Analytical Solutions for Some Simple Flows of a Binary Mixture of Incompressible Newtonian Fluids

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Abstract

The problems dealing with some simple flows of a mixture of two incompressible Newtonian fluids have been analysed. By using the theory of binary mixtures of Newtonian fluids, the equations governing the velocity fields are reduced to a system of coupled ordinary differential equations. In the case of non-inertial flow the analytical solutions of these equations have been obtained for the following three problems: (i) the parallel flow with a free surface; (ii) the flow between intersecting planes, one of which is moving; (iii) the flow between two coaxial moving cylinders.

Key words: Mixture, Newtonian fluid, Non-inertial flow.

Introduction

Recently, there has been remarkable interest in flows of fluid mixtures due to the occurrence of these flows in industrial processes, particularly in lubrication practice. A familiar example is an emulsion, which is the dispersion of one fluid within another fluid. Typical emulsions are oil dispersed within water or water within oil. Such emulsions are of considerable practical interest because synthetic fluids are more toxic than mineral oils and are uneconomical to use in applications requiring large quantities of lubricant, for example, metal working, mining, cutting and hydraulic fluids. Several problems relating to the mechanics of oil and water emulsions have been considered within the context of the mixture theory by Al-Sharif *et al.* (1993), Chamniprasart *et al.* (1993), and Wang *et al.* (1993). Another example where fluid mixtures play an important role is in multigrade oils. In order to enhance the lubrication

properties of mineral oils, such as the viscosity index, polymeric type fluids are added to the base oil (Dai and Khonsari, 1994).

The origin of the modern formulation of continuum thermomechanical theories of mixtures goes back to papers written by Truesdell (1957). He presented a comprehensive treatment of the thermomechanics of interacting continua which discussed the appropriate forms for the balance of mass, momentum, energy and also the possible structure for the second law of thermodynamics. This work gave impetus to many studies on the theory of interacting continua and a rigorous and firm mathematical foundation has been developed. We refer the reader to the works of Bowen (1976), Atkin and Craine (1976b), Bedford and Drumheller (1983), and Rajagopal and Tao (1995) regarding the historical development of the theory and detailed analysis of various results on this subject.

In the present paper a binary mixture, each con-

stituent of which is an incompressible inert Newtonian fluid, is considered. In the following section the balance laws and relevant constitutive equations are briefly presented and then the equations governing the motion of the binary mixture are stated for the case of non-inertial flow. In the subsequent sections, we obtain the exact solutions for some simple flows of the binary mixture under consideration.

Basic theory

(i) Kinematics and balance laws

The governing equations are summarized in this section, for more details the reader should consult Craine (1971) and Atkin and Craine (1976a,b). Consider a mixture of two continua, in motion relative to each other. Let $\mathbf{X}_\beta$ represent the position of a material point of the β th constituent $\Re_\beta$ in its reference configuration. At any time t each spatial point $\mathbf{x}$ in the mixture is occupied simultaneously by one particle from each $\Re_\beta$. The motion of a binary mixture $\Re$ of components $\Re_\beta$ is denoted by

$$\mathbf{x} = \phi_\beta(\mathbf{X}_\beta, t), \quad t \geq 0, \quad \beta = 1, 2 \qquad (1)$$

where the function ϕ_β is called the deformation function for the β th constituent and is assumed to be sufficiently smooth so as to make the necessary mathematical operations correct. Throughout this paper the subscript β takes the values 1, 2.

If $\mathbf{v}_\beta$ denotes the velocity of the β th constituent, the material derivative D_β/Dt is defined by

$$\frac{D_\beta}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_\beta \cdot \nabla \qquad (2)$$

where ∇ is the gradient operator.

Let the density of the β th constituent, after mixing, be ρ_β, then the total density ρ of the mixture is given by

$$\rho = \sum_\beta \rho_\beta \qquad (3)$$

and the mean velocity, $\mathbf{w}$, of the mixture is defined by

$$\mathbf{w} = \frac{1}{\rho} \sum_\beta \rho_\beta \mathbf{v}_\beta \quad . \qquad (4)$$

The basic equations for a binary mixture in which the constituents have a common temperature $\Im$ and do not interact chemically are the following:

Continuity equations

$$\frac{D_1\rho_1}{Dt} + \rho_1 (\nabla \cdot \mathbf{v}_1) = 0, \quad \frac{D_2\rho_2}{Dt} + \rho_2 (\nabla \cdot \mathbf{v}_2) = 0. \qquad (5)$$

Equations of motion

$$\rho_1 \frac{D_1\mathbf{v}_1}{Dt} = \nabla \cdot \boldsymbol{\sigma}_1 - \mathbf{f} + \rho_1\mathbf{F}_1,$$
$$\rho_2 \frac{D_2\mathbf{v}_2}{Dt} = \nabla \cdot \boldsymbol{\sigma}_2 + \mathbf{f} + \rho_2\mathbf{F}_2. \qquad (6)$$

Energy equation

$$\sum_\beta \rho_\beta \frac{D_\beta U_\beta}{Dt} = \rho r - \nabla \cdot \mathbf{q} + \mathbf{f} \cdot (\mathbf{v}_1 - \mathbf{v}_2)$$
$$+ \sum_\beta tr\left[\boldsymbol{\sigma}_\beta \cdot (\nabla \mathbf{v}_\beta)^T\right] \qquad (7)$$

where the superscript T and tr denote transpose and trace of a second-order tensor field, respectively. The quantities U_β, $\boldsymbol{\sigma}_\beta$ and $\mathbf{F}_\beta$ are in turn internal energy per unit mass, partial stress and external body force acting on per unit mass of the β th constituent. In addition, r, the heat supply per unit mass, and $\mathbf{q}$, the heat flux, refer to the mixture as a whole, and $\mathbf{f}$ denotes the diffusive force[1] . It is important to bear in mind that the ij th component of $\nabla \mathbf{v}_\beta$ taken as $v_{\beta j ; i}$, where the semicolon stands for covariant differentiation.

Consideration of the balance of angular momentum for $\Re_1$ and $\Re_2$ shows that $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ need not be symmetric although the balance of angular momentum for the mixture results in the symmetry of $\boldsymbol{\sigma}$, the total stress in the mixture, defined by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2. \qquad (8)$$

Admissible thermomechanical processes in the mixture must be compatible with an entropy production inequality. If S_1 and S_2 are the entropies per unit mass of the constituents, then the Clausius-Duhem inequality may be written as follows (Green and Naghdi, 1969; Bowen and Wiese, 1969):

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[1]The diffusive force $\mathbf{f}$ may be interpreted as the drag exerted on one constituent due to the motion of the other (Craine, 1971).

$$\sum_\beta \rho_\beta \frac{D_\beta S_\beta}{Dt} - \frac{\rho r}{\Im} + \nabla \cdot \left(\frac{\mathbf{q}}{\Im}\right) \geq 0. \qquad (9)$$

(ii) Constitutive equations

In this work we shall concern ourselves with a mixture of two incompressible Newtonian fluids. Let the densities of $\Re_1$ and $\Re_2$, before mixing, be ρ_{10} and ρ_{20} respectively, which in view of the assumed incompressibility are constants. Introducing a composition factor γ, defined as the proportion by volume of the constituent $\Re_1$, and assuming that the mixture does not contain voids, it follows that

$$\rho_1 = \gamma \rho_{10}, \quad \rho_2 = (1 - \gamma)\rho_{20} \qquad (10)$$

and hence

$$\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1 \quad . \qquad (11)$$

By using (3) and (11), it can be easily shown that

$$\rho_1 = \frac{\rho_{10}(\rho_{20} - \rho)}{\rho_{20} - \rho_{10}}, \quad \rho_2 = \frac{\rho_{20}(\rho - \rho_{10})}{\rho_{20} - \rho_{10}} \quad . \qquad (12)$$

