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Unsteady Flow to a Non-Penetrating Well in Extensive Aquifers

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The unsteady drawdown distribution around a non-penetrating well discharging from an extensively thick confined aquifer has been obtained. The solution is expressed in terms of dimensionless time factor and well function similar to the Theis method. However, the basic parameters are the hydraulic conductivity and specific storage coefficient. It is found that the type curves for non-penetrating wells is identical to the Theis type curve. Classical matching procedure leads to the determination of the specific storage and hydraulic conductivity coefficients.

Introduction

The main objective of studying groundwater movement is to find the ability of an aquifer to store and transmit water. These properties are characterized by two important parameters, namely, storage coefficient and transmissivity. They are both dependent on the thickness of the aquifer. Provided that the confined aquifer is of finite extent with completely penetrating well, direct determination of these parameters are obtained by matching time-drawdown observations from the field with Theis (1935) and Papadopulos and Cooper (1967) type curves for infinitesimally small and large diameter wells, respectively. However, for large times, irrespective of the diameter size, the Jacob straight line method yields the numerical values of the parameters (Cooper and Jacob 1946).

Construction of fully penetrating wells are difficult and expensive especially for preliminary groundwater investigations where the main purpose is to obtain useful data about the aquifer properties but not yet direct water supply. Therefore, construction of non-penetrating wells by penetrating the confining layer down to its lower boundary proves to be easier and more economic. These wells are encountered in India and Saudi Arabia in the crystalline and basaltic rocks as well as Quaternary alluvium deposits. Another economic advantage of non-penetrating wells is that they do not require a screen.

Earlier works concerning non-penetrating wells have considered steady state groundwater flow (Forchheimer 1930; Muskat 1937; and Aravin and Numerov 1965). Later, Basak (1977) presented steady state solution for non-penetrating wells by incorporating Forchheimer (1901) non-linear velocity gradient response. Unsteady flow to non-penetrating wells has been considered initially by Kanwar (1974). He defined a non-penetrating well function and tabulated its values for determining the aquifer constants by matching technique. This solution requires variable discharge pumping test which is difficult to control in practice. Kanwar et al (1976) obtained simplified solution for constant discharge pumping test. Another solution is presented by Kanwar et al (1979) which requires the pumping test data from a variable discharging of non-penetrating well similar to the solution proposed by Hantush (1964) for fully penetrating well. All of these solutions are rather tedious to apply in practice.

In this paper, a rigorous solution for the unsteady flow to a non-penetrating well with constant discharge is presented. The application of this method is simple and similar to Theis procedure.

Problem Definition and Theoretical Solution

A non-penetrating well is constructed simply by puncturing the confining layer as shown in Fig. 1. Groundwater enters the well bottom through a cavity formed in the aquifer just below the confining layer. The aquifer is assumed to be non-leaky, uniform, homogeneous, isotropic, infinite in areal extent, excessive in thickness and the parameters are temporally and spatially constant during the pumping test. The well has small diameter, in fact, it is thought as a point sink. The water is pumped at a predetermined rate, Q. During the pumping the equipotential surfaces in the aquifer are concentric hemispheres and the streamlines are straight lines intersecting at the bottom of the well.

Due to extensiveness of the aquifer, the transmissivity and the storage coefficient definitions become invalid and instead hydraulic conductivity, k, and specific storage, S_s, are effective. Consider an aquifer domain between two concentric hemispheres of radii r and $r+\Delta r$ centered at the well bottom center. During a time interval, Δt, the difference between water volumes leaving and entering this do-

