordering of layers, loading speed, and the ratio of radius to thickness on the critical parameters are investigated.

Key words: Buckling, Laminated, Orthotropic shell, Torsion, Dynamic critical load, Dynamic factor, Wave numbers.

Introduction

There is no need to argue that composite shells are important in modern technology. It is, therefore, not surprising that a sizeable number of investigations have been concerned with the vibration and static buckling of isotropic and anisotropic or, as a special but important case, orthotropic shells. The buckling of cylindrical shells under torsion (torsional buckling) has been much less studied in contrast to buckling under axial compression and external pressure, especially for anisotropic shells. The pioneering studies in the static buckling of shells under torsion are as follows: Donnell (1933) studied the stability of thin-walled tubes, Batdorf *et al.* (1947) worked on the buckling of circular cylindrical shell, Weingarten (1962) studied the stability of circular cylindrical shells with an elastic core, Seide (1962) and Yamaki and Tani (1969) studied the buckling problem of conical shells under torsion, and Kunukasseril (1967) studied the buckling problem of multi-layered anisotropic cylindrical shells. Volmir (1967) and Leissa (1973) collected and reviewed the comprehensive literature dealing with the buckling and vibration of homogeneous shells, made of either isotropic or orthotropic materials. Recently, some research studies have been published concerning the buckling and vibration of one-layered and laminated shells under torsion (Chandrasekaran, 1977; Tani, 1980; Tani, 1981; Kim, *et al.* 1999; Hui and Du, 1987; Tabiei and Simitses, 1994; Mao and Lu, 1999; Tan, 2000 and Park, *et al.* 2001).

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In practice, time-dependent compressive loads are encountered not only as static and periodic functions but also as linear (Khashba, 1975; Sachenkov and Baktieva 1978) and power functions (Aksogan and Sofiyev, 2000, 2002; Sofiyev and Aksogan, 2001, 2002). It is well known that due to the difficulties emerging during the solution and theoretical analysis of the dynamic buckling problems of shells under such torsional loads this subject has not been studied sufficiently. One such problem, not considered previously, is the dynamic buckling of laminated orthotropic cylindrical shells under the effect of torsion, which is a linear function of time.

The aim of the present research is to study the buckling problem of laminated cylindrical thin shells made of orthotropic composite materials subjected to torsion varying as a linear function of time by using the Ritz-type variational method.

Problem formulation

Consider a circular cylindrical shell of medium length as shown in Figure 1. It is assumed to be thin, laminated and composed of N layers of equal thickness δ of orthotropic composite materials perfectly bonded together. The shell is of the length L, thickness 2h and radius R. The origin of the coordinate system is taken as the midpoint of the length of the shell. The x-axis is taken along a generator and the y-axis is taken in tangential directions, and the z-axis is normal to them. The axes of orthotropy in all layers are parallel to the x- and y-axes (see Figure 1).

The contact condition between any two consecutive layers is one of perfectly rigid bonding, ensuring the satisfaction of the Kirchhoff-Love hypothesis for the whole shell, meaning that there is a single displacement and a single strain expression for the whole shell and that the pressures at the contact surfaces do not need any particular attention. During the deformation there is no slip and no loss of contact between the contact surfaces of the layers. The middle surface $z = 0$ is located at a layer interface for even values of N, whereas for odd values of N the middle surface is located at the center of the middle layer.

For a thin shell, after imposing the following three assumptions about the nonlinear terms (Volmir, 1967; Mao and Lu, 1999).