Substituting (12) into Eqs. (5) and eliminating $\partial \rho / \partial t$ between them gives the relation

$$(\rho_{20} - \rho)\, tr(\mathbf{d}_1) + (\rho - \rho_{10})\, tr(\mathbf{d}_2) - \xi \cdot \mathbf{a} = 0 \quad (13)$$

where

$$2\,\mathbf{d}_\beta = (\nabla \mathbf{v}_\beta)^T + \nabla \mathbf{v}_\beta \quad , \quad \xi = \nabla \rho, \quad \mathbf{a} = \mathbf{v}_1 - \mathbf{v}_2. \qquad (14)$$

The derivation of the constitutive equations appropriate to our binary mixture of incompressible Newtonian fluids has been outlined in Atkin and Craine (1976a, b). If the mixture is considered to be a purely mechanical system, that is, thermal effects are ignored, the relevant equations are

$$A_\beta = A_\beta(\rho), \quad A = A(\rho), \qquad (15)$$

$$\begin{aligned} p_1 &= (\rho - \rho_{20})\left(\rho_1 \frac{dA_1}{d\rho} + \lambda\right), \\ p_2 &= (\rho - \rho_{10})\left(\rho_2 \frac{dA_2}{d\rho} - \lambda\right), \end{aligned} \qquad (16)$$

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[2]See Beevers and Craine (1982).

$$\begin{aligned} \mathbf{f} &= \alpha \mathbf{a} - \lambda \xi, \\ \mathbf{q} &= -k' \mathbf{a}, \end{aligned} \qquad (17)$$

$$\begin{aligned} \boldsymbol{\sigma_1} = {}& [-p_1 + \lambda_1 tr(\mathbf{d}_1) + \lambda_3 tr(\mathbf{d}_2)]\, \mathbf{I} \\ & + 2\mu_1 \mathbf{d}_1 + 2\mu_3 \mathbf{d}_2 + \lambda_5 \boldsymbol{\Gamma}, \end{aligned} \qquad (18)$$

$$\begin{aligned} \boldsymbol{\sigma_2} = {}& [-p_2 + \lambda_4 tr(\mathbf{d}_1) + \lambda_2 tr(\mathbf{d}_2)]\, \mathbf{I} \\ & + 2\mu_4 \mathbf{d}_1 + 2\mu_2 \mathbf{d}_2 - \lambda_5 \boldsymbol{\Gamma} \end{aligned} \qquad (19)$$

where A_β denotes the partial Helmholtz free energy, and the Helmholtz free energy A of the mixture (total free energy) is defined by

$$A = \frac{1}{\rho}\sum_\beta \rho_\beta A_\beta \qquad (20)$$

and the coefficients $\alpha, \lambda_1, ..., \lambda_5, k', \mu_1, ..., \mu_4$ are functions of ρ and satisfy the inequalities

$$\alpha \geq 0 \quad , \quad \lambda_5 \geq 0 \quad , \quad \mu_1 \geq 0 \quad , \quad \mu_2 \geq 0 \quad ,$$

$$\lambda_1 + \frac{2}{3}\mu_1 \geq 0 \quad , \quad \lambda_2 + \frac{2}{3}\mu_2 \geq 0 \quad ,$$

$$\begin{aligned} (\mu_3 + \mu_4)^2 &\leq 4\mu_1\mu_2 \ , \ \left[\lambda_3 + \lambda_4 + \tfrac{2}{3}(\mu_3 + \mu_4)\right]^2 \\ &\leq 4\left(\lambda_1 + \tfrac{2}{3}\mu_1\right)\left(\lambda_2 + \tfrac{2}{3}\mu_2\right). \end{aligned} \qquad (21)$$

The quantity λ is a Lagrange multiplier[2] associated with the constraint (13) and $\boldsymbol{\Gamma}$ is given by

$$2\boldsymbol{\Gamma} = [(\nabla \mathbf{v}_1)^T - \nabla \mathbf{v}_1] - [(\nabla \mathbf{v}_2)^T - \nabla \mathbf{v}_2]. \qquad (22)$$

Finally, for the case of non-inertial flow $(D_\beta \mathbf{v}_\beta / Dt = 0)$, neglecting the body forces, we shall derive the equations governing the flow of a mixture of two Newtonian fluids. For this purpose, inserting $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$ and $\mathbf{f}$ from Eqs. (18), (19) and $(17)_1$ into Eqs. (6), with the help of Eqs. (14) and (22), one gets the following equations of motion:

$$M_1 \nabla^2 \mathbf{v}_1 + M_5 \nabla (\nabla \cdot \mathbf{v}_1) + (\nabla \cdot \mathbf{v}_1) \nabla \lambda_1 + (\nabla \mathbf{v}_1)^T \cdot (\nabla M_1) + (\nabla \mathbf{v}_1) \cdot (\nabla M_9)$$

$$+ M_2 \nabla^2 \mathbf{v}_2 + M_6 \nabla (\nabla \cdot \mathbf{v}_2) + (\nabla \cdot \mathbf{v}_2) \nabla \lambda_3 + (\nabla \mathbf{v}_2)^T \cdot (\nabla M_2) + (\nabla \mathbf{v}_2) \cdot (\nabla M_{10})$$

$$-\alpha (\mathbf{v}_1 - \mathbf{v}_2) = -\lambda \nabla \rho + \nabla p_1 \quad , \tag{23}$$

$$M_3 \nabla^2 \mathbf{v}_1 + M_7 \nabla (\nabla \cdot \mathbf{v}_1) + (\nabla \cdot \mathbf{v}_1) \nabla \lambda_4 + (\nabla \mathbf{v}_1)^T \cdot (\nabla M_3) + (\nabla \mathbf{v}_1) \cdot (\nabla M_{11})$$

$$+ M_4 \nabla^2 \mathbf{v}_2 + M_8 \nabla (\nabla \cdot \mathbf{v}_2) + (\nabla \cdot \mathbf{v}_2) \nabla \lambda_2 + (\nabla \mathbf{v}_2)^T \cdot (\nabla M_4) + (\nabla \mathbf{v}_2) \cdot (\nabla M_{12})$$

$$+\alpha (\mathbf{v}_1 - \mathbf{v}_2) = \lambda \nabla \rho + \nabla p_2 \tag{24}$$

where

$$M_1 = \mu_1 + \frac{\lambda_5}{2} \quad , \quad M_2 = \mu_3 - \frac{\lambda_5}{2} \quad , \quad M_3 = \mu_4 - \frac{\lambda_5}{2} \quad , \quad M_4 = \mu_2 + \frac{\lambda_5}{2} \quad ,$$

$$M_5 = \lambda_1 + \mu_1 - \frac{\lambda_5}{2}, \quad M_6 = \lambda_3 + \mu_3 + \frac{\lambda_5}{2}, \quad M_7 = \lambda_4 + \mu_4 + \frac{\lambda_5}{2}, \quad M_8 = \lambda_2 + \mu_2 - \frac{\lambda_5}{2},$$

$$M_9 = \mu_1 - \frac{\lambda_5}{2} \quad , \quad M_{10} = \mu_3 + \frac{\lambda_5}{2} \quad , \quad M_{11} = \mu_4 + \frac{\lambda_5}{2} \quad , \quad M_{12} = \mu_2 - \frac{\lambda_5}{2} \quad . \tag{25}$$

Note that, under isothermal conditions, the coefficients M_1 etc. appearing in (23) and (24) depend only on the total density ρ, and hence spatial coordinates. In the subsequent sections, we shall obtain the exact solutions of the above equations for some simple flows of a binary mixture of incompressible Newtonian fluids.

Parallel flow with a free surface

First, we examine the flow of a film of a binary mixture of incompressible Newtonian fluids of uniform thickness δ. The ambient air is assumed to be stationary and, therefore, the flow is driven by externally imposed pressure gradients $\partial p_1 / \partial x$ and $\partial p_2 / \partial x$. Let the y-axis be directed normally to the
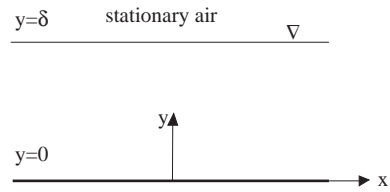
plate, and the x-axis along this plate (see Figure 1).