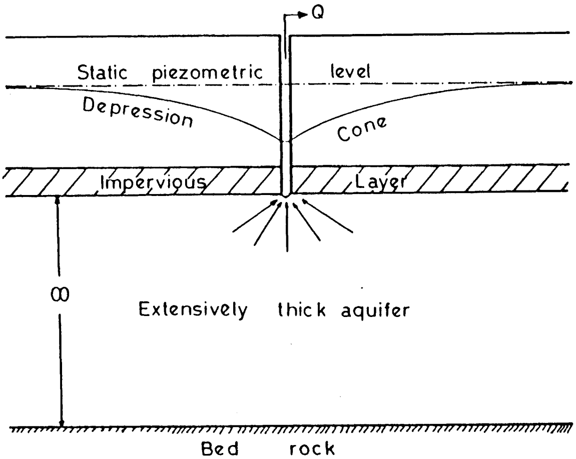

Fig. 1. Description of a non-penetrating well.

main comes from storage within this domain causing a decline of head $\Delta\varnothing(r,t) = \varnothing(r,t) - \varnothing(r,t+\Delta t)$. Hence, the continuity equation for the domain can be written as

$$\Delta t[Q(r,t) - Q(r+\Delta r,t)] = S_s 2\pi r^2 \Delta r[\varnothing(r,t) - \varnothing(r,t+\Delta t)] \tag{1}$$

As $\Delta r \to 0$ and $\Delta t \to 0$ this expression leads to a first order partial differential equation

$$\frac{\partial Q(r,t)}{\partial r} = S_s 2\pi r^2 \frac{\partial \varnothing(r,t)}{\partial t} \tag{2}$$

On the other hand, discharge can be written in terms of filter velocity, $v(r,t)$ as

$$Q(r,t) = 2\pi r^2 v(r,t) \tag{3}$$

Differentiation of this equation with respect to r leads to

$$\frac{\partial Q(r,t)}{\partial r} \equiv 4\pi r v(r,t) + 2\pi r^2 \frac{\partial v(r,t)}{\partial r} \tag{4}$$

Hence, the substitution of Eq. (4) into Eq. (3) gives after some algebra

$$\frac{\partial v(r,t)}{\partial r} + \frac{2}{r} v(r,t) \equiv S_s \frac{\partial \varnothing(r,t)}{\partial t} \tag{5}$$

This is, in fact, the continuity equation which relates the velocity to head change in unsteady flow towards non-penetrating well.

In addition, the Darcy law is simply

$$v(r,t) \equiv k \frac{\partial \varnothing(r,t)}{\partial r} \tag{6}$$

Eqs. (5) and (6) represent together completely unsteady flow to non-penetrating well. Indeed, substitution of Eq. (6) into Eq. (5) results in

$$\frac{\partial^2 \emptyset(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial \emptyset(r,t)}{\partial r} \equiv \frac{S_s}{k}\frac{\partial \emptyset(r,t)}{\partial t} \tag{7}$$

which has been invariably employed in all the previous papers by Kanwar (1974), Kanwar et al (1976, 1979). Their final results are based on the direct solution of Eq. (7) with relevant initial and boundary conditions through Laplace transformations.

However, herein, the simultaneous solution of Eqs. (5) and (6) is adopted leading to a simple and exact result. They can be reduced to ordinary differential equations with the use of Boltzmann transformation, η, which states that

$$\eta \equiv \frac{r}{2t^{\frac{1}{2}}} \tag{8}$$

By employing the chain rule for differentiation of composite functions it is possible to arrive at the following equalities

$$\frac{\partial \emptyset(r,t)}{\partial r} = \frac{\partial \emptyset(r,t)}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{\partial \emptyset(r,t)}{\partial \eta}\frac{\partial}{\partial r}\left(\frac{r}{2t^{\frac{1}{2}}}\right) = \frac{1}{2t^{\frac{1}{2}}}\frac{\partial \emptyset(r,t)}{\partial \eta} \tag{9}$$

similarly,

$$\frac{\partial v(r,t)}{\partial r} = \frac{1}{2t^{\frac{1}{2}}}\frac{\partial v(r,t)}{\partial \eta} \tag{10}$$

and

$$\frac{\partial \emptyset(r,t)}{\partial t} \equiv \frac{\partial \emptyset(r,t)}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial \emptyset(r,t)}{\partial \eta}\frac{\partial}{\partial r}\left(\frac{r}{2t^{\frac{1}{2}}}\right) \equiv -\frac{\eta}{2t}\frac{\partial \emptyset(r,t)}{\partial \eta} \tag{11}$$