(a) The only load is the boundary torques, so that all nonlinear terms not related to the torsional force n_{12}^0 are neglected;

(b) All nonlinear terms not related to w or its derivatives are neglected;

(c) All nonlinear terms not in the dynamic stability and compatibility equations of a laminated cylindrical shell are neglected;

the dynamic stability and compatibility equations simplifies to the following form:

$$\begin{aligned}&\frac{\partial^2 m_{11}}{\partial x^2}+2\frac{\partial^2 m_{12}}{\partial x\partial y}+\frac{\partial^2 m_{22}}{\partial y^2}+\frac{n_{22}}{R}+\\&n_{11}^0\frac{\partial^2 w}{\partial x^2}+2n_{12}^0\frac{\partial^2 w}{\partial x\partial y}+n_{22}^0\frac{\partial^2 w}{\partial y^2}=\rho_1\frac{\partial^2 w}{\partial t^2}\end{aligned} \quad (1)$$

$$\frac{\partial^2 \varepsilon_{11}^0}{\partial y^2}+\frac{\partial^2 \varepsilon_{22}^0}{\partial x^2}-2\frac{\partial^2 \varepsilon_{12}^0}{\partial x\partial y}=-\frac{1}{R}\frac{\partial^2 w}{\partial x^2} \quad (2)$$

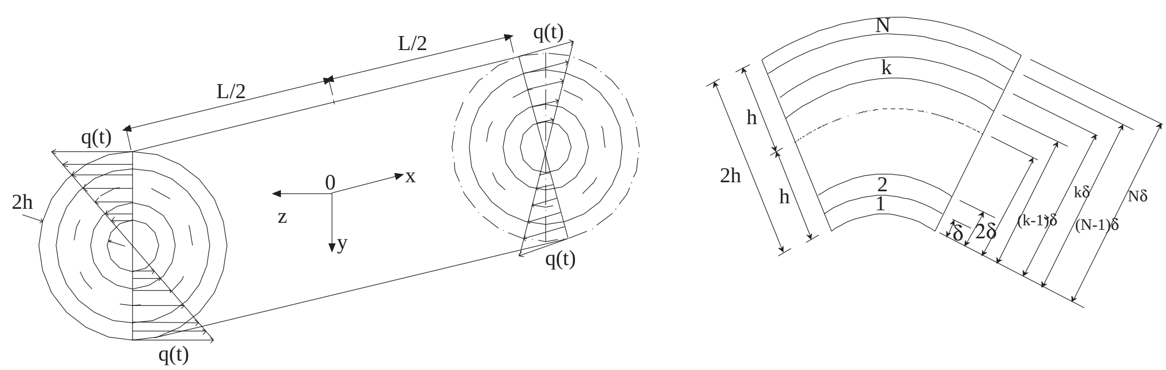

Figure 1. Geometry and the cross-section of a cylindrical thin shell with N layers.

where w is the small incremental displacement of the middle surface in the normal direction, $m_{ij}, (i, j = 1, 2)$ are the moment resultants, $n_{ij}, (i, j = 1, 2)$ are the force resultants, $n_{ij}^0, (i, j = 1, 2)$ are the membrane forces for the condition with zero initial moments, ε_{ij}^0, $(i, j = 1, 2)$are the strain components on the middle surface and t is time. Additionally,

$$\rho_1 = \frac{2h}{N} \sum_{k=0}^{N-1} \rho^{(k+1)} \tag{3}$$

where $\rho^{(k+1)}$ is the density of the material for the layer $k + 1$.

The ends of the cylindrical shell are subjected to uniform torsional stresses varying as a linear function of time in the following form:

$$n_{11}^0 = 0, \quad n_{22}^0 = 0 \quad , \quad n_{12}^0 = -S_0 t \tag{4}$$

where S_0 is the loading speed.

According to the shell theory, the stress-strain relations for a thin laminated layer are given as follows:

$$\begin{pmatrix} \sigma_{11}^{(k+1)} \\ \sigma_{22}^{(k+1)} \\ \sigma_{12}^{(k+1)} \end{pmatrix} = \begin{bmatrix} Q_{11}^{(k+1)} & Q_{12}^{(k+1)} & 0 \\ Q_{12}^{(k+1)} & Q_{22}^{(k+1)} & 0 \\ 0 & 0 & Q_{33}^{(k+1)} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix} \tag{5}$$

where $\sigma_{ij}^{(k+1)}$, $(i, j = 1, 2, 3)$ are the stresses in the layers. The quantities $Q_{ij}^{(k+1)}, (i, j = 1, 2, 3)$ for orthotropic lamina are