Figure 1. Basic geometry of the problem

We shall seek a solution of the form

$$v_{\beta x} = v_{\beta x}(y), \quad \rho = \rho(y) \tag{26}$$

where the function $v_{\beta x}$ denotes the velocity component of the βth fluid in the x direction. With this

assumption, it is shown that the equations of continuity (5) can be satisfied identically. Substituting Eq. (26) into the x- and y-components of the equations of motion (23) and (24), we get

$$M_1 v_{1x}'' \quad + M_2 v_{2x}'' + M_1' v_{1x}' + M_2' v_{2x}'$$
$$-\alpha(v_{1x} - v_{2x}) = \frac{\partial p_1}{\partial x}, \tag{27}$$

$$\lambda \rho' = \frac{\partial p_1}{\partial y}, \tag{28}$$

$$M_3 v_{1x}'' \quad + M_4 v_{2x}'' + M_3' v_{1x}' + M_4' v_{2x}'$$
$$+\alpha(v_{1x} - v_{2x}) = \frac{\partial p_2}{\partial x}, \tag{29}$$

$$-\lambda \rho' = \frac{\partial p_2}{\partial y}. \tag{30}$$

In the above equations, primes denote differentiation with respect to y. With the use of Eqs. (12), (16) and (20), elimination of $\partial \lambda / \partial y$ between Eqs. (28) and (30) gives

$$(\rho - \rho_{10})(\rho_{20} - \rho) \frac{d\rho}{dy} \frac{d^2(\rho A)}{d\rho^2} = 0 \tag{31}$$

and since, in general, $\rho \neq \rho_{10}$, $\rho \neq \rho_{20}$ and $d^2(\rho A)/d\rho^2 \neq 0$ we deduce that ρ is a constant. As a result, the coefficients M_1 etc. in (27) and (29) are constants. It also follows that the quantities p_1 and p_2 are not functions of y. Then, from Eqs. (27) and (29), it is evident that the pressure gradients are constants, i.e. $\partial p_1 / \partial x = -p_{10}$ and $\partial p_2 / \partial x = -p_{20}$. Thus, Eqs. (27) and (29) reduce to

$$M_1 v_{1x}'' + M_2 v_{2x}'' - \alpha (v_{1x} - v_{2x}) = p_{10}, \tag{32}$$

$$M_3 v_{1x}'' + M_4 v_{2x}'' + \alpha (v_{1x} - v_{2x}) = p_{20}. \tag{33}$$

It is convenient at this point to introduce dimensionless variables and material constants. If $\bar{f}$ is used to denote the dimensionless form of a quantity f , it follows that

$$\bar{y} = \frac{y}{\delta}, \quad \bar{M}_i = \frac{M_i}{\mu}, \quad \bar{v}_{\beta x} = \frac{v_{\beta x} \mu}{p_0 \delta^2},$$
$$\bar{\alpha} = \frac{\alpha \delta^2}{\mu}, \quad \bar{Q} = \frac{Q \mu}{p_0 \delta^3} \tag{34}$$

where μ is the viscosity of the mixture and Q is the volume flux of the mixture per unit distance normal to the plane of flow. In addition, it is assumed that the pressure gradients imposed on mixture components are the same, i.e. $p_{10} = p_{20} = p_0$. Thus the dimensionless governing equations are as follows:

$$\bar{M}_1 \bar{v}_{1x}'' + \bar{M}_2 \bar{v}_{2x}'' - \bar{\alpha} (\bar{v}_{1x} - \bar{v}_{2x}) = -1, \tag{35}$$

$$\bar{M}_3 \bar{v}_{1x}'' + \bar{M}_4 \bar{v}_{2x}'' + \bar{\alpha} (\bar{v}_{1x} - \bar{v}_{2x}) = -1. \tag{36}$$

Throughout this paper, henceforth for convenience, unless stated otherwise, we shall drop the bars that appear over the dimensionless quantities.

Subtracting M_4 times Eq. (35) from M_2 times Eq. (36), and M_3 times Eq. (35) from M_1 times Eq. (36), we get the following equations, respectively

$$\eta_1 v_{1x}'' - \alpha (M_2 + M_4)(v_{1x} - v_{2x}) = M_2 - M_4, \tag{37}$$

$$-\eta_1 v_{2x}'' - \alpha (M_1 + M_3)(v_{1x} - v_{2x}) = M_1 - M_3 \tag{38}$$

and the sum of above equations is

$$\eta_1 (v_{1x}'' - v_{2x}'') - \alpha \eta_2 (v_{1x} - v_{2x}) = \eta_3 \tag{39}$$

where

$$\eta_1 = M_1 M_4 - M_2 M_3, \quad \eta_2 = M_1 + M_2 + M_3 + M_4,$$

$$\eta_3 = M_1 + M_2 - M_3 - M_4. \tag{40}$$

Hereafter, we shall assume that $\alpha \neq 0$, $\eta_1 \neq 0$ and $\eta_2 \neq 0$.

The boundary conditions for the velocity fields are

$$v_{\beta x}(0) = 0, \quad v_{1x}(1) - v_{2x}(1) = W \tag{41}$$

where W is a constant to be determined later. Equation (39), which satisfies boundary conditions (41), is solved by the following simple analytical expression

$$v_{1x} - v_{2x} = C_1 cosh(\gamma_1 y) + C_2 sinh(\gamma_1 y) - \frac{\eta_3}{\alpha \eta_2} \tag{42}$$

where $\gamma_1 = \sqrt{\alpha \eta_2/\eta_1}$, and the constants C_1 and C_2 are

$$C_1 = \frac{\eta_3}{\alpha \eta_2}, \quad C_2 = \frac{W}{sinh(\delta \gamma_1)} - C_1 tanh\left(\frac{\delta \gamma_1}{2}\right). \tag{43}$$

Substituting Eq. (42) into Eqs. (37)-(38) and solving them, we have, respectively

$$v_{1x} = \frac{(M_2+M_4)}{\eta_2}[C_1 \cosh(\gamma_1 y) + C_2 \sinh(\gamma_1 y)]$$
$$+ C_3 y + C_4 - \frac{y^2}{\eta_2}, \tag{44}$$

$$v_{2x} = -\frac{(M_1+M_3)}{\eta_2}[C_1 \cosh(\gamma_1 y) + C_2 \sinh(\gamma_1 y)]$$
$$+ C_5 y + C_6 - \frac{y^2}{\eta_2} \tag{45}$$

where $C_3, ..., C_6$ are the constants of integration. Boundary conditions (41) are not sufficient for determining these constants in a unique way. It would thus appear that the additional boundary condition must be imposed. This is a free-surface condition, that is, the atmospheric shear stress, which is assumed to be negligible (no wind, negligible air viscosity), must be equal to the total shear stress, σ_{xy}, of the mixture at $y = 1$. Thus

$$\sigma_{xy}(1) = (M_1 + M_3)v'_{1x}(1) + (M_2 + M_4)v'_{2x}(1) = 0. \tag{46}$$

From conditions (41) and (46), we find that

$$C_3 = \frac{1}{\eta_2}\Big\{(M_2 + M_4)\big[C_1 + W - C_1 cosh(\gamma_1)$$
$$- C_2 sinh(\gamma_1)\big] + 2\Big\},$$

$$C_5 = \frac{1}{\eta_2}\Big\{-(M_1 + M_3)\big[C_1 + W - C_1 cosh(\gamma_1)$$
$$- C_2 sinh(\gamma_1)\big] + 2\Big\},$$

$$C_4 = -C_1\frac{(M_2 + M_4)}{\eta_2}, \quad C_6 = C_1\frac{(M_1 + M_3)}{\eta_2}. \tag{47}$$

The volume rate of flow per unit length in the z-direction is

$$Q = \int_0^1 v_{1x}dy + \int_0^1 v_{2x}dy. \tag{48}$$

Inserting v_{1x} and v_{2x} from Eqs. (44) and (45), with the aid of Eq. (47), into Eq. (48) yields

$$W = \gamma_1\left[C_1 + \frac{4 - 3Q\eta_2}{3(M_1 - M_2 + M_3 - M_4)}\right] \coth\left(\frac{\gamma_1}{2}\right) - 2C_1. \tag{49}$$

It is obvious from Eq. (49) that the value of constant W in Eqs. $(43)_2$ and $(47)_{1,2}$ can be determined by experimental measurement of Q.