The substitutions of these differentiations into Eqs. (5) and (6) gives

$$\frac{dv(\eta)}{d\eta} + \frac{2}{\eta}v(\eta) = -\frac{\eta}{t^{\frac{1}{2}}}\frac{d\emptyset(\eta)}{d\eta} \tag{12}$$

and

$$v(\eta) = \frac{k}{2t^{\frac{1}{2}}}\frac{d\emptyset(\eta)}{d\eta} \tag{13}$$

Elimination of $d\emptyset(\eta)/d\eta$ between these two last equations gives rise to a first order linear ordinary differential equation in terms of the velocity as

$$\frac{dv(\eta)}{d\eta} + 2\left(\frac{1}{\eta} + \frac{\eta}{k}S_s\right)v(\eta) \equiv 0 \tag{14}$$

By separation of variables it is possible to write $\frac{dv(\eta)}{v(\eta)} = -2\left(\frac{1}{\eta} + \frac{\nu}{k}S_s\right)d\eta$

Integration by parts gives $\mathrm{Ln}\, v(\eta) \equiv -2\,\mathrm{Ln}\,\eta - \frac{S_s}{k}\eta^2 + C$ or after some arrangement

$$v(\eta) \equiv C\frac{e^{-(S_s/k)\eta^2}}{\eta^2} \qquad (15)$$

At this stage it is necessary to consider initial and boundary conditions of the problem. These are

$$\emptyset(r,0) = \emptyset \qquad (\text{for all } r) \qquad (16)$$

$$\emptyset(\infty,0) = \emptyset \qquad (\text{for all } t > 0) \qquad (17)$$

$$\lim_{r\to 0}[2\pi r^2 v(r,t)] = Q \qquad (18)$$

However, in terms of Boltzmann's transformation Eq. (18) is

$$\lim_{\eta\to 0}[8\pi\eta^2 t v(\eta)] = Q \qquad (19)$$

Substitution of Eq. (15) into Eq. (19) gives $C = \frac{Q}{8\pi t}$
Hence, Eq. (15) can be rewritten as

$$v(\eta) = \frac{Q}{8\pi t\eta^2}\, e^{-(S_s/k)\eta^2}$$

or explicitly in terms of the original variables

$$v(r,t) = \frac{Q}{2\pi r^2}\, e^{-(r^2 S_s)/4kt} \qquad (20)$$

The head distribution around the well can be evaluated from Eq. (5) by substitution of Eq. (20)

$$\frac{\partial\emptyset(r,t)}{\partial t} \equiv -\frac{Q}{4\pi krt}\, e^{-(r^2 S_s)/4kt}$$

taking integration with respect to t and considering Eq. (16)

$$\emptyset_0 - \emptyset(r,t) \equiv -\frac{Q}{4\pi kr}\int_0^t \frac{e^{-(r^2 S_s)/4kt}}{t}\, dt \qquad (21)$$

The left hand side is equal to drawdown $s(r,t)$ at distance r and time t. Furthermore, by defining new variable as dimensionless time factor

$$u \equiv \frac{r^2 S_s}{4kt} \qquad (22)$$

reduces Eq. (21) after some algebraic manipulations to

$$s(r,t) = \frac{Q}{4\pi kr} \int_{u}^{\infty} \frac{e^{-u}}{u} \quad (23)$$

In obtaining Eq. (23) an important point is the distance r corresponding to an observation well was considered as constant. Eq. (23) can be written succinctly as

$$W(u) \equiv \int_{u}^{\infty} \frac{e^{-u}}{u} du \quad (24)$$

where also

$$W(u) = \frac{4\pi kr}{Q} s(r,t) \quad (25)$$

is the well function for non-penetrating well. Eq. (24) in its form, is nothing but the Theis solution. Therefore, an important conclusion is that type curve for non-penetrating well is, in fact, equivalent to Theis type curve.

