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# Finite Element Analysis of Square Aquifers Containing Pumped Wells and Comparison with Finite Difference Method

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The numerical solution of the behaviour of discrete time steps in digital computer analysis of square aquifers containing pumped wells is examined by using the finite element method with a 4 node linear quadrilateral isoparametric surface element. A wide range of time steps are used in the computation. The calculations show that discrete time steps can cause errors and oscillations in the calculations particularly when wells start and stop pumping. Comparison with known results obtained by theoretical and finite difference procedures has been considered. The main objective of this paper is to demonstrate comparison of the finite element and finite difference simulation results over a regular linear 4 node quadrilateral mesh suitable to represent the two numerical schemes with a marked similarity. The dimensionless time drawdown results of the finite element method agreed well with the finite difference and analytical results for small time increment. However, for large time increments, there are from slight to significant oscillations in the results and notable discrepancies are observed in the solutions of the two numerical methods.

## Introduction

Recent and ongoing research on the finite element method indicates a profuse literature describing the versatility and rich mathematical abstraction of the subject. The literature on the comparison of the finite element and finite difference procedures in subsurface hydrology is limited. The well known work of Witherspoon et al. (1968), Pinder and Frind (1972), Pinder and Gray (1976), Gray and Pinder (1976) and Pinder and Gray (1977) demonstrate an excellent account of the potential of the two numerical schemes for some classes of hydrologic problems. In order to make the comparison of the two numerical schemes in simulation of groundwater flow systems on a rational basis, it is essential to set up similar geometric representations of the field problem as well as identical aquifer input parameters.

Rushton (1973) used 3 implicit finite difference methods (the Crank-Nicholson, the alternating direction implicit and the backward difference) and examined the behavior of discrete time steps in digital computer analysis of square aquifers containing pumped wells and suggested guidelines for an optimum choice of discrete time steps. The square grids used by Rushton in the finite difference model can be well represented in the finite element method by the use of a four node linear isoparametric quadrilateral elements. In this work, in order to demonstrate the comparison between the two numerical schemes, it was found adequate to develop the above problem for the Crank-Nicholson method. Two types of solutions for the square aquifer problem were found by Rushton. An analytical solution has been obtained by using the image well theory. The Crank-Nicholson scheme was used to study the behavior of the numerical scheme where different time increments were used. It was found that the smallest $\Delta T/a^2S = 0.01$ is designed to give full details of the time drawdown curves, whereas $\Delta tT/a^2S > 1$ demonstrates the behavior of the numerical technique as manifested by oscillations in the numerical results.

This paper summarizes the result of a finite element analysis of discrete time step behavior in digital computer analysis of square aquifers containing pumped wells, and compares the result of the finite element solutions with both finite difference and analytical results. Finite element in space and centered -in-space time discretization (Crank-Nicholson) procedures have been chosen to illustrate the approach.

## Theoretical Formulation

The transient flow of groundwater in a confined aquifer can be expressed with the partial differential equation

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right) = S\frac{\partial h}{\partial t} + Q \qquad (1)$$

Initial and boundary conditions for the system are

$$h(x,y,0) = \bar{h}_0(x,y) \qquad (1a)$$

$$n_x T_x \frac{\partial h}{\partial x} + n_y T_y \frac{\partial h}{\partial y} + \bar{q} = 0 \quad \text{on} \quad \Gamma_1 \qquad (1b)$$

$$h = \bar{h} \quad \text{on} \quad \Gamma_2 \qquad (1c)$$

in which

$h$ – hydraulic head, $(L)$,
$T$ – transmissivity, $(L^2T^{-1})$,
$Q$ – pumpage rate from well, $(L^3T^{-1})$,
$S$ – storage coefficient,
$\bar{q}$ – the flux of water per unit area of boundary, $(L^2T^{-1})$,
$\bar{n}$ – is the unit outward normal to the boundary,
$\Gamma_1$ represents Neumann conditions and
$\Gamma_2$ represents the Dirichlet conditions on part of the system.

The development of the finite element equations has been described in previous works. Zienkiewicz and Parekh (1970), Pinder and Frind (1972). Although prime interest is focused in this work on the comparison of the results of the two numerical methods, it is essential to include the salient features of the finite element equations, so that the reader can compare the theoretical formulations and the numerical solutions of the two numerical procedures. For details on the finite difference formulation the reader is referred to Rushton's work.