Flow between intersecting planes, one of which is moving

In this section, we consider the slow motion of a mixture of two incompressible Newtonian fluids near a corner of plane rigid walls, one of which is stationary and the other moving. The flow is caused by the motion of the wall at $\theta = 0$. The moving wall is made of a porous material through which the fluids are injected with constant velocities V_1 and V_2 (see Figure 2).

It seems reasonable to assume that the velocity distribution and total density in planar polar coordinates (r, θ) are of the form

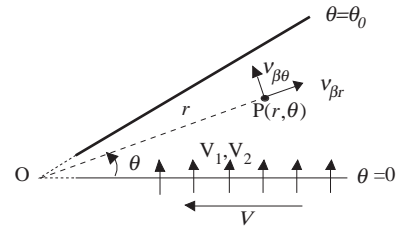$$\mathbf{v}_\beta = [v_{\beta r}(r, \theta), v_{\beta \theta}(r, \theta)], \quad \rho = \rho(r, \theta) \tag{50}$$

Figure 2. Sketch of flow geometry and coordinate system

where $v_{\beta r}$ and $v_{\beta \theta}$ denote the velocity components of the βth fluid in the directions of r and θ, respectively. By defining the stream function $\psi_\beta(r, \theta)$, such that

$$v_{\beta r} = \frac{1}{r}\frac{\partial \psi_\beta}{\partial \theta}, \quad v_{\beta \theta} = -\frac{\partial \psi_\beta}{\partial r} \tag{51}$$

the equation $\nabla \cdot \mathbf{v}_\beta = 0$ is satisfied automatically. In this case, we easily conclude from Eqs. (5) and (13), taking account of Eq. (12), that the density ρ of the mixture is a constant. Since ρ has been proved to be constant, all of the coefficients in Eqs. (23) and (24) are constants. Now we shall seek a solution of the following form (Riedler and Schneider, 1983):

$$\psi_\beta(r, \theta) = r\, f_\beta(\theta). \tag{52}$$

Inserting $v_{\beta r}$ and $v_{\beta \theta}$ from Eq. (51), with the aid of (52), into the r- and θ-components of the momentum equations (23) and (24) and eliminating the pressure terms by cross-differentiating yields

$$M_1\left(f_1^{iv} + 2f_1'' + f_1\right) + M_2\left(f_2^{iv} + 2f_2'' + f_2\right) - \alpha\, r^2\left(f_1'' - f_2'' + f_1 - f_2\right) = 0, \tag{53}$$

$$M_3\left(f_1^{iv} + 2f_1'' + f_1\right) + M_4\left(f_2^{iv} + 2f_2'' + f_2\right) + \alpha\, r^2\left(f_1'' - f_2'' + f_1 - f_2\right) = 0 \tag{54}$$

where primes denote differentiation with respect to θ.

Let us make the variables and material constants non-dimensional by the following substitutions:

$$\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha\, r^2}{\mu}, \quad \bar{f}_\beta(\theta) = \frac{f_\beta(\theta)}{V}. \tag{55}$$

Thus the non-dimensional governing equations become

$$\bar{M}_1\left(\bar{f}_1^{iv} + 2\bar{f}_1'' + \bar{f}_1\right) + \bar{M}_2\left(\bar{f}_2^{iv} + 2\bar{f}_2'' + \bar{f}_2\right) - \bar{\alpha}\left(\bar{f}_1'' - \bar{f}_2'' + \bar{f}_1 - \bar{f}_2\right) = 0, \tag{56}$$

$$\bar{M}_3\left(\bar{f}_1^{iv} + 2\bar{f}_1'' + \bar{f}_1\right) + \bar{M}_4\left(\bar{f}_2^{iv} + 2\bar{f}_2'' + \bar{f}_2\right) + \bar{\alpha}\left(\bar{f}_1'' - \bar{f}_2'' + \bar{f}_1 - \bar{f}_2\right) = 0. \tag{57}$$

The boundary conditions for the dimensionless velocity fields are as follows:

$$\bar{v}_{\beta r}(r, 0) = -1, \quad \bar{v}_{\beta r}(r, \theta_0) = 0,$$

$$\bar{v}_{\beta \theta}(r, 0) = V_\beta\, /\, V\, (V_\beta > 0, V > 0), \quad \bar{v}_{\beta \theta}(r, \theta_0) = 0. \tag{58}$$

From Eqs. (51), (52) and (58), it follows that the boundary conditions for the function f_β are

$$f_\beta(0) = -V_\beta\, /\, V, \quad f_\beta(\theta_0) = 0,$$

$$f_\beta'(0) = -1, \quad f_\beta'(\theta_0) = 0. \tag{59}$$

From Eqs. (56) and (57), making simple algebraic calculations as in the previous section, we can obtain the following equations:

461

$$\eta_1 \left(f_1^{iv} + 2f_1'' + f_1\right) - \alpha \ (M_2 + M_4)\left(f_1'' - f_2'' + f_1 - f_2\right) = 0, \tag{60}$$

$$-\eta_1 \left(f_2^{iv} + 2f_2'' + f_2\right) - \alpha \ (M_1 + M_3)\left(f_1'' - f_2'' + f_1 - f_2\right) = 0. \tag{61}$$

The sum of the above equations is

$$f_1^{iv} - f_2^{iv} + (1 - \gamma_2)\left(f_1'' - f_2''\right) - \gamma_2 \left(f_1 - f_2\right) = 0 \tag{62}$$

where $\gamma_2 = -1 + \alpha \, \eta_2/\eta_1$. The characteristic roots of Eq. (62) are $\pm i$ and $\pm\sqrt{\gamma_2}$. Hence, the general solution is

If $\gamma_2 < 0$,

$$f_1 - f_2 = C_1 \cos\theta + C_2 \sin\theta + C_3 cos(\sqrt{|\gamma_2|}\theta) + C_4 sin(\sqrt{|\gamma_2|}\,\theta). \tag{63}$$

If $\gamma_2 = 0$,

$$f_1 - f_2 = D_1 \cos\theta + D_2 \sin\theta + D_3\theta + D_4. \tag{64}$$

If $\gamma_2 > 0$,

$$f_1 - f_2 = E_1 \cos\theta + E_2 \sin\theta + E_3 cosh(\sqrt{\gamma_2}\theta) + E_4 sinh(\sqrt{\gamma_2}\theta). \tag{65}$$

Applying the boundary conditions (59) to Eqs. (63)-(65) separately, we find

$$C_1 = \frac{\sqrt{|\gamma_2|}(V_2 - V_1)}{C^*}\left[\cos\theta_0 \cos(\sqrt{|\gamma_2|}\theta_0) + \sqrt{|\gamma_2|} \sin\theta_0 \sin(\sqrt{|\gamma_2|}\theta_0) - 1\right] \quad ,$$

$$C_2 = \frac{\sqrt{|\gamma_2|}(V_1 - V_2)}{C^*}\left[\sqrt{|\gamma_2|} \cos\theta_0 \sin(\sqrt{|\gamma_2|}\theta_0) - \sin\theta_0 \cos(\sqrt{|\gamma_2|}\theta_0)\right] \quad ,$$

$$C_3 = \frac{(V_1 - V_2)}{C^*}\left\{\sqrt{|\gamma_2|}\left[1 - \cos\theta_0 \cos(\sqrt{|\gamma_2|}\theta_0)\right] - \sin\theta_0 \sin(\sqrt{|\gamma_2|}\theta_0)\right\},$$

$$C_4 = \frac{(V_1 - V_2)}{C^*}\left[\sin\theta_0 \cos(\sqrt{|\gamma_2|}\theta_0) - \sqrt{|\gamma_2|} \cos\theta_0 \sin(\sqrt{|\gamma_2|}\theta_0)\right] \quad ,$$

$$D_1 = D^*(\cos\theta_0 - 1), \quad D_2 = D^* \sin\theta_0, \quad D_3 = -D_2, \quad D_4 = D_1 + D^*\theta_0 \sin\theta_0,$$

