Nordic Hydrology, 1980, 83-92

Use of Cumulants to Estimate Coefficients in Chow-Kulandaiswamy's GHS Model

Vijay P. Singh and Roger C. McCann

Mississippi State University,
Mississippi State, Miss 39762, USA

The coefficients in the General Hydrologic System (GHS) model of Chow and Kulandaiswamy (1971) are estimated using the method of cumulants. Based on this method a rational criterion is developed to determine the number of terms to be retained in the GHS Model. The least squares method used by Chow and Kulandaiswamy (1971) and the method of cumulants proposed here are compared and are found to yield comparable values of the model coefficients on an example watershed.

Introduction

The so-called General Hydrologic System (GHS) model (Kulandaiswamy 1964; Chow 1964; Dooge 1973) developed by Chow and Kulandaiswamy (1971) has been utilized in several studies (Chaudhary 1976; Chaudhary, Simoes and Ferreira Filho 1976; Kulandaiswamy, Krishnaswamy and Ramalingam 1967; Kulandaiswamy and Subramanian 1967; Kulandaiswamy and Rao 1970, 1971a, 1971b) dealing with watershed runoff modeling and flood routing in open channels. The GHS model hypothesizes that storage S in the watershed can be expressed as a linear sum of surface runoff Q, rainfall excess I, and their time derivatives

$$S = \sum_{i=0}^{M} a_i \frac{d^i Q}{dt^i} + \sum_{j=0}^{N} b_j \frac{d^j I}{dt^j} \tag{1}$$

where a_i and b_j are coefficients as functions of Q and I. M and N are finite nonegative integer numbers with $M>N$. Eq. (1), when coupled with the spatially lumped form of continuity equation

$$\frac{dS}{dt} = I - Q \tag{2}$$

together with the initial conditions

$$Q(O) = \frac{d^i Q(O)}{dt^i} = 0, \quad i = 0,1,2, \ldots, M \tag{3}$$

constitutes the GHS model

$$Q + \sum_{i=0}^{M} a_i \frac{d^{i+1} Q}{dt^{i+1}} = I - \sum_{j=0}^{N} b_j \frac{d^{j+1} I}{dt^{j+1}} \tag{4}$$

Of practical expediency Chow and Kulandaiswamy (1971) introduced two simplifications in Eq. (4):

1) The coefficients a_i and b_j are constants, or only functions of some characteristic of Q and I such as mean or maximum. In this study we will consider them to be constant.

2) M and N equal 2 and 1 respectively. This choice was based on fitting various forms of Eq. (4) to observed watershed runoff data. They reported that higher derivatives of I and Q were insignificant for the cases which they studied and could therefore be ignored without causing appreciable error in model results.

Thus the GHS model reduces to

$$Q + a_0 \frac{dQ}{dt} + a_i \frac{d^2 Q}{dt^2} + a_2 \frac{d^3 Q}{dt^3} = I - b_0 \frac{dI}{dt} - b_1 \frac{d^2 I}{dt^2} \tag{5}$$

In fitting Eq. (5) to observed runoff data Chow and Kulandaiswamy (1971) used the least squares method to determine the coefficients a_i and b_j. This implies that the coefficients will depend on how accurately the derivatives of I and Q can be determined. Given I in a histogram form the accuracy of its derivatives is doubtful. Given skewed shape of Q, its derivatives can hardly be determined accurately (Conte 1965). It may therefore be desirable to develop a method to estimate a_i and b_j which is independent of the derivatives of I and Q. This study proposes the method of cumulants which satisfies this requirement. It may be remarked that this method offers an explanation for determining M and N. Utilizing, rainfall-runoff observations from a small agricultural watershed the model coefficients are determined for $M=1$ by the method of cumulants and are compared with those determined by the least squares method.

Simplification of Eq. (4)

Chow (1964), Kulandaiswamy (1964), and Chow and Kulandaiswamy (1971) derived the instantaneous unit hydrographs (IUH) for various cases depending upon the roots of the equation $a_2x^3 + a_1x^2 + a_0x + 1 = 0$. The use of IUH is consistent because a_i and b_j are constant and Eq. (5) is linear. There is, however, an inconsistency in their derivation recognition of which simplifies Eqs. (4) and (5) considerably as discussed by Singh and McCann (1980).

For an IUH rainfall excess is represented by a delta function $\delta(t)$. If an impulse of rainfall excess $\delta(t)$ takes place uniformly over the entire watershed then it is natural to assume that the IUH will be 0 at t=0 and will experience rise and fall with the progress of time. Singh and McCann (1980) have proved that in order for this to be true and Eq. (5) to have a unique solution, where I is $\delta(t)$, the coefficients b_0 and b_i must vanish. Therefore, Eq. (4) reduces to

$$u + \sum_{i=0}^{M} a_i \frac{d^{i+1} u}{dt^{i+1}} = \delta(t) \tag{6}$$

Eq. (5) to

$$u + a_0 \frac{du}{dt} + a_1 \frac{d^2 u}{dt^2} + a_2 \frac{du^3}{dt^3} = \delta(t) \tag{7}$$

and initial conditions in Eq. (3) to

$$u(0) = \frac{d^{i+1} u}{dt^{i+1}}(0) = 0, \qquad i = 0,1,2, \ldots, M \tag{8}$$

where u is the IUH. Thus it is seen that u will be independent of b_j. Chow and Kulandaiswamy (1971) incorrectly complicated their model. Runoff Q can be determined by convolution of u with I

$$Q(t) = \int_0^{t_*} u(t-\tau) I(\tau) d\tau \tag{9}$$

where $t_*=t$ for $t<T$, duration of I, and $t_*=T$ for $t \geq T$.