$$Q_{11}^{(k+1)} = \frac{E_1^{(k+1)}}{1 - \nu_{12}^{(k+1)} \nu_{21}^{(k+1)}}, \; Q_{22}^{(k+1)} = \frac{E_2^{(k+1)}}{1 - \nu_{12}^{(k+1)} \nu_{21}^{(k+1)}}, \tag{6}$$

$$\begin{aligned} Q_{12}^{(k+1)} &= \nu_{21}^{(k+1)} Q_{11}^{(k+1)} = \nu_{12}^{(k+1)} Q_{22}^{(k+1)}, \\ Q_{33}^{(k+1)} &= 2G^{(k+1)}, \quad k = 0, 1, 2, ..., (N-1) \end{aligned} \tag{7}$$

where $E_1^{(k+1)}$ and $E_2^{(k+1)}$ are the Young's moduli in the x and y directions for the layer k+1, respectively, $G^{(k+1)}$ is the shear modulus on the plane of the layer k+1, $\delta = 2h/N$ is the thickness of the layers, $\nu_{12}^{(k+1)}$ and $\nu_{21}^{(k+1)}$ are the Poisson's ratios, assumed to be constant.

By Love's first approximation theory the strain-displacement relations are given by

$$(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}) = \left(\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0\right) + z\left(\chi_{11}, \chi_{22}, \chi_{12}\right) \tag{8}$$

where χ_{11} and χ_{22} are the curvatures of the deformed shell in the directions x and y, respectively, whereas χ_{12} is the twist of the middle surface. The last three entities are given by

$$(\chi_{11}, \; \chi_{22}, \; \chi_{12}) = \left(-\frac{\partial^2 w}{\partial x^2}, \; -\frac{\partial^2 w}{\partial y^2}, \; -\frac{\partial^2 w}{\partial x \partial y}\right) \tag{9}$$

The force and moment resultants are defined in the following manner:

$$(n_{ij}, m_{ij}) = \sum_{k=0}^{N-1} \int_{-h+k\delta}^{-h+(k+1)\delta} \sigma_{ij}^{(k+1)}(1, z)\, dz \quad (i, j = 1, 2) \tag{10}$$

The relations between the forces n_{ij}, $(i, j = 1, 2)$ and the stress function ϕ are given by

$$(n_{11}, \; n_{22}, \; n_{12}) = \left(\frac{\partial^2 \phi}{\partial y^2}, \; \frac{\partial^2 \phi}{\partial x^2}, \; -\frac{\partial^2 \phi}{\partial x \partial y}\right) \tag{11}$$

Substituting expressions (4-11) in (1-2) a system of differential equations for the stress function ϕ and the normal displacement of the middle surface w can be obtained in the form

$$\begin{aligned} & c_{12} \frac{\partial^4 \phi}{\partial x^4} + (c_{11} - 2c_{31} + c_{22}) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + c_{21} \frac{\partial^4 \phi}{\partial y^4} - \\ & c_{13} \frac{\partial^4 w}{\partial x^4} - (c_{14} + 2c_{32} + c_{23}) \frac{\partial^4 w}{\partial x^2 \partial y^2} - \\ & c_{24} \frac{\partial^4 w}{\partial y^4} + \frac{1}{R} \frac{\partial^2 \phi}{\partial x^2} - S_0 t \frac{\partial^2 w}{\partial x \partial y} = \rho_1 \frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{12}$$

$$\begin{aligned} & b_{22} \frac{\partial^4 \phi}{\partial x^4} + (b_{12} + 2b_{31} + b_{21}) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + b_{11} \frac{\partial^4 \phi}{\partial y^4} - \\ & b_{23} \frac{\partial^4 w}{\partial x^4} - (b_{13} - 2b_{32} + b_{24}) \frac{\partial^4 w}{\partial x^2 \partial y^2} - \end{aligned}$$

$$b_{14}\frac{\partial^4 w}{\partial y^4}=-\frac{1}{R}\frac{\partial^2 w}{\partial x^2} \quad (13)$$

where b_{ij} and $c_{ij},(i,j=1,\,2,\,3,\,4)$ are given in the Appendix.

