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态射和的 Drazin 逆 *

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摘要：设 $\mathcal{C}$ 是加法范畴，态射 $\varphi,\ \eta: X \to X$ 是 $\mathcal{C}$ 上的态射．若 $\varphi,\ \eta$ 具有 Drazin 逆且 $\varphi\eta = 0$，则 $\varphi + \eta$ 也具有 Drazin 逆．若 φ 具有 Drazin 逆 φ^D 且 $1_X + \varphi^D\eta$ 可逆，作者讨论 $f = \varphi + \eta$ 的 Drazin 逆 (群逆) 并且给出 $f^D(f^{\#}) = (1_X + \varphi^D\eta)^{-1}\varphi^D$ 的充分必要条件．最后，把 Huylebrouck 的结果从群逆推广到了 Drazin 逆．

关键词：Drazin 逆；群逆；态射．

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1 引言

设 $P,\ Q$ 为复矩阵，则 P^D 和 Q^D 存在，若 $PQ = 0$, Hartwig, 王和魏 [4] 通过 P, Q, P^D 和 Q^D 给出了 $P + Q$ 的 Drazin 逆的表达式．很自然我们不禁要问上述结论是否可以推广到加法范畴 $\mathcal{C}$, 也就是说，假设 $\varphi, \eta : X \to X$ 为 $\mathcal{C}$ 中态射， φ^D 和 η^D 存在，若 $\varphi\eta = 0$, 那么 $\varphi + \eta$ 的 Drazin 逆是否存在？如果存在，是否同样可以用 φ, η, φ^D 和 η^D 来表示？本文回答了这些问题并且将文献 [10] 中的结论做了推广．

游和陈 [9] 讨论了态射和的群逆，但没有考虑 Drazin 逆的情况．本文将讨论 $\varphi + \eta$ 的 Drazin 逆并将 Huylebrouck 的结论推广到 Drazin 逆．

设 $B = A + E$ 为复矩阵. 魏 [7−8] 研究了群逆的扰动问题并给出了 $B^{\#} = (I + A^D E)^{-1}A^D$ 的充分必要条件. 同时若 E 是可容 A - 扰动矩阵， $I + A^D E$ 可逆， Castro González, Koliha, Straškraba 和魏 [2−3] 研究了 $B^D = (I + A^D E)^{-1}A^D$ 的充分必要条件．同样我们将把上述结论推广到范畴 $\mathcal{C}$ 中的态射上．

定义 1.1[1,5] 设 $\varphi,\ \xi : X \to X$ 为加法范畴 $\mathcal{C}$ 中的态射，满足

(1^k) $\varphi^k = \varphi^k\xi\varphi$;

(2) $\xi = \xi\varphi\xi$;

(5) $\varphi\xi = \xi\varphi$.

则 ξ 称为 φ 的 $(1^k, 2, 5)$ - 逆或 φ 的 Drazin 逆，记为 $\xi = \varphi^D$. 若 $k = 1$, 则 ξ 称为 φ 的群逆，记为 $\xi = \varphi^{\#}$.

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2 $\varphi\eta=0$ 时 $\varphi+\eta$ 的 Drazin 逆

定理 2.1 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi: X\to X$ 具有 $(1^{k_1},2,5)$ - 逆 φ^D, $\eta: X\to X$ 具有 $(1^{k_2},2,5)$ - 逆 η^D 满足 $\varphi\eta=0$. 则 $\varphi+\eta$ 有 $(1^m,2,5)$ - 逆且

$$(\varphi+\eta)^D=(1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D+\eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D),$$

其中 $m=k_1+k_2-1$.

证 令 $w_1=(1_X-\eta\eta^D)\sum\limits_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D$, $w_2=\eta^D\sum\limits_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)$. 记 $w=w_1+w_2$. 因为 $\varphi\eta=0$, 则有 $\varphi\eta^D=\varphi\eta(\eta^D)^2=0$, $\varphi^D\eta=(\varphi^D)^2\varphi\eta=0$. 故 $\varphi w_2=0$, 且

$$\varphi w_1=\varphi(1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D=\varphi\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D=\varphi\varphi^D,$$

$$\begin{aligned}\eta w_1&=\eta(1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D=(1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^{i+1}(\varphi^D)^{(i+1)}\\&=(1_X-\eta\eta^D)\sum_{i=1}^{k_2}\eta^i(\varphi^D)^i=(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i+(1_X-\eta\eta^D)\eta^{k_2}(\varphi^D)^{k_2}\\&=(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i,\\\eta w_2&=\eta\eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)=\eta\eta^D(1_X-\varphi\varphi^D)+\eta\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\\&=\eta\eta^D(1_X-\varphi\varphi^D)+\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D),\end{aligned}$$

因此

$$\begin{aligned}(\varphi+\eta)w&=\varphi w_1+\varphi w_2+\eta w_1+\eta w_2\\&=\varphi\varphi^D+(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i+\eta\eta^D(1_X-\varphi\varphi^D)+\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\\&=\varphi\varphi^D+\eta\eta^D-\eta\eta^D\varphi\varphi^D+(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i+\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D).\end{aligned}$$

另一方面

$$\begin{aligned}w_1\varphi&=(1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D\varphi=(1_X-\eta\eta^D)\varphi^D\varphi+(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D\varphi\\&=(1_X-\eta\eta^D)\varphi^D\varphi+(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i,\\w_2\varphi&=\eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\varphi=\sum_{j=0}^{k_1-1}(\eta^D)^{j+1}\varphi^{j+1}(1_X-\varphi\varphi^D)\end{aligned}$$

$$= \sum_{j=1}^{k_1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D) = \sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)+(\eta^D)^{k_1}\varphi^{k_1}(1_X-\varphi\varphi^D)$$
$$= \sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D),$$

同时有

$$w_1\eta = (1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D\eta = 0,$$

$$w_2\eta = \eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\eta = \eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j\eta = \eta^D\eta,$$

因此

$$\begin{aligned} w(\varphi+\eta) &= w_1\varphi+w_2\varphi+w_1\eta+w_2\eta \\ &= (1_X-\eta\eta^D)\varphi^D\varphi+(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i+\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)+\eta^D\eta \\ &= (\varphi+\eta)w. \end{aligned}$$

接下来我们证明 $w(\varphi+\eta)w = w$. 令

$$\tilde{w_1} = (1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i,\quad \tilde{w_2} = \sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D),$$

则

$$w_1\varphi^D\varphi = w_1,$$

$$w_1\eta\eta^D(1_X-\varphi\varphi^D) = (1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D\eta\eta^D(1_X-\varphi\varphi^D) = 0,$$

$$\varphi^D\tilde{w_1} = \varphi^D(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i = \varphi^D\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i = 0.$$

因此 $w_1\tilde{w_1}=0$. 由 $\varphi^D\eta^D=(\varphi^D)^2\varphi\eta^D=0$ 可得 $\varphi^D\tilde{w_2}=0$. 因此 $w_1\tilde{w_2}=0, w_2\varphi^D\varphi=0$, 且有

$$w_2\eta = \eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\eta = \eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j\eta = \eta^D\eta,$$