$$E_1 = \frac{\sqrt{\gamma_2}(V_2 - V_1)}{E^*}[1 - \cos\theta_0 \cosh(\sqrt{\gamma_2}\theta_0) + \sqrt{\gamma_2} \sin\theta_0 \sinh(\sqrt{\gamma_2}\theta_0)] \quad ,$$

$$E_2 = \frac{\sqrt{\gamma_2}(V_1 - V_2)}{E^*}[\sin\theta_0\cosh(\sqrt{\gamma_2}\theta_0) + \sqrt{\gamma_2}\cos\theta_0\sinh(\sqrt{\gamma_2}\theta_0)] \quad ,$$

$$E_3 = \frac{(V_1 - V_2)}{E^*}\{\sin\theta_0\sinh(\sqrt{\gamma_2}\theta_0) + \sqrt{\gamma_2}[\cos\theta_0\cosh(\sqrt{\gamma_2}\theta_0) - 1]\},$$

$$E_4 = \frac{(V_2 - V_1)}{E^*}[\sin\theta_0\cosh(\sqrt{\gamma_2}\theta_0) + \sqrt{\gamma_2}\cos\theta_0\sinh(\sqrt{\gamma_2}\theta_0)] \tag{66}$$

where

$$C^* = V\Big\{2\sqrt{|\gamma_2|}[\cos\theta_0\cos(\sqrt{|\gamma_2|}\theta_0) - 1] + (1 + |\gamma_2|)\sin\theta_0\sin(\sqrt{|\gamma_2|}\theta_0)\Big\},$$

$$D^* = (V_2 - V_1)/[V(2\cos\theta_0 + \theta_0\sin\theta_0 - 2)],$$

$$E^* = V\{2\sqrt{\gamma_2}[1 - \cos\theta_0\cosh(\sqrt{\gamma_2}\theta_0)] + (\gamma_2 - 1)\sin\theta_0\sinh(\sqrt{\gamma_2}\theta_0)\}. \tag{67}$$

Having substituted Eqs. (63)-(65) into Eqs. (60) and (61), we integrate the resulting equations and obtain the following solutions for $f_1(\theta)$ and $f_2(\theta)$, respectively

If $\gamma_2 < 0$,

$$f_1(\theta) = (C_5 + C_6\theta)\cos\theta + (C_7 + C_8\theta)\sin\theta - \frac{\alpha(M_2 + M_4)}{\eta_1(|\gamma_2| - 1)}\Big[C_3\cos(\sqrt{|\gamma_2|}\theta) + C_4\sin(\sqrt{|\gamma_2|}\theta)\Big],$$

$$f_2(\theta) = (C_9 + C_{10}\theta)\cos\theta + (C_{11} + C_{12}\theta)\sin\theta + \frac{\alpha(M_1 + M_3)}{\eta_1(|\gamma_2| - 1)}\Big[C_3\cos(\sqrt{|\gamma_2|}\theta) + C_4\sin(\sqrt{|\gamma_2|}\theta)\Big]. \tag{68}$$

If $\gamma_2 = 0$,

$$f_1(\theta) = (D_5 + D_6\theta)\cos\theta + (D_7 + D_8\theta)\sin\theta + \frac{\alpha(M_2 + M_4)}{\eta_1}(D_4 + D_3\theta),$$

$$f_2(\theta) = (D_9 + D_{10}\theta)\cos\theta + (D_{11} + D_{12}\theta)\sin\theta - \frac{\alpha(M_1 + M_3)}{\eta_1}(D_4 + D_3\theta). \tag{69}$$

If $\gamma_2 > 0$,

$$f_1(\theta) = (E_5 + E_6\theta)\cos\theta + (E_7 + E_8\theta)\sin\theta + \frac{M_2 + M_4}{\eta_2}[E_3\cosh(\sqrt{\gamma_2}\theta) + E_4\sinh(\sqrt{\gamma_2}\theta)],$$

$$f_2(\theta) = (E_9 + E_{10}\theta)\cos\theta + (E_{11} + E_{12}\theta)\sin\theta - \frac{M_1 + M_3}{\eta_2}[E_3\cosh(\sqrt{\gamma_2}\theta) + E_4\sinh(\sqrt{\gamma_2}\theta)] \tag{70}$$

463

With the help of Eq. (59), the constants of integration $C_5, ..., C_{12}$, $D_5, ..., D_{12}$ and $E_5, ..., E_{12}$ can be expressed as

$$C_5 = -\frac{V_1}{V} + \frac{\alpha(M_2 + M_4)}{\eta_1(|\gamma_2| - 1)}C_3,$$

$$C_6 = \left\{ [\alpha(M_2 + M_4)(V_1 - V_2) + \eta_1 V_1(|\gamma_2| - 1)][\sin(2\theta_0) + 2\theta_0] + 2V\eta_1(|\gamma_2| - 1)\sin^2\theta_0 \right\} \Big/ \hat{C},$$

$$\begin{aligned}
C_7 &= 2\alpha V(M_2 + M_4)\left\{ C_3\left[\left(\cos\theta_0 - \cos(\sqrt{|\gamma_2|}\theta_0) - \sqrt{|\gamma_2|}\sin(\sqrt{|\gamma_2|}\theta_0)\right)\sin\theta_0 \right. \right. \\
&\quad + \left(1 - \cos\theta_0\cos(\sqrt{|\gamma_2|}\theta_0)\right)\theta_0 \Big] + C_4\Big[\left(\sqrt{|\gamma_2|}\theta_0\cos(\sqrt{|\gamma_2|}\theta_0) - \sin(\sqrt{|\gamma_2|}\theta_0)\right)\sin\theta_0 \\
&\quad + \left. \left(\sqrt{|\gamma_2|}\theta_0 - \cos\theta_0\sin(\sqrt{|\gamma_2|}\theta_0)\right)\theta_0 \Big] \right\} \Big/ \hat{C} \\
&\quad + [\eta_1 V_1(1 - |\gamma_2|)\sin(2\theta_0) + 2\eta_1\theta_0(1 - |\gamma_2|)(V_1 + \theta_0 V)] \Big/ \hat{C},
\end{aligned}$$

$$C_8 = \left\{ 2[\alpha(M_2 + M_4)(V_1 - V_2) + \eta_1 V_1(|\gamma_2| - 1)]\sin^2\theta_0 + \eta_1 V(|\gamma_2| - 1)[2\theta_0 - \sin(2\theta_0)] \right\} \Big/ \hat{C},$$

$$C_9 = C_5 - C_1, \quad C_{10} = C_6, \quad C_{11} = C_7 - C_2, \quad C_{12} = C_8,$$

$$D_5 = -\frac{V_1}{V} - \frac{\alpha(M_2 + M_4)}{\eta_1}D_4,$$

$$D_6 = \left\{ [\alpha(M_2 + M_4)(V_2 - V_1) + \eta_1 V_1][\sin(2\theta_0) + 2\theta_0] + 2\eta_1 V\sin^2\theta_0 \right\} \Big/ \hat{D},$$

$$\begin{aligned}
D_7 &= 2\alpha V(M_2 + M_4)\left\{ D_3(\cos\theta_0 - 1)\theta_0^2 + D_4[(1 - \cos\theta_0)(\sin\theta_0 - \theta_0)] \right\} \Big/ \hat{D} \\
&\quad - \eta_1\left\{ V_1[\sin(2\theta_0) + 2\theta_0] + 2V\theta_0^2 \right\} \Big/ \hat{D},
\end{aligned}$$

$$D_8 = \left\{ 2[\alpha(M_2 + M_4)(V_2 - V_1) + \eta_1 V_1]\sin^2\theta_0 + \eta_1 V[2\theta_0 - \sin(2\theta_0)] \right\} \Big/ \hat{D},$$

$$D_9 = D_5 - D_1, \quad D_{10} = D_6, \quad D_{11} = D_7 - D_2, \quad D_{12} = D_8,$$

$$E_5 = -\frac{V_1}{V} - \frac{(M_2 + M_4)}{\eta_2}E_3,$$

$$E_6 = \left\{ [(M_2 + M_4)(V_2 - V_1) + \eta_2 V_1][\sin(2\theta_0) + 2\theta_0] + 2\eta_2 V\sin^2\theta_0 \right\} \Big/ \hat{E},$$