Solution of the differential equations

In many practical cases, however, the real physical boundaries may be described better by mixed boundary conditions than by completely clamped boundary conditions. The boundary conditions of this problem are mixed boundary conditions. The solution of the system of Eqs. (11-12) is sought in the following form:

$$\begin{aligned} w&=\xi_1(t)\sin\tfrac{\pi x}{L}\sin\tfrac{n}{R}(y+\gamma x),\\ \phi&=\xi_2(t)\sin\tfrac{\pi x}{L}\sin\tfrac{n}{R}(y+\gamma x)\end{aligned} \quad (14)$$

where n is the wave number in the direction of the y axis, γ is the tangent of the angle between the waves and x axes, and $\xi_1(t)$ and $\xi_2(t)$ are the time-dependent amplitudes.

When $x=\pm L/2$ in expressions (14), although $w=0$ and $\phi=0$ conditions are satisfied, neither simple support nor clamped boundary conditions are satisfied. Both boundary conditions are satisfied when integrated from 0 to $2\pi R$, if $x=\pm L/2$ (Volmir, 1967; Mao and Lu, 1999).

Substituting expressions (14) in the equation set (12-13) and eliminating $\xi_2(t)$, applying Galerkin's method in the ranges $-L/2\le x\le L/2$ and $0\le y\le 2\pi R$, the differential equation

$$\frac{d^2\xi_1(\tau)}{d\tau^2}+(\lambda-\lambda_0\tau)\,\xi_1(\tau)=0 \quad (15)$$

is obtained, where $t=\tau t_{cr}$, in which t_{cr} is the critical time and the dimensionless time parameter τ satisfies $0\le\tau\le 1$ and the following definitions apply:

$$\lambda_0=\frac{S_0 t_{cr}^3}{\rho_1}\frac{m_1 n}{R^2},\quad m_1=n\gamma+\frac{\pi R}{L} \quad (16)$$

$$\lambda=\frac{t_{cr}^2}{\rho_1 R^4}\left\{\left[c_{13}m_1^4+(c_{14}+2c_{32}+c_{23})\,m_1^2n^2+c_{24}n^4\right]+\frac{\left[m_1^2R-c_{12}m_1^4-(c_{11}-2c_{31}+c_{22})\,m_1^2n^2-c_{21}n^4\right]}{b_{22}m_1^4+(b_{12}+2b_{31}+b_{21})\,m_1^2n^2+b_{11}n^4}\times\left[b_{23}m_1^4+(b_{13}-2b_{32}+b_{24})\,m_1^2n^2+b_{14}n^4+m_1^2R\right]\right\} \quad (17)$$

An approximating function will be chosen as $\xi_1(\tau)=Ae^{2\tau}\tau[4/3-\tau]$satisfying the initial conditions $\xi_1(0)=0,\quad \partial\xi_1(1)/\partial\tau=0$, where the displacement amplitude A is found from the condition of transition to the static condition.

After multiplying (15) by $\xi_1'(\tau)$ and integrating it with respect to τ, from 0 to τ and from 0 to 1, in that order (i.e., Ritz-type variational method yields) for cylindrical shells of medium length, the wave number n satisfies the inequality $n\gamma<<1$ and $n^4>>m_1^4$; after some mathematical operations the following equation is found for determining the critical load and corresponding wave number (Sofiyev and Aksogan, 2002):

$$S_{crd}=S_0t_{cr}=7.3q_1^{5/8}q_2^{3/8}m_2^{1/2}\Omega^{3/10} \quad (18)$$

$$n_d=\left(q_2m_2^4/q_1\right)^{1/8}\Omega^{1/10} \quad (19)$$

in which the new constants are defined as follows:

$$\begin{aligned}\Omega&=\tfrac{S_0^2R^2\rho_1}{50q_1^{7/4}q_2^{5/4}m_2^3},\quad m_2=\tfrac{\pi R}{L},\\ q_1&=\tfrac{c_{24}b_{11}-c_{21}b_{14}}{b_{11}R^2},\quad q_2=\tfrac{1}{b_{11}}\end{aligned} \quad (20)$$