或

$$w_2\eta\eta^D(1_X-\varphi\varphi^D) = \eta^D\eta\eta^D(1_X-\varphi\varphi^D) = \eta^D(1_X-\varphi\varphi^D)$$

和

$$\begin{aligned} \sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\tilde{w_1} &= \sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i \\ &= \sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^{j-1}(1_X-\varphi\varphi^D)\varphi\eta(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^{i-1}(\varphi^D)^i \\ &= 0, \end{aligned}$$

$$
\begin{aligned}
w_2\tilde{w_1} &= [\eta^D(1_X-\varphi\varphi^D)+\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)]\tilde{w_1}=\eta^D(1_X-\varphi\varphi^D)\tilde{w_1}\\
&=\eta^D(1_X-\varphi\varphi^D)(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i\\
&=\eta^D(1_X-\varphi\varphi^D)\eta(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^{i-1}(\varphi^D)^i\\
&=\eta^D\eta(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^{i-1}(\varphi^D)^i=0,\\
w_2\tilde{w_2} &= [\eta^D(1_X-\varphi\varphi^D)+\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)]\tilde{w_2}\\
&=\eta^D(1_X-\varphi\varphi^D)\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\\
&\quad+\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\\
&=\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\\
&\quad+\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^{j-1}\varphi(1_X-\varphi\varphi^D)\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^{j-1}\varphi^j(1_X-\varphi\varphi^D)\\
&=\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D),
\end{aligned}
$$

$$
\begin{aligned}
w(\varphi+\eta)w &= (w_1+w_2)[(\varphi+\eta)w]\\
&=(w_1+w_2)[\varphi\varphi^D+\eta\eta^D(1_X-\varphi\varphi^D)+\tilde{w_1}+\tilde{w_2}]\\
&=w_1[\varphi\varphi^D+\eta\eta^D(1_X-\varphi\varphi^D)]+w_1\tilde{w_1}+w_1\tilde{w_2}\\
&\quad+w_2[\varphi\varphi^D+\eta\eta^D(1_X-\varphi\varphi^D)]+w_2\tilde{w_1}+w_2\tilde{w_2}\\
&=w_1+\eta^D(1_X-\varphi\varphi^D)+\eta^D\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\\
&=w_1+\eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)=w_1+w_2=w.
\end{aligned}
$$

最后我们证明 $(\varphi+\eta)^m w(\varphi+\eta)=(\varphi+\eta)^m$, 其中 $m=k_1+k_2-1$. 由 $\varphi\eta=0$, 得

$$
\begin{aligned}
(\varphi+\eta)^m &= \sum_{s=0}^{m}\eta^s\varphi^{m-s},\\
(\varphi+\eta)^m\eta\eta^D(1_X-\varphi\varphi^D) &= \sum_{s=0}^{m}\eta^s\varphi^{m-s}\eta\eta^D(1_X-\varphi\varphi^D)\\
&=\eta^m\eta\eta^D(1_X-\varphi\varphi^D)=\eta^m(1_X-\varphi\varphi^D),
\end{aligned}
$$

$$\begin{aligned}(\varphi+\eta)^m\tilde{w_1} &= (\varphi+\eta)^m(1_X-\eta\eta^D)\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i\\ &= (\varphi+\eta)^m\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i-(\varphi+\eta)^m\eta\eta^D\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i\\ &= \sum_{s=0}^{m}\eta^s\varphi^{m-s}\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i-\sum_{s=0}^{m}\eta^s\varphi^{m-s}\eta\eta^D\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i\\ &= \eta^m\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i-\eta^m\eta\eta^D\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i\\ &= \eta^m\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i-\eta^m\sum_{i=1}^{k_2-1}\eta^i(\varphi^D)^i=0.\end{aligned}$$

现在对 $j=1,2,\cdots,k_1-1$, 有

$$\begin{aligned}\eta^m(\eta^D)^j &= \eta^{m-1}\eta(\eta^D)^2(\eta^D)^{j-2}=\eta^{m-1}(\eta^D)^{j-1}=\cdots\cdots\\ &= \eta^{m-(j-1)}(\eta^D)^{j-(j-1)}=\eta^{m-(j-1)}\eta^D\\ &= \eta^{m-k_2-j}\eta^{k_2+1}\eta^D=\eta^{m-k_2-j}\eta^{k_2}=\eta^{m-j}.\end{aligned}$$

所以

$$\begin{aligned}(\varphi+\eta)^m\tilde{w_2} &= \sum_{s=0}^{m}\eta^s\varphi^{m-s}\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)=\eta^m\sum_{j=1}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)\\ &= \sum_{j=1}^{k_1-1}\eta^m(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D)=\sum_{j=1}^{k_1-1}\eta^{m-j}\varphi^j(1_X-\varphi\varphi^D),\end{aligned}$$

$$\begin{aligned}(\varphi+\eta)^m\varphi\varphi^D &= \sum_{s=0}^{m}\eta^s\varphi^{m-s}\varphi\varphi^D=\sum_{s=0}^{m}\eta^{m-s}\varphi^s\varphi\varphi^D\\ &= \eta^m\varphi\varphi^D+\sum_{s=1}^{k_1-1}\eta^{m-s}\varphi^s\varphi\varphi^D+\sum_{s=k_1}^{m}\eta^{m-s}\varphi^s\varphi\varphi^D\\ &= \eta^m\varphi\varphi^D+\sum_{s=1}^{k_1-1}\eta^{m-s}\varphi^s\varphi\varphi^D+\sum_{s=k_1}^{m}\eta^{m-s}\varphi^s,\end{aligned}$$

$$\begin{aligned}(\varphi+\eta)^m w(\varphi+\eta) &= (\varphi+\eta)^m[\varphi\varphi^D+\eta\eta^D(1_X-\varphi\varphi^D)+\tilde{w_1}+\tilde{w_2}]\\ &= (\varphi+\eta)^m\varphi\varphi^D+(\varphi+\eta)^m\eta\eta^D(1_X-\varphi\varphi^D)+(\varphi+\eta)^m\tilde{w_1}+(\varphi+\eta)^m\tilde{w_2}\\ &= \eta^m\varphi\varphi^D+\sum_{s=1}^{k_1-1}\eta^{m-s}\varphi^s\varphi\varphi^D+\sum_{s=k_1}^{m}\eta^{m-s}\varphi^s+\eta^m(1_X-\varphi\varphi^D)\\ &\quad+\sum_{j=1}^{k_1-1}\eta^{m-j}\varphi^j(1_X-\varphi\varphi^D)\\ &= \eta^m+\sum_{s=1}^{k_1-1}\eta^{m-s}\varphi^s+\sum_{s=k_1}^{m}\eta^{m-s}\varphi^s=\eta^m+\sum_{s=1}^{m}\eta^{m-s}\varphi^s=(\varphi+\eta)^m.\end{aligned}$$

因此， w 是 $\varphi+\eta$ 的 $(1^m,2,5)$ - 逆，即， $(\varphi+\eta)^D$ 存在且有

$$(\varphi+\eta)^D=w=(1_X-\eta\eta^D)\sum_{i=0}^{k_2-1}\eta^i(\varphi^D)^i\varphi^D+\eta^D\sum_{j=0}^{k_1-1}(\eta^D)^j\varphi^j(1_X-\varphi\varphi^D).$$ ▎

注 定理 2.1 回答了文献 [10] 中提出的问题.