Mathematical Solutions

We can solve Eq. (6) subject to the initial condition in Eq. (8) for some special cases using the Laplace transform. Let the Laplace transform of u be defined as

$$y(s) = \int_0^{\infty} e^{-st} u(t)\, dt \tag{10}$$

Then, taking the Laplace transform of Eq. (6) subject to Eq. (8)

$$y(s) = \frac{1}{1 + \sum_{i=0}^{M} a_i \, s^{i+1}} \tag{11}$$

Hence, $u(t)$ is the inverse Laplace transform of $(1 + \sum_{i=0}^{M} a_i \, s^{i+1})^{-1}$. When $M = 0$

$$u(t) = \frac{1}{a_0} e^{-t/a_0}, \quad t > 0 \tag{12}$$

When $M = 1$

$$u(t) = \frac{\alpha\beta}{\beta-\alpha} [e^{\beta t} - e^{\alpha t}] \tag{13}$$

Where α, β are the roots of

$$f(x) = a_1 x^2 + a_0 x + 1 \tag{14}$$

When $M = 2$

$$u(t) \equiv - abc \left[\frac{e^{at}}{(a-b)(a-c)} + \frac{e^{bt}}{(b-a)(b-c)} + \frac{e^{ct}}{(c-a)(c-b)} \right] \tag{15}$$

where a,b,c are the roots of

$$g(x) = a_2 x^3 + a_1 x^2 + a_0 x + 1 \tag{16}$$

Similarly we can solve Eq. (6) for $M = 3$, 4, and so on, though with increasing complexity. In general it is difficult to determine the inverse Laplace transform of Eq. (11), but this is not needed from a practical standpoint as noted by Chow and Kulandaiswamy (1971).

Method of Cumulants to Determine Coefficients a_i

The coefficients a_i in Eq. (6) can be estimated using either moments or cumulants. A distinct advantage of cumulants is that they, except for the first, remain unaffected by the change of origin (Dooge 1973). We, therefore, resorted to the use of cumulants.

The R-th cumulant of IUH, u, can be defined as

$$K_R = (-1)^R \frac{d^R}{ds^R} [\log y(s)]_{s=0} \tag{17}$$

Substituting Eq. (11) into Eq. (17)

$$K_R \equiv (-1)^{R+1} \frac{d^R}{ds^R} \log [1 + \sum_{i=0}^{M} a_i s^{i+1}]_{s=0} \tag{18}$$

The notation$]_{s=0}$ means that the associated quantity is to be evaluated at $s=0$. Eq. (18) yields

$$\begin{aligned} K_1 &= a_0 \\ K_2 &= a_0^2 - 2a_1 \\ K_3 &= 2a_0^3 - 6a_0 a_1 + 6a_2 \\ &\vdots \\ K_M &= \text{function of } a_0, a_1, a_2 \ \ldots\ a_{M-1} \end{aligned} \tag{19}$$

Hence, Eq. (19) yields

$$\begin{aligned} a_0 &= K_1 \\ a_1 &= \frac{1}{2}(K_1^2 - K_2) \\ a_2 &= \frac{1}{6}(K_3 + K_1^3) - \frac{1}{2}K_1 K_2 \\ &\vdots \\ a_M &= \text{function of } K_1, K_2, K_3, \ \ldots\ K_{M+1} \end{aligned} \tag{20}$$

The cumulants K_1, K_2, K_3, ... of u can be determined from rainfall excess-runoff data using the theorem of cumulants for linear, time-invariant hydrologic systems, Dooge 1973)

$$K_R(Q) = K_R(I) + K_R(u) \tag{21}$$

Thus it can be seen from Eq. (20) that the coefficients a_i can be determined without using derivatives of Q. This advantage coupled with simplicity of computation might make the method of cumulants more attractive and preferable in some cases than the least squares method as used by Chow and Kulandaiswamy (1971). A method of computing cumulants for a given set of data is discussed in the appendix. To compare these two methods a_0 and a_1 for the case $M=1$ were computed for six rainfall excess-runoff events from watersheds SW-17, Riesel (Waco), Texas (U. S. Department of Agriculture 1965; Singh 1976) as shown in Table 1. The two methods yielded comparable values of the coefficients.

Table 1 - Estimation of the coefficients a_O and a_1 for the case $M = 1$ using the least squares method and the method of cumulants for some rainfall excess - runoff events on watershed SW-17, Riesel (Waco), Texas.