The minimization of Eq. (1) subject to the initial and boundary conditions yields a system of first order differential equation of the type

$$\underset{\sim}{A}\,\underset{\sim}{H} + \underset{\sim}{B}\frac{\partial}{\partial t}\underset{\sim}{H} = \underset{\sim}{C} \tag{2}$$

The element matrices in Eq. (2) have the following form

$$\underset{\sim}{A}^e_{ij} = \int_{A^e} T(\nabla \underset{\sim}{N}^e)^T \underset{\sim}{\nabla}\,\underset{\sim}{N}^e \, dA \qquad i,j = 1,2,\ldots,n \tag{3}$$

$$\underset{\sim}{B}^e_{ij} = \int_{A^e} S\, N_i\, N_j \, dA \qquad i,j = 1,2,\ldots,n \tag{4}$$

$$\underset{\sim}{C}^e_{ij} = \int_{A^e} \underset{\sim}{N}^T Q \, dA - \int \underset{\sim}{N}^T \bar{q} \, dS \qquad i = 1,2,\ldots,n \tag{5}$$

$$\underset{\sim}{H} = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix} \tag{6}$$

where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are square matrices which represent the transmissivity and storage coefficient respectively. $\underset{\sim}{C}$ is a column matrix representing the dicharge, $\underset{\sim}{H}$ is a column matrix of nodal value of the hydraulic head.

To perform integrations over the isoparametric element, the following transformations are essential to change the derivatives in the global $(x,y)$ coordinates to the local $(\xi, \eta)$ coordinates. The two dimensional nabla operator is transformed as

$$\nabla = \underset{\sim}{J}^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \tag{7}$$

where $J$ is the Jacobian matrix for the coordinate transformation. The Jacobian matrix $J$ becomes

$$J = \begin{bmatrix} \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} & \frac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_1}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_4 & y_4 \end{bmatrix} \tag{8}$$

The element area has to be changed

$$dA = \det \underset{\sim}{J}\, d\xi d\eta \tag{9}$$

Eqs. (3-5) have to be transformed accounting Eqs. (7-9) and change the limitations of the surface integrals to $-1$ and $+1$ in both integrals in accordance with the isoparametric rule. Zienkiewicz (1971), Connor and Brebbia (1976), Aalto (1978).

$$\underset{\sim}{A}^{e}_{ij} = \int_{-1}^{1}\int_{-1}^{1} T \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & \frac{\partial N_i}{\partial \eta} \end{bmatrix} (\underset{\sim}{J}^{-1})^T \underset{\sim}{J}^{-1} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \det \underset{\sim}{J}\, d\xi d\eta \tag{10}$$

$$\underset{\sim}{B}^{e}_{ij} = \int_{-1}^{1}\int_{-1}^{1} S N_i N_j \det \underset{\sim}{J}\, d\xi d\eta \tag{11}$$

$$\underset{\sim}{C}^{e}_{ij} = \int_{-1}^{1}\int_{-1}^{1} Q N_i \det \underset{\sim}{J}\, d\xi d\eta + \int_{-1}^{1} \bar{q} N_i \left[ \left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2 \right]^{\frac{1}{2}} d\xi$$
$$+ \int_{-1}^{1} \bar{q} N_i \left[ \left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2 \right]^{\frac{1}{2}} d\eta \tag{12}$$

The element boundary on side $\xi$ = constant and $\eta$ = constant is accounted by the second and third expression of Eq. (12). The integrations in Eqs. (10-12) are performed by a Gaussian quadrature scheme, Zienkiewicz (1971).

The time marching process is accomplished by using the temporal operator Crank-Nicolson time stepping scheme in interval $t$ to $t+\Delta t$. Matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ at the mid point of the time interval can be expressed as

$$\underset{\sim}{A}\frac{1}{2}(\underset{\sim}{H}_t+\underset{\sim}{H}_{t+\Delta t}) + \frac{\underset{\sim}{B}(\underset{\sim}{H}_{t+\Delta t} - \underset{\sim}{H}t)}{\Delta t} = \frac{\underset{\sim}{C}_{t+\Delta t} + \underset{\sim}{C}_t}{t} \tag{13}$$