推论 2.2 设 $\mathcal{C}$ 为加法范畴. 若态射 $\varphi,\eta:X\to X$ 分别有群逆 $\varphi^\#$ 和 $\eta^\#$ 且满足 $\varphi\eta=0$, 则 $\varphi+\eta$ 有群逆且 $(\varphi+\eta)^\#=(1_X-\eta\eta^\#)\varphi^\#+\eta^\#(1_X-\varphi\varphi^\#)$.

定理 2.3 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi:X\to X$ 具有 $(1^{k_1},2,5)$ - 逆 φ^D 且满足 $\varphi\varphi^D\eta=\eta$. 若 $\gamma=(\varphi+\eta)\varphi\varphi^D$ 有 $(1^{k_2},2,5)$ - 逆 γ^D. 则 $\varphi+\eta$ 有 $(1^{k_1+k_2},2,5)$ - 逆且

$$(\varphi+\eta)^D=\gamma^D+\sum_{i=0}^{k_1-1}(\gamma^D)^{i+2}\eta\varphi^i(1_X-\varphi\varphi^D).$$

证 记 $q=\varphi^2\varphi^D+\eta$, $s=\eta(1_X-\varphi\varphi^D),\gamma=(\varphi+\eta)\varphi\varphi^D$, 则 $q=s+\gamma$ 且有

$$s\gamma=\eta(1_X-\varphi\varphi^D)(\varphi+\eta)\varphi\varphi^D=\eta(1_X-\varphi\varphi^D)\varphi\varphi\varphi^D+\eta(1_X-\varphi\varphi^D)\eta\varphi\varphi^D=0,$$

$$s^2=\eta(1_X-\varphi\varphi^D)\eta(1_X-\varphi\varphi^D)=0.$$

因此 s 有 $(1^2,2,5)$ - 逆和 $s^D=0$. 由定理 2.1 得 q 有 $(1^{k_2+1},2,5)$ - 逆,

$$q^D=\gamma^D(1_X+\gamma^Ds)=\gamma^D+(\gamma^D)^2s.$$

令 $p=\varphi(1_X-\varphi\varphi^D)$, 则 $\varphi+\eta=p+q$,

$$pq=\varphi(1_X-\varphi\varphi^D)(\varphi^2\varphi^D+\eta)=\varphi(1_X-\varphi\varphi^D)\varphi^2\varphi^D+\varphi(1_X-\varphi\varphi^D)\eta=0,$$

$$p^{k_1}=[\varphi(1_X-\varphi\varphi^D)]^{k_1}=\varphi^{k_1}(1_X-\varphi\varphi^D)=0.$$

所以， p 有 $(1^{k_1},2,5)$ - 逆， $p^D=0$. 又由定理 2.1 得 $\varphi+\eta=p+q$ 有 $(1^m,2,5)$ - 逆 (其中 $m=k_1+k_2$) 且有

$$\begin{aligned}(\varphi+\eta)^D=(p+q)^D&=q^D\sum_{j=0}^{k_1-1}(q^D)^jp^j(1_X-pp^D)\\&=q^D\sum_{j=0}^{k_1-1}(q^D)^jp^j=q^D+\sum_{j=1}^{k_1-1}(q^D)^{j+1}p^j.\end{aligned}$$

由 $s\gamma=0$, 我们得 $s\gamma^D=0$, 故

$$(q^D)^2=[\gamma^D+(\gamma^D)^2s][\gamma^D+(\gamma^D)^2s]=(\gamma^D)^2+(\gamma^D)^3s.$$

更进一步， $(q^D)^i=(\gamma^D)^i+(\gamma^D)^{i+1}s$, $i\geq 1$. 注意到

$$sp=\eta(1_X-\varphi\varphi^D)\varphi(1_X-\varphi\varphi^D)=\eta\varphi(1_X-\varphi\varphi^D)=\eta p,$$

$$\gamma p=(\varphi+\eta)\varphi\varphi^D\varphi(1_X-\varphi\varphi^D)=0,\quad \gamma^Dp=0,$$

$$(q^D)^{i+1}p=[(\gamma^D)^{i+1}+(\gamma^D)^{i+2}s]p=(\gamma^D)^{i+1}p+(\gamma^D)^{i+2}\eta p=(\gamma^D)^{i+2}\eta p,$$

$$(q^D)^{i+1}p^i = (\gamma^D)^{i+2}\eta p^i = (\gamma^D)^{i+2}\eta[\varphi(1_X - \varphi\varphi^D)]^i = (\gamma^D)^{i+2}\eta\varphi^i(1_X - \varphi\varphi^D).$$

因此得

$$\begin{aligned}(\varphi+\eta)^D &= q^D + \sum_{i=1}^{k_1-1}(q^D)^{i+1}p^i = \gamma^D + (\gamma^D)^2 s + \sum_{i=1}^{k_1-1}(\gamma^D)^{i+2}\eta\varphi^i(1_X - \varphi\varphi^D)\\ &= \gamma^D + \sum_{i=0}^{k_1-1}(\gamma^D)^{i+2}\eta\varphi^i(1_X - \varphi\varphi^D).\end{aligned}$$ ▌

推论 2.4[10] 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi: X \to X$ 有 $(1^{k_1}, 2, 5)$ - 逆 φ^D 且态射 $\eta: X \to X$ 满足 $\varphi\varphi^D\eta = \eta$, $1_X + \varphi^D\eta$ 可逆, 则 $\varphi+\eta$ 有 $(1^{k_1+1}, 2, 5)$ - 逆, 且有

$$(\varphi+\eta)^D = \xi + \sum_{i=0}^{k_1-1}\xi^{i+2}\eta\varphi^i(1_X - \varphi\varphi^D),$$

其中 $\xi = (1_X + \varphi^D\eta)^{-1}\varphi^D$.