$$E_7 = 2V(M_2 + M_4) \left\{ E_3 \left[\left(\cosh(\sqrt{\gamma_2}\theta_0) - \cos\theta_0 - \sqrt{\gamma_2}\,\theta_0 \sinh(\sqrt{\gamma_2}\theta_0) \right) \sin\theta_0 \right.\right.$$
$$+ \left(\cos\theta_0 \cosh(\sqrt{\gamma_2}\theta_0) - 1 \right) \theta_0 \right] + E_4 \left[\left(\sinh(\sqrt{\gamma_2}\theta_0) - \sqrt{\gamma_2}\,\theta_0 \cosh(\sqrt{\gamma_2}\theta_0) \right) \sin\theta_0 \right.$$
$$\left.\left. + \left(\cos\theta_0 \sinh(\sqrt{\gamma_2}\theta_0) - \sqrt{\gamma_2}\,\theta_0 \right) \theta_0 \right] \right\} \Big/ \hat{E} - \left[\eta_2 V_1 \sin(2\theta_0) + 2\eta_2\theta_0(V_1 + \theta_0 V) \right] \Big/ \hat{E},$$

$$E_8 = \left\{ 2\left[(M_2 + M_4)(V_2 - V_1) + \eta_2 V_1 \right] \sin^2\theta_0 + \eta_2 V[2\theta_0 - \sin(2\theta_0)] \right\} \Big/ \hat{E},$$

$$E_9 = E_5 - E_1, \quad E_{10} = E_6, \quad E_{11} = E_7 - E_2, \quad E_{12} = E_8 \tag{71}$$

where

$$\hat{C} = \eta_1 V(|\gamma_2| - 1)[\cos(2\theta_0) + 2\theta_0^2 - 1], \quad \hat{D} = \eta_1 V[\cos(2\theta_0) + 2\theta_0^2 - 1],$$

$$\hat{E} = \eta_2 V[\cos(2\theta_0) + 2\theta_0^2 - 1]. \tag{72}$$

Flow between two coaxial moving cylinders

Finally, we study the fully developed flow of a binary mixture of incompressible Newtonian fluids between an inner cylinder of radius r_1 rotating at a constant rate of w_1 as well as translating at uniform velocity V and outer concentric cylinder of radius r_2 rotating at a constant rate of w_2, as sketched in Figure 3. The flow is driven by a combination of externally applied pressure gradients $(\partial p_1/\partial z, \partial p_2/\partial z)$ and the motion of cylinders. Cylindrical coordinates (r, θ, z), with the z-axis coinciding with the common axis of the cylinders, are introduced.

We look for a solution, compatible with the mass balance equations (5), of the form

$$\mathbf{v}_\beta = [0, v_{\beta\theta}(r), v_{\beta z}(r)], \quad \rho = \rho(r) \tag{73}$$

where $v_{\beta\theta}$ and $v_{\beta z}$ denote the velocity components of the β th fluid in the directions of θ and z, respectively. Substituting velocity components and total density from Eq. (65) into equations of motion (23) and (24) gives

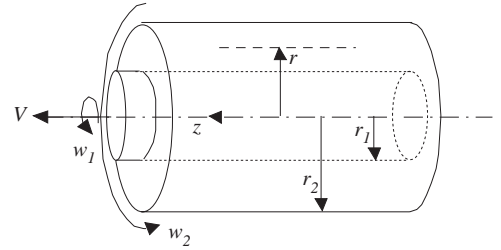$$\lambda\rho' = \frac{\partial p_1}{\partial r}, \tag{74}$$

Figure 3. Schematic diagram of flow

$$M_1(r^2 v_{1\theta}'' + r\,v_{1\theta}' - v_{1\theta}) + M_2(r^2 v_{2\theta}'' + r\,v_{2\theta}' - v_{2\theta}) - \alpha\,r^2(v_{1\theta} - v_{2\theta})$$

$$+ r^2(M_1' v_{1\theta}' + M_2' v_{2\theta}') - r(M_9' v_{1\theta} + M_{10}' v_{2\theta}) = r\frac{\partial p_1}{\partial\theta}, \tag{75}$$

$$M_1 v_{1z}'' + M_2 v_{2z}'' + \frac{1}{r}\left(M_1 v_{1z}' + M_2 v_{2z}'\right) + M_1' v_{1z}' + M_2' v_{2z}' - \alpha\,(v_{1z} - v_{2z}) = \frac{\partial p_1}{\partial z}, \tag{76}$$

465

$$-\lambda \rho' = \frac{\partial p_2}{\partial r}, \tag{77}$$

$$M_3(r^2 v_{1\theta}'' + r v_{1\theta}' - v_{1\theta}) + M_4(r^2 v_{2\theta}'' + r v_{2\theta}' - v_{2\theta}) + \alpha \, r^2(v_{1\theta} - v_{2\theta})$$

$$+r^2(M_3' v_{1\theta}' + M_4' v_{2\theta}') - r(M_{11}' v_{1\theta} + M_{12}' v_{2\theta}) = r\frac{\partial p_2}{\partial \theta}, \tag{78}$$

$$M_3 v_{1z}'' + M_4 v_{2z}'' + \frac{1}{r}\left(M_3 v_{1z}' + M_4 v_{2z}'\right) + M_3' v_{1z}' + M_4' v_{2z}' + \alpha \, (v_{1z} - v_{2z}) = \frac{\partial p_2}{\partial z}. \tag{79}$$

The primes here indicate differentiation with respect to r. Elimination of $\partial \lambda / \partial r$ between Eqs. (74) and (77), with the help of Eqs. (12), (16) and (20), leads to

$$(\rho - \rho_{10})\,(\rho_{20} - \rho)\,\frac{d\rho}{dr}\frac{d^2(\rho A)}{d\rho^2} = 0. \tag{80}$$

Here, in general, $\rho \neq \rho_{10}$, $\rho \neq \rho_{20}$ and $d^2(\rho A)/d\rho^2 \neq 0$, and hence we arrive at the conclusion that the total density ρ is a constant. Since ρ has been proved to be constant, the coefficients

M_1 etc. appearing in equations of motion become constants. It also follows that the quantities p_1 and p_2 are not functions of r. From Eqs. (75) and (78), it is clear that $\partial p_1 / \partial \theta$ and $\partial p_2 / \partial \theta$ are constants. Since p_1 and p_2 are periodic functions of θ, these constants must be equal to zero. Consequently, p_1 and p_2 can be at most functions of z. Then Eqs. (76) and (79) imply that the pressure gradients are constants, i.e. $\partial p_1 / \partial z = p_{10}$ and $\partial p_2 / \partial z = p_{20}$. It is assumed that $p_{10} = p_{20} = p_0$. In the light of these arguments, the equations of motion reduce to

$$\bar{M}_1(\bar{r}^2 \bar{v}_{1\theta}'' + \bar{r} \bar{v}_{1\theta}' - \bar{v}_{1\theta}) + \bar{M}_2(\bar{r}^2 \bar{v}_{2\theta}'' + \bar{r} \bar{v}_{2\theta}' - \bar{v}_{2\theta}) - \bar{\alpha} \, \bar{r}^2(\bar{v}_{1\theta} - \bar{v}_{2\theta}) = 0, \tag{81}$$

$$\bar{M}_1 \bar{v}_{1z}'' + \bar{M}_2 \bar{v}_{2z}'' + \frac{1}{\bar{r}}\left(\bar{M}_1 \bar{v}_{1z}' + \bar{M}_2 \bar{v}_{2z}'\right) - \bar{\alpha} \, (\bar{v}_{1z} - \bar{v}_{2z}) = p^*, \tag{82}$$

$$\bar{M}_3(\bar{r}^2 \bar{v}_{1\theta}'' + \bar{r} \bar{v}_{1\theta}' - \bar{v}_{1\theta}) + \bar{M}_4(\bar{r}^2 \bar{v}_{2\theta}'' + \bar{r} \bar{v}_{2\theta}' - \bar{v}_{2\theta}) + \bar{\alpha} \, \bar{r}^2(\bar{v}_{1\theta} - \bar{v}_{2\theta}) = 0, \tag{83}$$

$$\bar{M}_3 \bar{v}_{1z}'' + \bar{M}_4 \bar{v}_{2z}'' + \frac{1}{\bar{r}}\left(\bar{M}_3 \bar{v}_{1z}' + \bar{M}_4 \bar{v}_{2z}'\right) + \bar{\alpha} \, (\bar{v}_{1z} - \bar{v}_{2z}) = p^* \tag{84}$$

where

$$\bar{r} = \frac{r}{r_2}, \quad \bar{M}_i = \frac{M_i}{\mu}, \quad \bar{v}_{\beta\theta} = \frac{v_{\beta\theta}}{r_2 w_2}, \quad \bar{v}_{\beta z} = \frac{v_{\beta z}}{V}, \quad \bar{\alpha} = \frac{\alpha \, r_2^2}{\mu}, \quad p^* = \frac{p_0 r_2^2}{\mu \, V}. \tag{85}$$