In the static case ($t_{cr}\to\infty, S_0\to 0$), the static critical load and corresponding wave number is found to be

$$S_{cr}^{st}=1.938q_1^{5/8}q_2^{3/8}m_2^{1/2} \quad (21)$$

$$n_{st}^8=1.66q_2m_2^4/q_1 \quad (22)$$

The pre-buckling shear stress in the cylindrical shell can be determined using the static equilibrium condition as follows:

$$T=2\pi R^2S_{cr}^{st}=26.5828q_1^{5/8}q_2^{3/8}R^{5/2}/L^{1/2} \quad (23)$$

From $K_d=S_{crd}/S_{cr}^{st}$, the dynamic factor is given by

$$K_d=0.4158\left(\frac{S_0^2L^3\rho_1}{q_1^{7/4}q_2^{5/4}R}\right)^{3/10} \quad (24)$$

When $N = 1$ the appropriate formulas for a one-layered cylindrical shell made of a homogeneous orthotropic material are found to be a special case of expressions (18-19,21-24).

Numerical computations and results

The numerical computations for cross-ply laminated cylindrical shells up to 10 layers are considered (Figure 2). The cross-ply laminates are composed of lamina (plies) with their principal material directions (one being the fiber direction) aligned with the axial x-axis and the circumferential y-axis of the shell. That is, the fibers in one layer are aligned in the axial direction, whereas the fibers in the next layer are aligned in the circumferential direction. Theoretically any sequence of orientations between 0° (x-direction) and 90° (y-direction) can be considered (Jones and Morgan, 1975; Reddy, 1997).

The numerical computations were carried out for graphite/epoxy composites with the following material properties (Jones and Morgan, 1975; Ng, *et al.,* 1997; Greenberg and Stavsky, 1998), shell parameters (Mao and Lu, 1999; Tan, 2000) and loading (Sachenkov and Baktieva, 1978; Khashba, 1975):

$$E_1^{(k+1)} = 1.724 \times 10^5 MPa,$$

$$E_2^{(k+1)} = 7.79 \times 10^3 MPa,$$

$$\nu_{12}^{(k+1)} = 0.35, \quad \nu_{21}^{(k+1)} = 0.016,$$

$$\rho^{(k+1)} = 1.53 \times 10^3 kg/m^3,$$

$$R/(2h) = 100;\ 200, \quad L/R = 4;\ 5,$$

$$S_0 = 31.6 \div 6.325 \times 10^2 MPa/s$$

Table 1 shows the variation of the dynamic critical load and dynamic factor, with the ratio R/(2h), number and ordering of layers. When the number of layers increases, odd numbers of layers with (90°/0°/...) ordering shells, the values of the critical load decrease and the values of dynamic factor increase. The roles of the odd numbers of layers with (90°/0°/...) and (0°/90°/...) ordering shells are interchanged. In the cases of shells having even numbers of layers, when the number of layers increase, the values of the dynamic critical load increase and the values of dynamic factor decrease. For the shells that have more than 10 layers, the values of the dynamic critical load and dynamic factor are almost the same without considering the order of the layers. The values of the dynamic critical load increase with an increase in the ratio $R/(2h)$, whereas the values of the dynamic factor decrease. As pointed out above, for an even number of layers the values of the critical parameters are independent of the stacking sequences (90°/0°/ . . .) and (0°/90°/ . . .).

For different numbers and orderings of layers variations of dynamic critical load and dynamic factor versus loading speed S_0 are tabulated in Table 2. When loading speed S_0 increases, the values of the critical parameters increase. It is considered that in (0°/90°...) ordered shells, the effect of the variation of loading speed to dynamic factor is more important.