证 令 $\gamma = (\varphi+\eta)\varphi\varphi^D$, 接下来我们将证明 $\gamma^{\#} = \xi$.

$$\begin{aligned}\gamma\xi &= (\varphi+\eta)\varphi\varphi^D(1_X+\varphi^D\eta)^{-1}\varphi^D = (\varphi+\eta)\varphi\varphi^D\varphi^D(1_X+\eta\varphi^D)^{-1}\\ &= (\varphi+\eta)\varphi^D(1_X+\eta\varphi^D)^{-1} = (\varphi\varphi^D+\eta\varphi^D)(1_X+\eta\varphi^D)^{-1}\\ &= (\varphi\varphi^D+\varphi\varphi^D\eta\varphi^D)(1_X+\eta\varphi^D)^{-1} = \varphi\varphi^D,\\ \xi\gamma &= (1_X+\varphi^D\eta)^{-1}\varphi^D(\varphi+\eta)\varphi\varphi^D\\ &= (1_X+\varphi^D\eta)^{-1}\varphi^D\varphi + (1_X+\varphi^D\eta)^{-1}\varphi^D\eta\varphi\varphi^D\\ &= (1_X+\varphi^D\eta)^{-1}(1_X+\varphi^D\eta)\varphi\varphi^D = \varphi\varphi^D.\end{aligned}$$

故

$$\gamma\xi = \xi\gamma,$$

$$\gamma\xi\gamma = \varphi\varphi^D\gamma = \varphi\varphi^D(\varphi+\eta)\varphi\varphi^D = \varphi^2\varphi^D + \eta\varphi\varphi^D = (\varphi+\eta)\varphi\varphi^D = \gamma,$$

$$\xi\gamma\xi = (1_X+\varphi^D\eta)^{-1}\varphi^D\varphi\varphi^D = (1_X+\varphi^D\eta)^{-1}\varphi^D = \xi.$$

则有 $\gamma^{\#} = \xi$, 由定理 2.3 知 $\varphi+\eta$ 有 $(1^{k_1+1}, 2, 5)$ - 逆

$$(\varphi+\eta)^D = \gamma^D + \sum_{i=0}^{k_1-1}(\gamma^D)^{i+2}\eta\varphi^i(1_X - \varphi\varphi^D) = \xi + \sum_{i=0}^{k_1-1}\xi^{i+2}\eta\varphi^i(1_X - \varphi\varphi^D).$$ ▌

定理 2.5 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi: X \to X$ 具有 $(1^{k_1}, 2, 5)$ - 逆 φ^D 且态射 $\eta: X \to X$ 满足 $\varphi^D\eta = 0$. 若 $\gamma = (\varphi+\eta)(1_X - \varphi\varphi^D)$ 有 $(1^{k_2}, 2, 5)$ - 逆 γ^D, 则 $\varphi+\eta$ 有 $(1^{k_2+1}, 2, 5)$ - 逆且有

$$(\varphi+\eta)^D = \gamma^D + (1_X - \gamma^D\eta)\varphi^D + (1_X - \gamma\gamma^D)\sum_{i=0}^{k_2-1}(\varphi+\eta)^i\eta(\varphi^D)^{i+2}.$$

证 令 $p = \varphi^2\varphi^D$, 则 p 有群逆 $p^{\#} = \varphi^D$, 令 $q = \varphi(1_X - \varphi\varphi^D) + \eta$, 则有

$$\varphi + \eta = p + q,$$

$$pq = \varphi^2\varphi^D\varphi(1_X - \varphi\varphi^D) + \varphi^2\varphi^D\eta = 0,$$

令 $s = \eta\varphi\varphi^D$, 则 $q = s + \gamma$, 且 $s^2 = \eta\varphi\varphi^D\eta\varphi\varphi^D = 0$. 故 s 有 $(1^2, 2, 5)$ - 逆 $s^D = 0$,

$$s\gamma = \eta\varphi\varphi^D[(\varphi + \eta)(1_X - \varphi\varphi^D)] = \eta\varphi\varphi^D\varphi(1_X - \varphi\varphi^D) = 0.$$

则有 $s\gamma^D = 0$. 由定理 2.1 知 q 有 $(1^k, 2, 5)$ - 逆，其中 $k = k_2 + 1$.

$$q^D = (s + \gamma)^D = \gamma^D + (\gamma^D)^2 s,$$

$$qq^D = (s + \gamma)[\gamma^D + (\gamma^D)^2 s] = \gamma[\gamma^D + (\gamma^D)^2 s] = \gamma\gamma^D + \gamma^D s.$$

由定理 2.1 知， $\varphi + \eta$ 有 $(1^k, 2, 5)$ - 逆，且

$$\begin{aligned}(\varphi + \eta)^D = (p + q)^D &= (1_X - qq^D)\sum_{i=0}^{k-1} q^i (p^D)^i p^D + q^D(1_X - pp^D) \\ &= (1_X - qq^D)\sum_{i=0}^{k-1} q^i (p^\sharp)^i p^\sharp + q^D(1_X - pp^\sharp) \\ &= (1_X - qq^D)\sum_{i=0}^{k-1} q^i (\varphi^D)^i \varphi^D + q^D(1_X - \varphi\varphi^D).\end{aligned}$$

故有 $q\varphi^D = [\varphi(1_X - \varphi\varphi^D) + \eta]\varphi^D = \eta\varphi^D$, $q^2\varphi^D = (\varphi + \eta - \varphi^2\varphi^D)\eta\varphi^D = (\varphi + \eta)\eta\varphi^D$. 注意到 $\varphi^D(\varphi + \eta) = \varphi^D\varphi$, $\varphi^D(\varphi + \eta)^2 = \varphi^D\varphi(\varphi + \eta) = \varphi^2\varphi^D$. 因为 $\varphi^D(\varphi + \eta)^i = \varphi^i\varphi^D$, 所以 $\varphi^D(\varphi + \eta)^i\eta = \varphi^i\varphi^D\eta = 0$. 由归纳法知 $q^i\varphi^D = (\varphi + \eta)^{i-1}\eta\varphi^D, i \geq 1$. 所以

$$q^i(\varphi^D)^{i+1} = (\varphi + \eta)^{i-1}\eta(\varphi^D)^{i+1}.$$

又

$$s(\varphi + \eta)^{i-1}\eta = \eta\varphi\varphi^D(\varphi + \eta)^{i-1}\eta = 0,$$

$$s(1_X - \varphi\varphi^D) = \eta\varphi\varphi^D(1_X - \varphi\varphi^D) = 0,$$

$$s\varphi^D = \eta\varphi\varphi^D\varphi^D = \eta\varphi^D,$$

$$\gamma\varphi^D = (\varphi + \eta)(1_X - \varphi\varphi^D)\varphi^D = 0.$$

即得

$$\gamma^D\varphi^D = 0,$$

$$q^D(1_X - \varphi\varphi^D) = [\gamma^D + (\gamma^D)^2 s](1_X - \varphi\varphi^D) = \gamma^D(1_X - \varphi\varphi^D) = \gamma^D,$$

$$(1_X - qq^D)\varphi^D = (1_X - \gamma\gamma^D - \gamma^D s)\varphi^D = \varphi^D - \gamma\gamma^D\varphi^D - \gamma^D s\varphi^D = \varphi^D - \gamma^D\eta\varphi^D,$$

$$\begin{aligned}(1_X - qq^D)q^i(\varphi^D)^{i+1} &= (1_X - \gamma\gamma^D - \gamma^D s)q^i(\varphi^D)^{i+1} \\ &= (1_X - \gamma\gamma^D)q^i(\varphi^D)^{i+1} - \gamma^D s q^i(\varphi^D)^{i+1} \\ &= (1_X - \gamma\gamma^D)(\varphi + \eta)^{i-1}\eta(\varphi^D)^{i+1} - \gamma^D s(\varphi + \eta)^{i-1}\eta(\varphi^D)^{i+1} \\ &= (1_X - \gamma\gamma^D)(\varphi + \eta)^{i-1}\eta(\varphi^D)^{i+1}.\end{aligned}$$