From the above equations, after a little algebra as in the previous sections, we get

$$\eta_1 \left(v_{1z}'' + \frac{1}{r}v_{1z}'\right) - \alpha \, (M_2 + M_4) \, (v_{1z} - v_{2z}) = (M_4 - M_2) \, p^*, \tag{86}$$

$$-\eta_1 \left(v''_{2z} + \frac{1}{r} v'_{2z} \right) - \alpha \ (M_1 + M_3) \ (v_{1z} - v_{2z}) = (M_3 - M_1) \ p^*, \tag{87}$$

$$\eta_1 (r^2 v''_{1\theta} + r v'_{1\theta} - v_{1\theta}) - \alpha \ r^2 \ (M_2 + M_4) \ (v_{1\theta} - v_{2\theta}) = 0, \tag{88}$$

$$-\eta_1 (r^2 v''_{2\theta} + r v'_{2\theta} - v_{2\theta}) - \alpha \ r^2 \ (M_1 + M_3) \ (v_{1\theta} - v_{2\theta}) = 0. \tag{89}$$

Adding Eqs. (86) and (87) gives

$$\eta_1 \ (v''_{1z} - v''_{2z}) + \frac{1}{r} \eta_1 \ (v'_{1z} - v'_{2z}) - \alpha \, \eta_2 \ (v_{1z} - v_{2z}) = \eta_4 \tag{90}$$

where $\eta_4 = -\eta_3 \, p^*$. Integration of this equation yields

$$v_{1z} - v_{2z} = C_1 I_0 \left(\gamma_1 r \right) + C_2 K_0 \left(\gamma_1 r \right) - \frac{\eta_4}{\alpha \, \eta_2} \tag{91}$$

where I_0 and K_0 are modified Bessel functions of order zero.

The no-slip boundary conditions of the problem are

$$v_{\beta\theta}(r_1/r_2) = \frac{r_1 w_1}{r_2 w_2}, \quad v_{\beta\theta}(1) = 1, \tag{92}$$

$$v_{\beta z}(r_1/r_2) = 1, \quad v_{\beta z}(1) = 0. \tag{93}$$

The boundary conditions on velocity given by Eq. (93) require

$$C_1 = \frac{\eta_4 [K_0(\gamma_1) - K_0(\gamma_1 r_1/r_2)]}{\alpha \, \eta_2 [I_0(\gamma_1 r_1/r_2) K_0(\gamma_1) - I_0(\gamma_1) K_0(\gamma_1 r_1/r_2)]},$$

$$C_2 = \frac{\eta_4 [I_0(\gamma_1) - I_0(\gamma_1 r_1/r_2)]}{\alpha \, \eta_2 [K_0(\gamma_1 r_1/r_2) I_0(\gamma_1) - K_0(\gamma_1) I_0(\gamma_1 r_1/r_2)]}. \tag{94}$$

Substituting Eq. (91) into Eqs. (86) and (87), and integrating these differential equations, we have, respectively

$$v_{1z} = \frac{M_2 + M_4}{\eta_2} \{ C_1 [I_0 \left(\gamma_1 r \right) - 1] + C_2 K_0 \left(\gamma_1 r \right) \} + \frac{p^*}{2\eta_2} r^2 + C_3 \ln r + C_4, \tag{95}$$

$$v_{2z} = -\frac{M_1 + M_3}{\eta_2} \{ C_1 [I_0 \left(\gamma_1 r \right) - 1] + C_2 K_0 \left(\gamma_1 r \right) \} + \frac{p^*}{2\eta_2} r^2 + C_5 \ln r + C_6. \tag{96}$$

Here $C_3, ..., C_6$ are constants of integration. Boundary conditions (93) allow us to express these constants as

$$C_3 = \frac{M_2 + M_4}{\eta_2 \ln(r_1/r_2)} \{C_1[I_0(\gamma_1) - I_0(\gamma_1 r_1/r_2)] + C_2[K_0(\gamma_1) - K_0(\gamma_1 r_1/r_2)]\}[1em] + \frac{p^* (r_2^2 - r_1^2) + 2 r_2^2 \eta_2}{2 r_2^2 \eta_2 \ln(r_1/r_2)},$$

$$C_4 = -\frac{M_2 + M_4}{\eta_2} \{C_1[I_0(\gamma_1) - 1] + C_2 K_0(\gamma_1)\} - \frac{p^*}{2\eta_2},$$

$$C_5 = \frac{M_1 + M_3}{\eta_2 \ln(r_1/r_2)} \{C_1[I_0(\gamma_1 r_1/r_2) - I_0(\gamma_1)] + C_2[K_0(\gamma_1 r_1/r_2) - K_0(\gamma_1)]\} + \frac{p^* (r_2^2 - r_1^2) + 2 r_2^2 \eta_2}{2 r_2^2 \eta_2 \ln(r_1/r_2)},$$

$$C_6 = \frac{M_1 + M_3}{\eta_2} \{C_1[I_0(\gamma_1) - 1] + C_2 K_0(\gamma_1)\} - \frac{p^*}{2\eta_2}. \tag{97}$$

The sum of Eqs. (88) and (89)

$$r^2(v_{1\theta}'' - v_{2\theta}'') + r(v_{1\theta}' - v_{2\theta}') - (1 + \gamma_1^2 r^2)(v_{1\theta} - v_{2\theta}) = 0. \tag{98}$$

The solution of this equation is

$$v_{1\theta} - v_{2\theta} = C_7 I_1(\gamma_1 r) + C_8 K_1(\gamma_1 r) \tag{99}$$

where I_1 and K_1 are modified Bessel functions of order one. On application of conditions (92), we get

$$v_{1\theta} - v_{2\theta} = 0. \tag{100}$$

Substituting Eq. (100) into Eqs. (88) and (89), we obtain

$$r^2 v_{\beta\theta}'' + r v_{\beta\theta}' - v_{\beta\theta} = 0. \tag{101}$$

It is easy to see that Ω_β, under the boundary conditions (92), has the form

$$v_{\beta\theta}(r) = \frac{[(r^2 - 1)w_1 + w_2] r_1^2 - r^2 w_2 r_2^2}{r(r_1^2 - r_2^2)w_2}. \tag{102}$$

Discussion

In this paper some steady and slow flows of a mixture of two incompressible inert Newtonian fluids have been studied theoretically. Exact solutions have been obtained for the problems under consideration. We infer from these solutions that the presence of externally applied pressure gradients or the difference between boundary conditions for each fluid brings about the relative motion between the fluids, i.e. $\mathbf{v}_1 - \mathbf{v}_2 \neq 0$.