Before the comparison, we must take the following points into consideration in this study, the aim is to find the values of dynamic critical parameters. The dynamic factor can be found by finding the upper values of static critical load. The approximate formula for the upper values of static critical load is obtained from the dynamic case, particularly. Consequently, finding static critical load is not taken up alone such as in other similar studies. Despite this, the upper values of static critical load are consistent with other studies, with which we compared the results.

Table 3 presents the comparison of the present results with those of Vinson and Sierakowski (1986), and Tan (2000). Comparisons were carried out for the following material properties and one-layered (0°) shell parameters:

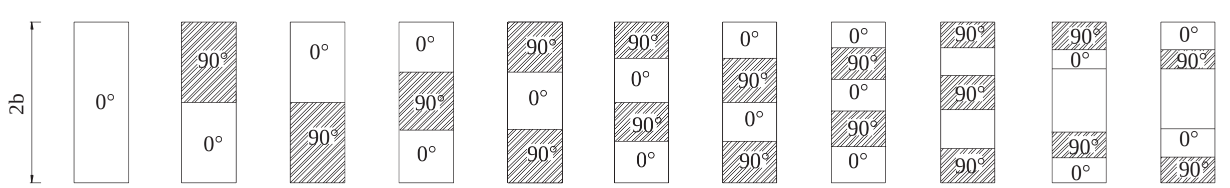

Figure 2. Contrast between symmetric and anti-symmetric cross-ply laminates.

Table 1. Variations of the dynamic critical load and dynamic factor for $R/(2h) = 100;\ 200$, $S_0 = 20000 MPa/s$, $L/R = 4$ and for $k = 0, 1, 2, ..., 9$.

N	Stacking sequence	$R/(2h)$=100		$R/(2h)$=200	
		$S_{crd}\,(MPa)$	K_d	$S_{crd}\,(MPa)$	K_d
1	(0°)	27.206	11.276	41.239	3.593
1	(90°)	37.085	7.086	56.210	2.258
2	(0°/90°)	31.495	6.660	47.737	2.122
2	(90°/0°)	31.495	6.660	47.737	2.122
3	(0°/90°/0°)	28.827	9.594	43.693	9.594
3	(90°/0°/90°)	36.951	3.298	56.008	1.051
4	(0°/90°/...)	34.167	4.343	51.788	1.384
4	(90°/0°/...)	34.167	4.343	51.788	1.384
5	(0°/90°/...)	32.203	5.565	48.810	1.773
5	(90°/0°/...)	36.272	3.422	54.979	1.090
6	(0°/90°/...)	34.504	4.124	52.299	1.314
6	(90°/0°/...)	34.504	4.124	52.299	1.314
7	(0°/90°/...)	33.099	4.900	50.168	1.561
7	(90°/0°/...)	35.908	3.526	54.426	1.223
8	(0°/90°/...)	34.616	4.055	52.468	1.292
8	(90°/0°/...)	34.616	4.055	52.468	1.292
9	(0°/90°/ ...)	33.527	4.625	50.817	1.474
9	(90°/0°/...)	35.684	3.599	54.086	1.147
10	(0°/90°/ ...)	34.666	4.024	52.544	1.282
10	(90°/0°/...)	34.666	4.024	52.544	1.282

Table 2. Variations of the dynamic critical load and dynamic factor for different loading speed and for different number and ordering of layers ($R/(2h) = 200$, $L/R = 4$).

S_0	$S_{crd}\,(MPa)$				K_d			
(MPa / s)	(0°)	(0°/90°)	(0°/90°/0°)	(90°/0°/90°)	(0°)	(0°/90°)	(0°/90°/0°)	(90°/0°/90°)
1000	4.509	5.2194	4.7773	6.1237	1.869	1.2037	1.5899	0.5466
1500	5.751	6.6570	6.0930	7.8103	2.383	1.4076	2.0277	0.6972
2000	6.834	7.9111	7.2410	9.2817	2.832	1.6728	2.4098	0.8285
2500	7.813	9.0445	8.2784	10.6497	3.238	1.9125	2.7551	0.9472

Isotropic material properties.......................... $E = 10^5\,MPa$, $\nu = 0.3$
Orthotropic material properties......................$E_1 = 10^5\,MPa$, $E_2 = 2x10^4\,MPa$, $\nu = 0.3$
Shell properties............................. ... ... $2h = 0.00001m$, $R = 0.1m$, $L = 0.5\,m$.