因此

$$\begin{aligned}(\varphi+\eta)^D &= (1_X-qq^D)\sum_{i=0}^{k-1}q^i(\varphi^D)^{i+1}+q^D(1_X-\varphi\varphi^D)\\ &= (1_X-qq^D)\varphi^D+q^D(1_X-\varphi\varphi^D)+\sum_{i=1}^{k-1}(1_X-qq^D)q^i(\varphi^D)^{i+1}\\ &= \varphi^D-\gamma^D\eta\varphi^D+\gamma^D+\sum_{i=1}^{k-1}(1_X-\gamma\gamma^D)(\varphi+\eta)^{i-1}\eta(\varphi^D)^{i+1}\\ &= \gamma^D+(1_X-\gamma^D\eta)\varphi^D+\sum_{i=0}^{k-2}(1_X-\gamma\gamma^D)(\varphi+\eta)^i\eta(\varphi^D)^{i+2}\\ &= \gamma^D+(1_X-\gamma^D\eta)\varphi^D+(1_X-\gamma\gamma^D)\sum_{i=0}^{k_2-1}(\varphi+\eta)^i\eta(\varphi^D)^{i+2}.\end{aligned}$$

∎

3 $1_X+\varphi^D\eta$ 可逆时 $\varphi+\eta$ 的 Drazin 逆

定理 3.1 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi: X\to X$ 有 $(1^k,2,5)$ - 逆 φ^D 且态射 $\eta: X\to X$ 使得 $1_X+\varphi^D\eta$ 可逆. 记

$$\gamma=\alpha(1_X-\varphi^D\varphi)\eta\varphi^D\beta,$$

$$\delta=\alpha\varphi^D\eta(1_X-\varphi\varphi^D)\beta,$$

$$\varepsilon=(1_X-\varphi\varphi^D)\eta\alpha(1_X-\varphi^D\varphi),$$

其中 $\alpha=(1_X+\varphi^D\eta)^{-1}$, $\beta=(1_X+\eta\varphi^D)^{-1}$. 若 $1_X-\gamma$ 和 $1_X-\delta$ 可逆且满足

$$\eta(\varphi^D\varphi-1_X)\varphi=0,\quad \varphi(\varphi\varphi^D-1_X)\eta=0.$$

则 $f=\varphi+\eta-\varepsilon$ 有 $(1^k,2,5)$ - 逆.

证 令 $f_0^{(2)}=\alpha\varphi^D$. 易知 $\alpha\varphi^D=\varphi^D\beta$, $\varphi^D\varepsilon=\varepsilon\varphi^D=0$. 则有

$$\begin{aligned}f_0^{(2)}ff_0^{(2)} &= \alpha\varphi^D(\varphi+\eta-\varepsilon)\alpha\varphi^D=\alpha\varphi^D\varphi\alpha\varphi^D+\alpha\varphi^D\eta\alpha\varphi^D\\ &= \alpha\varphi^D\varphi\varphi^D\beta+\alpha\varphi^D\eta\alpha\varphi^D=\alpha\varphi^D\beta+\alpha\varphi^D\eta\alpha\varphi^D\\ &= \alpha\varphi^D(\beta+\eta\alpha\varphi^D)=\alpha\varphi^D(\beta+\beta\eta\varphi^D)\\ &= \alpha\varphi^D\beta(1_X+\eta\varphi^D)=\alpha\varphi^D=f_0^{(2)},\\ ff_0^{(2)}f &= f\alpha\varphi^Df=(\varphi+\eta-\varepsilon)\alpha\varphi^D(\varphi+\eta-\varepsilon)=(\varphi+\eta)\alpha\varphi^D(\varphi+\eta)\\ &= \varphi\alpha\varphi^D(\varphi+\eta)+\eta\alpha\varphi^D(\varphi+\eta)\\ &= \varphi(1_X-\alpha\varphi^D\eta)\varphi^D(\varphi+\eta)+\eta\alpha\varphi^D(\varphi+\eta)\\ &= \varphi\varphi^D(\varphi+\eta)-\varphi\alpha\varphi^D\eta\varphi^D(\varphi+\eta)+\eta\alpha\varphi^D(\varphi+\eta)\\ &= \varphi\varphi^D\varphi+\eta-\eta+\varphi\varphi^D\eta-\varphi\varphi^D\eta\alpha\varphi^D(\varphi+\eta)+\eta\alpha\varphi^D(\varphi+\eta)\\ &= \varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta+(1_X-\varphi\varphi^D)\eta\alpha\varphi^D(\varphi+\eta)\\ &= \varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta[1_X-\alpha\varphi^D(\varphi+\eta)]\\ &= \varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta(1_X-\alpha\varphi^D\varphi-\alpha\varphi^D\eta)\\ &= \varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta\alpha(1_X-\varphi\varphi^D)\\ &= \varphi\varphi^D\varphi+\eta-\varepsilon=(\varphi\varphi^D\varphi-\varphi)+f.\end{aligned}$$

因为

$$\varepsilon\varphi=(1_X-\varphi\varphi^D)\eta\alpha(1_X-\varphi\varphi^D)\varphi=(1_X-\varphi\varphi^D)\beta\eta(1_X-\varphi\varphi^D)\varphi=0.$$

类似可得， $\varphi\varepsilon=0$.

$$\begin{aligned}f^2f_0^{(2)}f&=f(f+\varphi\varphi^D\varphi-\varphi)=f^2+(\varphi+\eta-\varepsilon)(\varphi^D\varphi-1_X)\varphi\\&=f^2+(\varphi^D\varphi-1_X)\varphi^2+\eta(\varphi^D\varphi-1_X)\varphi-\varepsilon\varphi(\varphi^D\varphi-1_X)\\&=f^2+(\varphi^D\varphi-1_X)\varphi^2.\end{aligned}$$

由归纳法知 $f^sf_0^{(2)}f=f^s+(\varphi^D\varphi-1_X)\varphi^s,\ \ s\geq 1$. 特别地

$$f^kf_0^{(2)}f=f^k+(\varphi^D\varphi-1_X)\varphi^k=f^k.$$

类似，由 $\varphi(\varphi\varphi^D-1_X)\eta=0$ 可得 $f^k=ff_0^{(2)}f^k$ 和

$$\varphi^Df=\varphi^D(\varphi+\eta-\varepsilon)=\varphi^D\varphi(1_X+\varphi^D\eta)=\varphi^D\varphi\alpha^{-1},$$

$$f_0^{(2)}f=\alpha\varphi^Df=\alpha\varphi^D\varphi\alpha^{-1}.$$

同样有

$$f\varphi^D=\beta^{-1}\varphi\varphi^D,\quad ff_0^{(2)}=\beta^{-1}\varphi\varphi^D\beta,$$

$$1_X-ff_0^{(2)}=\beta^{-1}(1_X-\varphi\varphi^D)\beta=(1_X+\eta\varphi^D)(1_X-\varphi\varphi^D)\beta=(1_X-\varphi\varphi^D)\beta,$$

$$1_X-f_0^{(2)}f=\alpha(1_X-\varphi\varphi^D)\alpha^{-1}=\alpha(1_X-\varphi\varphi^D)(1_X+\varphi^D\eta)=\alpha(1_X-\varphi\varphi^D),$$

$$\begin{aligned}1_X-f_0^{(2)}f(1_X-ff_0^{(2)})&=1_X-\alpha\varphi^D\varphi\alpha^{-1}(1_X-\varphi\varphi^D)\beta\\&=1_X-\alpha\varphi^D\varphi(1_X+\varphi^D\eta)(1_X-\varphi\varphi^D)\beta\\&=1_X-\alpha(\varphi^D\varphi+\varphi^D\eta)(1_X-\varphi\varphi^D)\beta\\&=1_X-\alpha\varphi^D\eta(1_X-\varphi\varphi^D)\beta=1_X-\delta.\end{aligned}$$