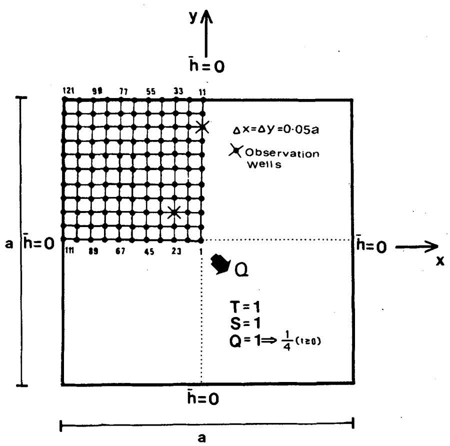
which can be rearranged to yield

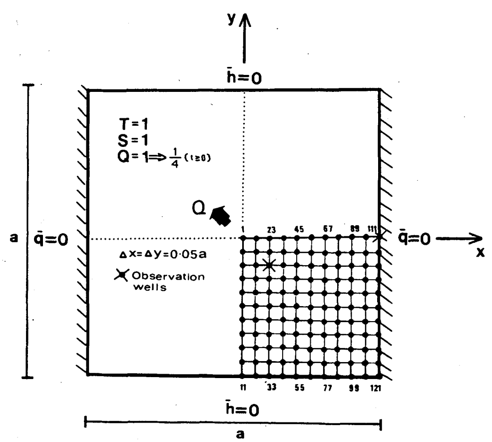
$$(\underset{\sim}{A} + \frac{2}{\Delta t}\underset{\sim}{B})\ \underset{\sim}{H}_{t+\Delta t} = (-\underset{\sim}{A} + \frac{2}{\Delta t}\underset{\sim}{B})\underset{\sim}{H}_t + 2\underset{\sim}{C} \tag{14}$$

Starting from the initial condition $\underset{\sim}{H}_o = \underset{\sim}{H}(t_o)$, $\underset{\sim}{H}$ at subsequent time steps $t+\Delta t$ can be solved recursively by using Eq. (14).

## Finite Element Discretization

The square aquifers in Figs. 1 and 2 correspond exactly to that of the models of Rushton (1973). Each square aquifer of sides, *a,* with a well positioned at the centre of the aquifer is considered. Rushton (1973) divided the sides of the aquifers into 20 mesh intervals. In this work, because of symmetry of the square aquifers, only, a quadrant of the aquifers with 100 elements and 121 nodal points is simulated to save computing time. In example 1, Dirichlet conditions ($\bar{h}$ = 0) are maintained on all boundaries. The pumping well is set at nodal point 1 and observation wells at the nodes 9 and 25 are located at a distance of 0.4a and 0.1414a respectively from the pumping well. In example 2, Dirichlet conditions ($\bar{h}$ = 0) are maintained on the two opposite boundaries. The remaining boundaries have Neumann type ($\bar{q}$=0) conditions. The pumping well is set at nodal point 1 and observation wells at the nodes 25 and 111 are located at a distance of 0.1414a and 0.5a, respectively, from the pumping well.




Finite element mesh of square aquifer, with a pumping well at the centre.

Fig. 1. Example 1. Fig. 2. Example 2.

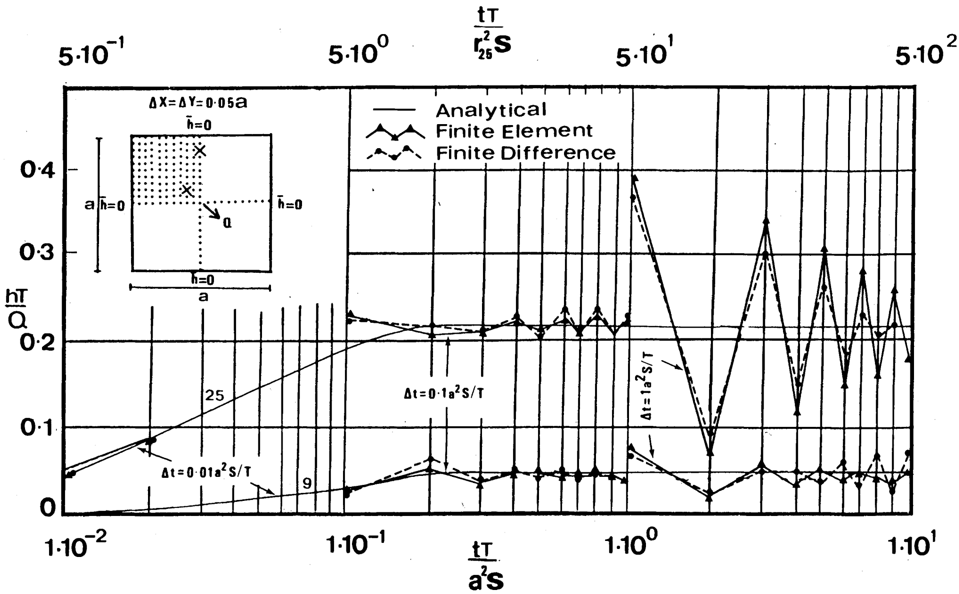



Fig. 3. Comparison of finite element and finite difference solution. Example 1.