In order to make predictions based on the forego-

ing analysis, it is necessary to know all of the material functions in the constitutive equations. Determination of these functions for a mixture is much more difficult than that for a single continuum, owing to the large number of response functions appearing in the constitutive equations. On the other hand, a significant body of literature has grown up around the problem of determining these functions due to the fact that the flow of mixtures is of great technical importance. For example, employing results obtained from the kinetic theory of fluids, Sampaio and Williams (1977) were able to derive formulae for

μ_1, μ_2, μ_3 and μ_4 in terms of the viscosities of the unmixed fluids and the volume fractions in the case of $\lambda_5 = 0$. In this work, we benefit from the formulae suggested by Sampaio and Williams (1977) with the intention of assigning reasonable values to $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$ and $\bar{M}_4$. To achieve this for a mixture composed of water and oil, at the outset we assume that the densities of unmixed fluids and the volume fractions are known. With the aid of Eqs. (3) and (10), knowledge of these quantities enables ρ_1, ρ_2 and $\rho = \rho_0$ to be calculated. Later, the viscosity coefficients can be determined by using the formulae proposed in the work of Sampaio and Williams (1977). For the purpose of simulations, the following values are given to the dimensionless parameters:

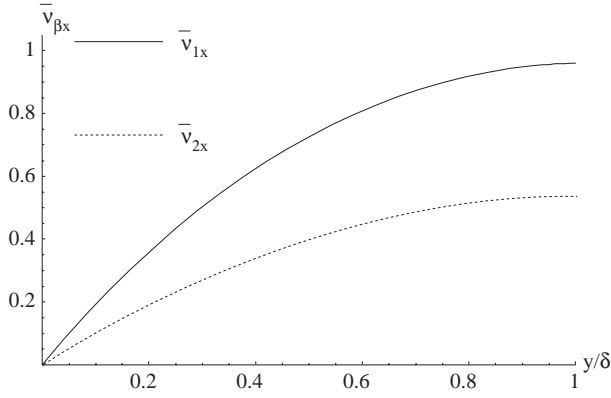$$\bar{M}_1 = 0.32, \quad \bar{M}_2 = \bar{M}_3 = 0.22, \quad \bar{M}_4 = 0.68. \tag{103}$$

Figure 4. Velocity components in the x-direction for $\bar{Q} = 1$, $\bar{\alpha} = 0.45$
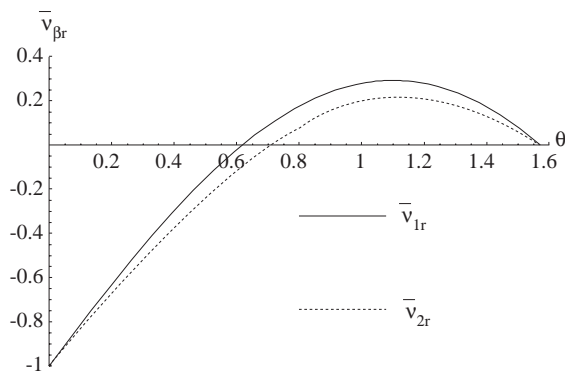

Figure 5. Radial velocity components for $V_1/V = -0.1$, $V_2/V = -0.2$, $\bar{\alpha} = 1$, $\theta_0 = \pi/2$
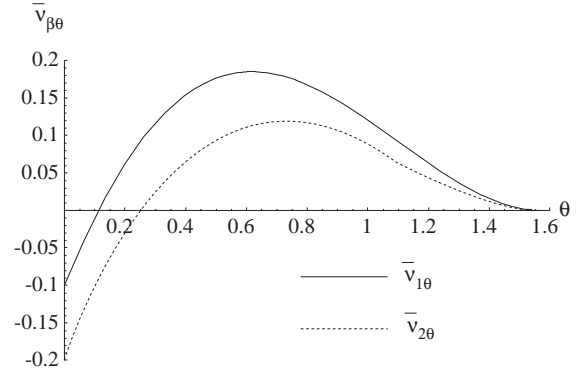

Figure 6. Tangential velocity components for $V_1/V = -0.1$, $V_2/V = -0.2$, $\bar{\alpha} = 1$, $\theta_0 = \pi/2$
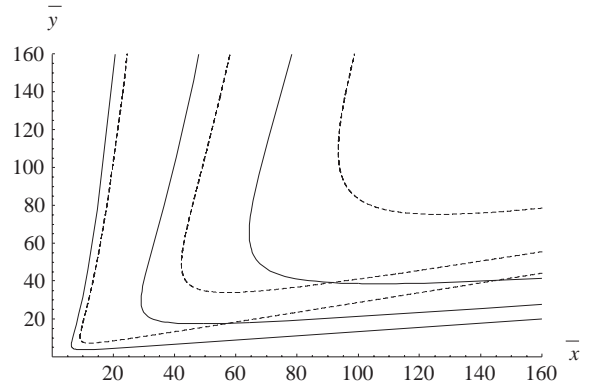

Figure 7. Streamline patterns for $V_1/V = -0.1$, $V_2/V = -0.2$, $\bar{\alpha} = 1$, $\theta_0 = \pi/2$
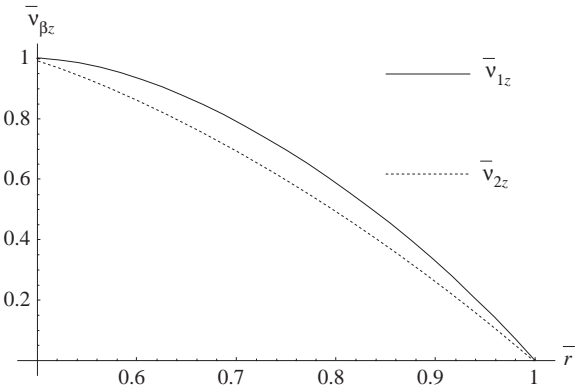

Figure 8. Axial velocity components for $\bar{\alpha} = 10$, $p^* = -5$
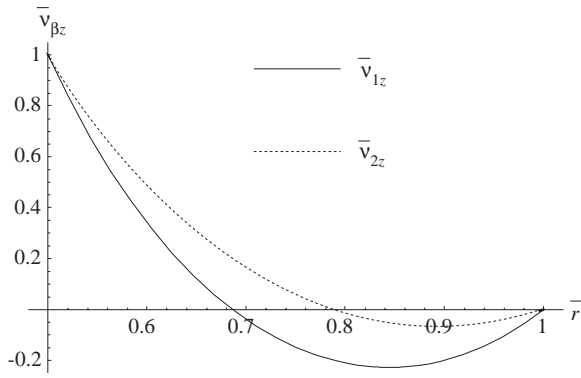
469

Figure 9. Axial velocity components for $\bar{\alpha} = 10$, $p^* = 10$

In Figures 4 to 9, the velocity distributions for constituents of binary mixture under consideration are plotted as a function of position for various values of the parameters, keeping the material constants fixed at the values given in Eq. (103). From these figures, we arrive at the conclusion that the particles of each constituent move independently with velocities $\mathbf{v}_1$, $\mathbf{v}_2$ at a given point in the mixture, but the velocity profiles of mixture components are generally similar to those of pure Newtonian fluids. For the solutions corresponding to pure Newtonian fluid we refer the reader to the books of Batchelor (1967) and Papanastasiou *et al.* (2000).

Nomenclature

A	Helmholtz free energy, L^2T^{-2}
F	external body force, LT^{-2}
f	diffusive force, $ML^{-2}T^{-2}$
I	identity tensor
I_m, K_m	modified Bessel functions of order m
k'	thermal coefficient, $ML^{-1}T^{-2}$
p	pressure, $ML^{-1}T^{-2}$
Q	volumetric flow rate per unit width, L^2T^{-1}
q	heat flux vector, MT^{-3}
r	heat supply, L^2T^{-3}
S	entropy, $L^2T^{-2}\theta^{-1}$
t	time, T
U	internal energy, L^2T^{-2}
V	translation velocity, LT^{-1}
V_1, V_2	injection velocities, LT^{-1}
v	velocity vector, LT^{-1}
w	mean velocity, LT^{-1}
w_1, w_2	angular velocities of the cylinders, T^{-1}
X	reference position of a typical particle, L
x	position vector of the material particle, L
α	diffusive coefficient, $ML^{-3}T^{-1}$
γ	composition factor, dimensionless
$\lambda_1, ..., \mu_1, ...$	material constants, $ML^{-1}T^{-1}$
ρ	density, ML^{-3}
$\boldsymbol{\sigma}$	stress tensor, $ML^{-1}T^{-2}$
ψ	stream function, L^2T^{-1}
$\Im$	temperature, θ

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