类似有

$$1_X-(1_X-f_0^{(2)}f)ff_0^{(2)}=1_X-\gamma.$$

$$\begin{aligned}f^{k+1}f_0^{(2)}&=f^kff_0^{(2)}=f^k-f^k(1_X-ff_0^{(2)})=f^k-f^kf_0^{(2)}f(1_X-ff_0^{(2)})\\&=f^k[1_X-f_0^{(2)}f(1_X-ff_0^{(2)})]=f^k(1_X-\delta).\end{aligned}$$

$$f_0^{(2)}f^{k+1}=(1_X-\gamma)f^k.$$

由 $1_X-\gamma$ 和 $1_X-\delta$ 可逆，得

$$f^k=f^{k+1}f_0^{(2)}(1_X-\delta)^{-1}=f^{k+1}x,\quad x=f_0^{(2)}(1_X-\delta)^{-1},$$

$$f^k=(1_X-\gamma)^{-1}f_0^{(2)}f^{k+1}=yf^{k+1},\quad y=(1_X-\gamma)^{-1}f_0^{(2)}.$$

因此， f 有 $(1^k,2,5)$ - 逆

$$\begin{aligned}f^D&=y^kf^kx=[(1_X-\gamma)^{-1}f_0^{(2)}]^kf^kf_0^{(2)}(1_X-\delta)^{-1}\\&=(1_X-\gamma)^{-1}[f_0^{(2)}(1_X-\gamma)^{-1}]^{k-1}f_0^{(2)}f^kf_0^{(2)}(1_X-\delta)^{-1}.\end{aligned}$$ ∎

推论 3.2 [5] 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi: X \to X$ 有群逆 $\varphi^{\#}$ 且态射 $\eta: X \to X$ 使得 $1_X + \varphi^{\#}\eta$ 可逆. 记

$$\gamma = \alpha(1_X - \varphi^{\#}\varphi)\eta\varphi^{\#}\beta,$$

$$\delta = \alpha\varphi^{\#}\eta(1_X - \varphi\varphi^{\#})\beta,$$

$$\varepsilon = (1_X - \varphi\varphi^{\#})\eta\alpha(1_X - \varphi^{\#}\varphi),$$

其中 $\alpha = (1_X + \varphi^{\#}\eta)^{-1}$, $\beta = (1_X + \eta\varphi^{\#})^{-1}$. 若 $1_X - \gamma$ 和 $1_X - \delta$ 可逆, 则 $f = \varphi + \eta - \varepsilon$ 有群逆且有 $f^{\#} = (1_X - \gamma)^{-1}\alpha\varphi^{\#}(1_X - \delta)^{-1}$.

推论 3.3 R 是有单位元的环, $J(R)$ 为 Jacobson 根. 设 R 中元素 a 有 $(1^k, 2, 5)$ - 逆 a^D, j 为 $J(R)$ 中元素. 若 $j(a^Da - 1)a = 0$, $a(a^Da - 1)j = 0$, 则 $a + j - \varepsilon$ 有 $(1^k, 2, 5)$ - 逆, 其中 $\varepsilon = (1 - aa^D)j(1 + a^Dj)^{-1}(1 - a^Da)$

证 因为 $j \in J(R)$, 则 $1 + a^{\#}j$, $1 - \gamma$, $1 - \delta$ 可逆, 由定理 3.1 知, 推论 3.3 成立. ∎

定理 3.4 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi: X \to X$ 有 Drazin 逆 φ^D, 态射 $\eta: X \to X$ 使得 $1_X + \varphi^D\eta$ 可逆且满足

$$\varphi\varphi^D\eta = \eta\varphi\varphi^D, \quad \varphi(1_X - \varphi\varphi^D)\eta = \eta\varphi(1_X - \varphi\varphi^D).$$

记 $f = \varphi + \eta$, 则下列叙述等价

(1) f 有 Drazin 逆, 且 $f^D = (1_X + \varphi^D\eta)^{-1}\varphi^D$;

(2) $\varphi\varphi^D\eta^m = \eta^m$ 其中 m 为正整数;

(3) $\eta^m\varphi\varphi^D = \eta^m$ 其中 m 为正整数.

此时, $ff^D = \varphi\varphi^D$.

证 令 $f_0^{(2)} = (1_X + \varphi^D\eta)^{-1}\varphi^D$, 与定理 3.1 证明类似,

$$1_X - f_0^{(2)}f = \alpha(1_X - \varphi^D\varphi), \quad 1_X - ff_0^{(2)} = (1_X - \varphi^D\varphi)\beta,$$

由 $\varphi\varphi^D\eta = \eta\varphi^D\varphi$ 可得

$$(1_X - \varphi\varphi^D)\eta = \eta(1_X - \varphi^D\varphi). \tag{3.1}$$

于是

$$\begin{aligned}\alpha^{-1}(1_X - \varphi^D\varphi) &= (1_X + \varphi^D\eta)(1_X - \varphi^D\varphi) = 1_X - \varphi^D\varphi + \varphi^D\eta(1_X - \varphi^D\varphi)\\ &= 1_X - \varphi^D\varphi + \varphi^D(1_X - \varphi^D\varphi)\eta = 1_X - \varphi^D\varphi.\end{aligned}$$

类似有 $(1_X - \varphi^D\varphi)\beta^{-1} = 1_X - \varphi^D\varphi$. 故

$$\alpha(1_X - \varphi^D\varphi) = (1_X - \varphi^D\varphi)\beta = 1_X - \varphi^D\varphi.$$

因此 $1_X - f_0^{(2)}f = 1_X - ff_0^{(2)} = 1_X - \varphi^D\varphi$, 即得

$$f_0^{(2)}f = ff_0^{(2)} = \varphi^D\varphi. \tag{3.2}$$

与定理 3.1 证明类似有, $ff_0^{(2)}f = f + (\varphi\varphi^D\varphi - \varphi - \varepsilon)$, 其中

$$\begin{aligned}\varepsilon &= (1_X - \varphi\varphi^D)\eta\alpha(1_X - \varphi\varphi^D) = \eta(1_X - \varphi\varphi^D)\alpha(1_X - \varphi\varphi^D)\\ &= \eta(1_X - \varphi\varphi^D)(1_X - \varphi\varphi^D) = \eta(1_X - \varphi\varphi^D).\end{aligned}$$