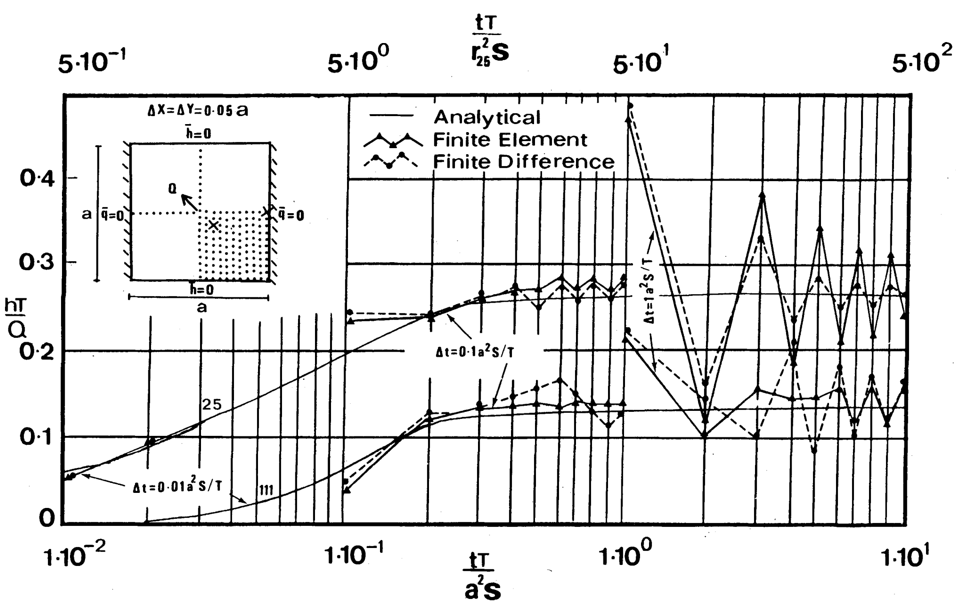



Fig. 4. Comparison of finite element and finite difference solution. Example 2.

## Results and Discussion

In the analysis, both time and drawdown were expressed as dimensionless factors. The computed dimensionless drawdown $hT/Q$ for both examples with dimensionless time $tT/a^2S$ are illustrated in Figs. 3 and 4. The upper time scale $tT/r_{25}^2S$ refers to the nearest observation well, at nodal point 25 in both examples while the lowest time scale $tT/a^2S$ refers to the breadth of the aquifer. The importance of the two nondimensional parameters in finite difference analysis was discussed in detail by Rushton (1973). For the sake of brevity they are not treated here. The discussion will be limited to comparison of the results of the two numerical schemes which appear to be of particular importance in this work. In Figs. 3 and 4 the analytical results are indicated by solid lines. The dots and triangles represent the finite difference and finite element solutions, respectively, when they differ from the analytical results. The finite element results show that for small time increments $\Delta tT/a^2S = 0.01$ the solutions agree very closely with the analytical and finite difference results. However, for $\Delta tT/a^2S = 0.1$ and 1, the solutions show significant and very large osciallations, respectively, about the analytical results as illustrated in Figs. 3 and 4 and have suffered the same degree of oscillation as the finite difference solutions. For $\Delta tT/a^2S = 0.1$ the oscillations die out after 15 time steps in both examples, versus 30 time steps in the finite difference analysis. For time steps $\Delta tT/a^2S > 1$, neither numerical scheme shows the true response of the pumping test, but demonstrates the behavior of the numerical technique when large time steps are used. Apparently, the two methods do not agree.

Each example problem needed 40.48 sec of CPU time, 1.24 min for execution and required 256K bytes of core storage on a UNIVAC 1108 computer using the FORTRAN V compiler. Rushton's paper does not address the question of computation time, core storage requirements, or the type of computer used for the finite difference work. A judicious choice of a numerical method is commonly based on the accuracy, runtimes and computer storage requirements of a model. In the absence of such requirements for the finite difference scheme, this paper has made the comparison only on the basis of the accuracy issue, and falls short of establishing the rather simplistic, but very essential conclusion on the relative efficency and performance of the two numerical procedures.

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