Table 3. Comparison of critical parameters with those of Vinson and Sierakowski (1986), and of Tan (2000),

$T\,(Nm)$					
Isotropic material			Orthotropic material		
Tan (2000)	Vinson and Sierakowski (1986)	Present	Tan (2000)	Vinson and Sierakowski (1986)	Present
0.2092	0.2154	0.2166	0.0713	0.0756	0.07922

The laminated cylindrical shell studied by Tabiei and Simitses (1994) and Mao and Lu (1999) have the ply properties

$$E_1 = 149619\,MPa, \quad E_2 = 9928\,MPa, \quad \nu_{12} = 0.28.$$

and the stacking sequence (0°/90°/0°). For instance, for R/(2h) = 100 and L/R = 5, the present theory gives static buckling loads (in $10^6\,N/m$) $S^*_{cr} = 2\pi R S^{st}_{cr} = 0.0847$ and $S^*_{cr} = 0.0757$ given by Tabiei and Simitses (1994), $S^*_{cr} = 0.0751$ given by Mao and Lu (1999).

For $N = 1$, $E_1 = E_2 = E$ and $\nu_{12} = \nu_{21} = \nu$ the appropriate formulas for a one-layered cylindrical shell made of isotropic material are found as a special case of Eqs. (18, 24), see as an example Sachenkov and Baktieva (1978). Therefore, the numerical calculations done for the formulas (18) were compared with the numerical results of Sachenkov and Baktieva (1978) for an isotropic cylindrical shell subjected to torsion varying linearly with time and a good match was observed.

Conclusions

In the present study the buckling of laminated orthotropic composite cylindrical thin shells under torsion, which is a linear function of time, was studied. At first, the fundamental relations and modified Donnell type dynamic buckling equations were written for a shell subject to torsion, which is a linear function of time. Then, applying Galerkin's method, a time-dependent differential equation with variable coefficients was obtained. Finally, the critical parameters were found analytically by applying the Ritz-type variational method. The effects of the variations of the numbers and ordering of layers, loading speed, and the ratio of radius to thickness on the critical parameters were studied numerically.

Appendix

The coefficients c_{ij} and b_{ij}, $(i, j = 1, 2, 3, 4)$ appearing in some equations in this paper are defined as follows:

$$c_{11} = A^1_{11} b_{11} + A^1_{12} b_{21}, \quad c_{12} = A^1_{11} b_{12} + A^1_{12} b_{22}, \quad c_{13} = A^1_{11} b_{13} + A^1_{12} b_{23} + A^2_{11},$$

$$c_{14} = A^1_{11} b_{14} + A^1_{12} b_{24} + A^2_{12}, \quad c_{21} = A^1_{21} b_{11} + A^1_{22} b_{21}, \quad c_{22} = A^1_{21} b_{12} + A^1_{22} b_{22},$$

$$c_{23} = A^1_{21} b_{13} + A^1_{22} b_{23} + A^2_{21}, \quad c_{24} = A^1_{21} b_{14} + A^1_{22} b_{24} + A^2_{22}, \quad c_{31} = A^1_{33} b_{31},$$

$$c_{32} = A^1_{33} b_{32} + A^2_{33}, \quad b_{11} = A^0_{22} D, \quad b_{12} = -A^0_{12} D, \quad b_{13} = \left(A^0_{12} A^1_{21} - A^1_{11} A^0_{22}\right) D,$$

$$b_{14} = \left(A^0_{12} A^1_{22} - A^1_{12} A^0_{22}\right) D, \quad b_{21} = -A^0_{21} D, \quad b_{22} = A^0_{11} D,$$

$$b_{23} = \left(A^0_{21} A^1_{11} - A^1_{21} A^0_{11}\right) D, \quad b_{24} = \left(A^0_{21} A^1_{12} - A^1_{22} A^0_{11}\right) D, \quad b_{31} = 1/A^0_{33},$$

$$b_{32} = -A^1_{33}/A^0_{33}, \quad D = 1/\left(A^0_{11} A^0_{22} - A^0_{21} A^0_{12}\right).$$