由 (3.1) 式和假设得 $\varphi\eta(1_X-\varphi\varphi^D)=\eta\varphi(1_X-\varphi\varphi^D)$. 由归纳法知

$$\varphi^s\eta(1_X-\varphi^D\varphi)=\eta\varphi^s(1_X-\varphi^D\varphi),\ \forall s\geq 0. \tag{3.3}$$

接下来证明

$$f^s(1_X-\varphi\varphi^D)=\sum_{i=0}^{s}C_s^i\varphi^{s-i}\eta^i(1_X-\varphi^D\varphi). \tag{3.4}$$

事实上，由 (3.1) 式

$$(1_X-\varphi\varphi^D)\eta^j=\eta^j(1_X-\varphi\varphi^D),\ \forall j\geq 1, \tag{3.5}$$

$$\begin{aligned}\varphi^{s-j}\eta^{j+1}(1_X-\varphi^D\varphi)&=\varphi^{s-j}\eta\eta^j(1_X-\varphi^D\varphi)\\&=\varphi^{s-j}\eta(1_X-\varphi\varphi^D)\eta^j\qquad \text{由 (3.5) 式}\\&=\eta\varphi^{s-j}(1_X-\varphi\varphi^D)\eta^j\qquad \text{由 (3.3) 式}\\&=\eta\varphi^{s-j}\eta^j(1_X-\varphi\varphi^D).\qquad \text{由 (3.5) 式}\end{aligned} \tag{3.6}$$

容易验证 $s=1$ 时 (3.4) 式成立，假设结论在 s 时成立

$$\begin{aligned}f^{s+1}(1_X-\varphi^D\varphi)&=ff^s(1_X-\varphi^D\varphi)=(\varphi+\eta)\sum_{i=0}^{s}C_s^i\varphi^{s-i}\eta^i(1_X-\varphi^D\varphi)\\&=\sum_{i=0}^{s}C_s^i\varphi^{s-i+1}\eta^i(1_X-\varphi^D\varphi)+\sum_{i=0}^{s}C_s^i\eta\varphi^{s-i}\eta^i(1_X-\varphi^D\varphi)\\&=\varphi^{s+1}(1_X-\varphi^D\varphi)+\sum_{i=1}^{s}C_s^i\varphi^{s-i+1}\eta^i(1_X-\varphi^D\varphi)\\&\quad+\sum_{i=0}^{s-1}C_s^i\eta\varphi^{s-i}\eta^i(1_X-\varphi^D\varphi)+\eta^{s+1}(1_X-\varphi^D\varphi)\\&=\varphi^{s+1}(1_X-\varphi^D\varphi)+\sum_{i=0}^{s-1}C_s^{i+1}\varphi^{s-i}\eta^{i+1}(1_X-\varphi^D\varphi)\\&\quad+\sum_{i=0}^{s-1}C_s^i\varphi^{s-i}\eta^{i+1}(1_X-\varphi^D\varphi)+\eta^{s+1}(1_X-\varphi^D\varphi)\qquad \text{由 (3.6) 式}\\&=\varphi^{s+1}(1_X-\varphi^D\varphi)+\sum_{i=0}^{s-1}C_{s+1}^{i+1}\varphi^{s-i}\eta^{i+1}(1_X-\varphi^D\varphi)+\eta^{s+1}(1_X-\varphi^D\varphi)\\&=\sum_{i=0}^{s+1}C_{s+1}^i\varphi^{(s+1)-i}\eta^i(1_X-\varphi^D\varphi).\end{aligned}$$

因此对任意 s, (3.4) 式成立.

$$\begin{aligned}ff_0^{(2)}f&=f+\varphi\varphi^D\varphi-\varphi-\varepsilon=f-\varphi(1_X-\varphi^D\varphi)-\eta(1_X-\varphi^D\varphi)\\&=f-(\varphi+\eta)(1_X-\varphi^D\varphi)=f-f(1_X-\varphi^D\varphi)\end{aligned}$$

故

$$f^sf_0^{(2)}f=f^s-f^s(1_X-\varphi^D\varphi). \tag{3.7}$$

(2) ⇔ (3) 由 (3.5) 式即得.

(2) ⇒ (1) 若存在 $m \geq 1$, 使得 $\varphi\varphi^D\eta^m = \eta^m$, 则

$$(1_X - \varphi^D\varphi)\eta^m = 0. \tag{3.8}$$

因为 φ^D 是 φ 的 $(1^k, 2, 5)$ - 逆. 令 $s = m + k - 1$, 则有

$$\begin{aligned} f^s(1_X - \varphi^D\varphi) &= \varphi^s(1_X - \varphi^D\varphi) + C_s^1\varphi^{s-1}(1_X - \varphi^D\varphi)\eta + \cdots + C_s^{s-k}\varphi^k(1_X - \varphi^D\varphi)\eta^{s-k} \\ &\quad + C_s^{s-k+1}\varphi^{k-1}(1_X - \varphi^D\varphi)\eta^{s-k+1} + \cdots + (1_X - \varphi^D\varphi)\eta^s \\ &= 0 \qquad \text{由 (3.8) 式,} \end{aligned}$$

因此由 (3.7) 式知

$$f^s = f^s f_0^{(2)} f, \quad f_0^{(2)} f = f f_0^{(2)}, \quad f_0^{(2)} f f_0^{(2)} = f_0^{(2)},$$

故 $f_0^{(2)}$ 是 $(1^s, 2, 5)$ - 逆, $f^D = f_0^{(2)} = (1_X + \varphi^D\eta)^{-1}\varphi^D$. 由 (3.2) 式得 $ff^D = \varphi\varphi^D$.

(1) ⇒ (2) 若 $f^D = (1_X + \varphi^D\eta)^{-1}\varphi^D$ 为 f 的 $(1^s, 2, 5)$ - 逆, 因为 φ^D 为 φ 的 $(1^k, 2, 5)$ - 逆. 令

$$u = f - f^2 f^D, \quad v = \varphi - \varphi^2\varphi^D,$$

则 $u^s = 0$, $v^k = 0$, 并由 (3.2) 式可得 $ff^D = \varphi\varphi^D$,

$$u = f(1_X - ff^D) = f(1_X - \varphi\varphi^D),$$

$$v = \varphi(1_X - \varphi\varphi^D).$$

于是

$$\begin{aligned} uv &= f(1_X - \varphi\varphi^D)\varphi(1_X - \varphi\varphi^D) = (\varphi + \eta)\varphi(1_X - \varphi\varphi^D) \\ &= \varphi^2(1_X - \varphi\varphi^D) + \eta\varphi(1_X - \varphi\varphi^D) = \varphi^2(1_X - \varphi\varphi^D) + \varphi(1_X - \varphi\varphi^D)\eta, \\ vu &= \varphi(1_X - \varphi\varphi^D)f(1_X - \varphi\varphi^D) \\ &= \varphi(1_X - \varphi\varphi^D)(\varphi + \eta)(1_X - \varphi\varphi^D) \\ &= \varphi^2(1_X - \varphi\varphi^D) + \varphi(1_X - \varphi\varphi^D)\eta(1_X - \varphi\varphi^D) \\ &= \varphi^2(1_X - \varphi\varphi^D) + \varphi(1_X - \varphi\varphi^D)(1_X - \varphi\varphi^D)\eta \qquad \text{由 (3.1) 式} \\ &= \varphi^2(1_X - \varphi\varphi^D) + \varphi(1_X - \varphi\varphi^D)\eta, \end{aligned}$$