On the other hand, for large time instances the velocity can be approximated from Eq. (20) as

$$v(r,\infty) = \frac{Q}{2\pi r^2} \quad (26)$$

Furthermore, substitution of the velocity in Darcy's law gives the spatial head distribution as,

$$\frac{\partial \varnothing(r,\infty)}{\partial r} = \frac{Q}{2\pi kr^2}$$

or integration between two observation wells at distance r_1 and r_2 ($>r_1$) and corresponding heads $\varnothing(r_1)$ and $\varnothing(r_2)$ leads to

$$\varnothing(r_2) - \varnothing(r_1) \equiv \frac{Q}{2\pi k} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \quad (27)$$

This is the steady state solution of groundwater flow towards a non-penetrating well (Muskat 1937; Aravin and Numerov 1965).

Application

All of the available data in the literature concerning non-penetrating wells are obtained with variable type of discharge. The nearest to a constant discharge pumping test is presented by Kanwar et al (1976). However, as they state, their simplified solution involves approximation at the first stage only for a constant-discharge pumping test in non-penetrating well. Their data gives the impression

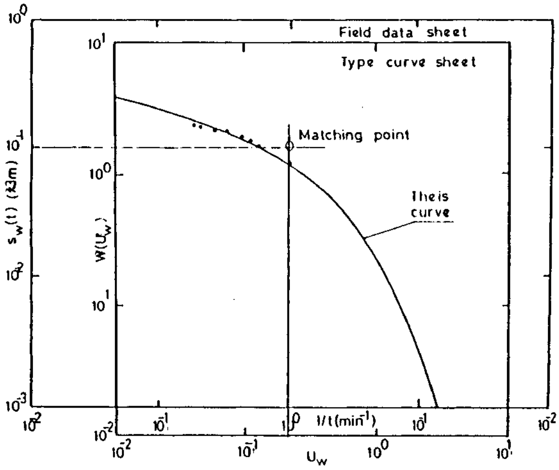

Fig. 2. Type curve matching.

that the flow reaches the steady state position, which is not possible in confined aquifers. Therefore, three late – stage drawdown readings in their Table 1 are not considered in this paper.

A graphical method proposed by Theis has been used for solving Eqs. (22) and (25) which leads to the determination of hydraulic conductivity and then the specific storage. This method requires a type curve that is the plot of $W(u)$ versus u on a double logarithmic paper. The plot of drawdown versus $1/t$ is made on the same scale double logarithmic paper. Then, the pumping test data sheet is overlain on the type curve sheet and they are superimposed such that the best matching is obtained (Fig. 2). Finally, a matching point is selected which gave the values: $s_m(r,t) = 0.30$ m; $(1/t)_m = 10^0$ 1/min; $u_m = 2.3\text{x}10^{-3}$; and $W_m(u) = 1.5\text{x}10^0$.

With these values at hand, the hydraulic conductivity can be calculated from Eq. (25) as,

$$k = \frac{QW_m(u)}{4\pi r s_m(r,t)} = \frac{0.59\times1.5}{4\times3.14\times9.8\times0.30} = 0.024 \text{ m/min}$$

The specific storage can be calculated from Eq. (22) as

$$S_s = \frac{4\,ku_m}{r^2(1/t)_m} = \frac{4\times0.024\times1.00}{(9.8)^2\times1.00} = 10^{-3} \text{ 1/m}$$

Conclusions

Unsteady flow solution has been obtained to determine the hydraulic conductivity and the specific storage from drawdown measurements around a non-penetrating well tapping a confined but semi-extensive aquifer with constant discharge. A graphical procedure similar to Theis method is suggested to calculate the aquifer properties. The application of the methodology is possible for very small diameter wells only.

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