Finally, the expressions for the factors A_{ij}^q, $(i,j=1,2,3)$, $q=0,\,1,\,2$ are:

$$A_{11}^q = h^{q+1}\sum_{k=0}^{N-1}\frac{E_1^{(k+1)}\tilde{h}^{(k+1)}}{1-\nu_{12}^{(k+1)}\nu_{21}^{(k+1)}},\quad A_{12}^q = h^{q+1}\sum_{k=0}^{N-1}\frac{\nu_{21}^{(k+1)}E_1^{(k+1)}\tilde{h}^{(k+1)}}{1-\nu_{12}^{(k+1)}\nu_{21}^{(k+1)}},$$

$$A_{22}^q = h^{q+1}\sum_{k=0}^{N-1}\frac{E_2^{(k+1)}\tilde{h}^{(k+1)}}{1-\nu_{12}^{(k+1)}\nu_{21}^{(k+1)}},\quad A_{21}^q = h^{q+1}\sum_{k=0}^{N-1}\frac{\nu_{12}^{(k+1)}E_2^{(k+1)}\tilde{h}^{(k+1)}}{1-\nu_{12}^{(k+1)}\nu_{21}^{(k+1)}},$$

$$A_{33}^q = 2h^{q+1}\sum_{k=0}^{N-1}G^{(k+1)}\tilde{h}^{(k+1)},\quad \tilde{h}^{(k+1)} = \frac{\left(-1+2\,(k+1)\,/N\right)^{q+1}-\left(-1+2k/N\right)^{q+1}}{q+1}.$$

Nomenclature

$E_1^{(k+1)}, E_2^{(k+1)}$	Young's moduli of the orthotropic materials in the layers
E_1, E_2	Young's moduli of the orthotropic material in a single layer shell
E	Young's modulus of the isotropic material in a single layer shell
$G^{(k+1)}$	shear moduli of the materials in the layers
ϕ	stress function
$2h$	thickness of the shell
K_d	dynamic factor
L	length of the shell
$m_{ij}, (i,j=1,2)$	internal moments per unit length of the cross-section of the shell
$n_{ij}, (i,j=1,2)$	internal forces per unit length of the cross-section of the shell
$n_{ij}^0, (i,j=1,2)$	membrane forces prior to buckling
N	number of layers
n	wave number in the circumferential direction
n_{st}, n_d	wave number corresponding to the static and dynamic critical loads
R	radius of the shell
S_{cr}^{st}, S_{crd}	static and dynamic critical loads
S_0	loading speed
t	time
t_{cr}	critical time
w	displacement of the middle surface in the inwards normal direction z
$\chi_{ij}, (i,j=1,2)$	curvatures of the middle surface
$\varepsilon_{ij}, (i,j=1,2)$	strains in the curvilinear coordinate directions
$\varepsilon_{ij}^0, (i,j=1,2)$	strain components on the middle surface of the conical shell
τ	dimensionless time parameter
$\rho^{(k+1)}$	density of the materials in the layers
$\nu_{12}^{(k+1)}, \nu_{21}^{(k+1)}$	Poisson's ratios of the orthotropic materials in the layers
ν_{12}, ν_{21}	Poisson's ratios of the orthotropic material in a single layer shell
ν	Poisson's ratio of the isotropic material in a single layer shell
$\sigma_{ij}, (i,j=1,2)$	stress components
$\xi_1(t), \xi_2(t)$	time-dependent amplitudes

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