即得 $uv = vu$. 取 $m = s + k - 1$, 则 $(u - v)^m = 0$. 但

$$u - v = \eta(1_X - \varphi\varphi^D).$$

注意到 (3.1) 式, $\eta(1_X - \varphi\varphi^D) = (1_X - \varphi\varphi^D)\eta$. 因此

$$0 = (u - v)^m = [\eta(1_X - \varphi\varphi^D)]^m = \eta^m(1_X - \varphi\varphi^D)^m = \eta^m(1_X - \varphi\varphi^D)$$

即存在 $m \geq 1$, 使得 $\eta^m(1_X - \varphi\varphi^D) = 0$, $\eta^m = \eta^m\varphi\varphi^D$. ∎

注 定理 3.4 将文献 [2] 中的结论推广到了加法范畴中的态射上.

定理 3.5 设 $\mathcal{C}$ 为加法范畴. 假设态射 $\varphi: X \to X$ 有 Drazin 逆 φ^D, $\eta: X \to X$ 为 $\mathcal{C}$ 中态射, 则以下等价

(1) $\varphi\varphi^D\eta = \eta\varphi\varphi^D = \varphi - \varphi^2\varphi^D + \eta$;

(2) $f=\varphi+\eta$ 有群逆并且 $f^{\#}=(1_X+\varphi^D\eta)^{-1}\varphi^D$.

证 (1) $\Rightarrow$ (2) 令 $f_0^{(2)}=(1_X+\varphi^D\eta)^{-1}\varphi^D$. 由条件 (1) 得

$$f=\varphi+\eta=\varphi+(\varphi\varphi^D\eta+\varphi^2\varphi^D-\varphi)=\varphi(1_X+\varphi^D\eta)+\varphi(\varphi\varphi^D-1_X),$$

$$\begin{aligned}ff_0^{(2)}&=\varphi(1_X+\varphi^D\eta)f_0^{(2)}+\varphi(\varphi\varphi^D-1_X)f_0^{(2)}\\&=\varphi(1_X+\varphi^D\eta)(1_X+\varphi^D\eta)^{-1}\varphi^D+\varphi(\varphi\varphi^D-1_X)\varphi^D(1_X+\eta\varphi^D)^{-1}\\&=\varphi\varphi^D.\end{aligned}$$

类似地，由 $\eta=\eta\varphi^D\varphi+\varphi^2\varphi^D-\varphi$ 可得 $f_0^{(2)}f=\varphi^D\varphi$. 故

$$ff_0^{(2)}=f_0^{(2)}f,$$

$$\begin{aligned}f-f^2f_0^{(2)}&=f(1_X-ff_0^{(2)})=f(1_X-\varphi\varphi^D)=\varphi(1_X-\varphi\varphi^D)+\eta(1_X-\varphi\varphi^D)\\&=\varphi(1_X-\varphi\varphi^D)+\varphi(\varphi^D\varphi-1_X)\qquad\text{由条件 (1)}\\&=0.\end{aligned}$$

因此， $f=ff_0^{(2)}f$, $f_0^{(2)}ff_0^{(2)}=f_0^{(2)}$, 即 $f_0^{(2)}$ 为 f 的群逆.

$$f^{\#}=f_0^{(2)}=(1_X+\varphi^D\eta)^{-1}\varphi^D,$$

且有 $f^{\#}f=\varphi^D\varphi$.

(2) $\Rightarrow$ (1) 与定理 3.1 的证明相类似，

$$1_X-ff_0^{(2)}=(1_X-\varphi^D\varphi)\beta,\quad 1_X-f_0^{(2)}f=\alpha(1_X-\varphi^D\varphi).$$

由于 $f_0^{(2)}=f^{\#}$, 则有 $(1_X-\varphi^D\varphi)\beta=\alpha(1_X-\varphi^D\varphi)$. 因此

$$\alpha^{-1}(1_X-\varphi^D\varphi)=(1_X-\varphi^D\varphi)\beta^{-1}.\tag{3.9}$$

(3.9) 式右乘 φ^D, 得

$$0=(1_X-\varphi^D\varphi)\beta^{-1}\varphi^D=(1_X-\varphi^D\varphi)(1_X+\eta\varphi^D)\varphi^D=(1_X-\varphi^D\varphi)\eta(\varphi^D)^2,$$

亦即

$$(1_X-\varphi^D\varphi)\eta\varphi^D=0.\tag{3.10}$$

同样， (3.9) 式左乘 φ^D, 得

$$\varphi^D\eta(1_X-\varphi^D\varphi)=0.$$

于是有， $\eta\varphi^D=\varphi^D\varphi\eta\varphi^D$， $\varphi^D\eta=\varphi^D\eta\varphi\varphi^D$, 故

$$\varphi\varphi^D\eta=\varphi\varphi^D\eta\varphi\varphi^D=\varphi^D\varphi\eta\varphi^D\varphi=\eta\varphi^D\varphi=\eta\varphi\varphi^D.\tag{3.11}$$

接下来我们计算 $ff_0^{(2)}f$, 由定理 3.1 的证明过程知

$$\begin{aligned}ff_0^{(2)}f&=\varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta+(1_X-\varphi\varphi^D)\eta\alpha\varphi^D(\varphi+\eta)\\&=\varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta+(1_X-\varphi\varphi^D)\eta\varphi^D\beta(\varphi+\eta)\\&=\varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta,\qquad\text{由 (3.10) 式}\end{aligned}$$

但 $f_0^{(2)}=f^{\#}$， $ff_0^{(2)}f=f$， $\varphi\varphi^D\varphi+\eta-(1_X-\varphi\varphi^D)\eta=\varphi+\eta$, 即 $\varphi-\varphi^2\varphi^D+\eta=\varphi\varphi^D\eta$ 由 (3.11) 式即得 $\varphi\varphi^D\eta=\eta\varphi\varphi^D=\varphi-\varphi^2\varphi^D+\eta$. ∎

注 定理 3.5 将文献 [8] 中的结论推广到了加法范畴中的态射上.

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The Drazin Inverse of a Sum of Morphisms

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Abstract: Let $\mathcal{C}$ be an additive category. Suppose that φ and $\eta : X \to X$ are two morphisms of $\mathcal{C}$. If φ and η have the Drazin inverses such that $\varphi\eta = 0$, then $\varphi + \eta$ has the Drazin inverse. If φ has the Drazin inverse φ^D such that $1_X + \varphi^D\eta$ is invertible. We study the Drazin inverse (resp. group inverse) of $f = \varphi + \eta$ and give the necessary and sufficient condition for f^D(resp. $f^{\#}$) $= (1_X + \varphi^D\eta)^{-1}\varphi^D$. Finally, we extend the Huylebrouck's result from the group inverse to the Drazin inverse.

Key words: Drazin inverse; Group inverse; Morphism.

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