	Rainfall-runoff event			Method of cumulants		Least squares method	
Serial Number	Date	Rainfall volume (cm)	Runoff volume (cm)	a_O	a_1	a_O	a_1
1	3-12-1953	2.11	1.7	1.0458	0.2468	0.4276	0.0253
2	4-24-1957	4.45	4.39	.2863	.0054	.2504	.0014
3	6-24-1959	5.05	3.86	0.34	0.0043	0.2830	0.0046
4	7-16-1961	4.45	4.39	.2863	.0054	.2504	.0014
5	6- 9-1962	5.28	4.24	.4287	.0267	.3661	.0183
6	3-29-1965	12.67	8.9	1.0381	0.062	0.8766	0.0415

Note: Units used in computation were cm/hour

Criterion for the Determination of *M*

Eq. (20) shows that the coefficients a_i are functions of the cumulants. Since cumulants, especially higher order, are sensitive to small changes in the function we expect the coefficients a_i to be sensitive to small errors in the rainfall excess-runoff data. Therefore, choosing M higher than 2 does not usually increase the accuracy of the model. In fact, a very high value of M may lead to entirely unrealistic results due to unreliable estimates of the coefficients.

Moreover, it can be shown that if, for example, the second order equation (M=1) accurately represents the system then the coefficients $a_2, a_3, \ldots a_M$ in Eq. (20) will all be zero. Similarly, if the third order equation accurately represents the system then the coefficients $a_3, a_4, \ldots a_M$ in Eq. (20) will all be zero, and so on.

To illustrate, if M=1 then a short calculation shows that a_2 the leading coefficient of the third order equation vanishes; that is

$$\frac{1}{6}(K_3 + K_1^3) - \frac{1}{2}K_1 K_2 = 0 \tag{22}$$

Similar calculations can be carried out for third and higher order systems. It then follows that if the second order equation »closely« approximates the system, small errors in measuring the data may be relatively large compared to a_2, the leading coefficient of the third order equation. This, coupled with the observation that sensitivity is related to the sensitivity of the cumulants, suggests that M=1 is the best choice for M in this case. These observations on the choice of M also help to

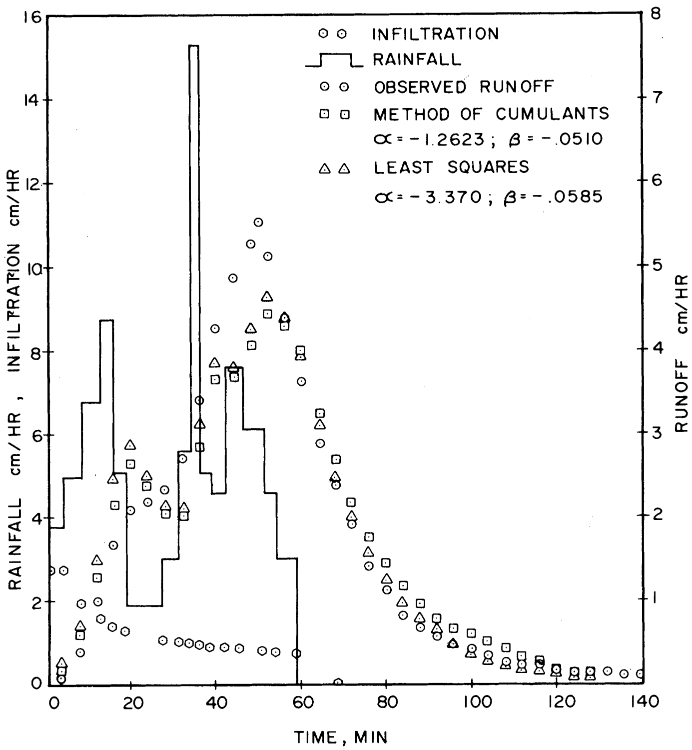

Fig. 1. Comparison of observed surface runoff hydrograph with hydrograph computed by method of moments and least squares method for rainfall-runoff event of 6-24-1959 on watershed SW-17, Riesel (Waco), Texas.

explain the occurrence of complex roots and oscillations in the results of Chow and Kulandaiswamy (1971).

Thus a rational criterion for determining M consists in a) sensitivity of cumulants, and b) relative magnitudes of the coefficients a_i. If these coefficients are accurately determined then it would normally follow that $a_i > a_{i+1}$, $i=0, 1, 2 \ldots M$. If the value of a_M is very small relative to a_{M-1} for specified M then this value of M may usually be appropriate. Our experience (Singh and McCann 1980), coupled with these observations, suggests that on most watersheds M equal to 1 or at most 2 will be satisfactory for the GHS model. To illustrate, the GHS model with $M=1$ was utilized to generate surface runoff hydrographs for the rainfall-runoff events of watershed SW-17, Riesel (Waco), Texas, reported in Table 1. In each case it was found that important hydrograph characteristics were adequately reproduced as shown in sample Figs. 1-2. Thus the second order system may be acceptable from a practical standpoint.

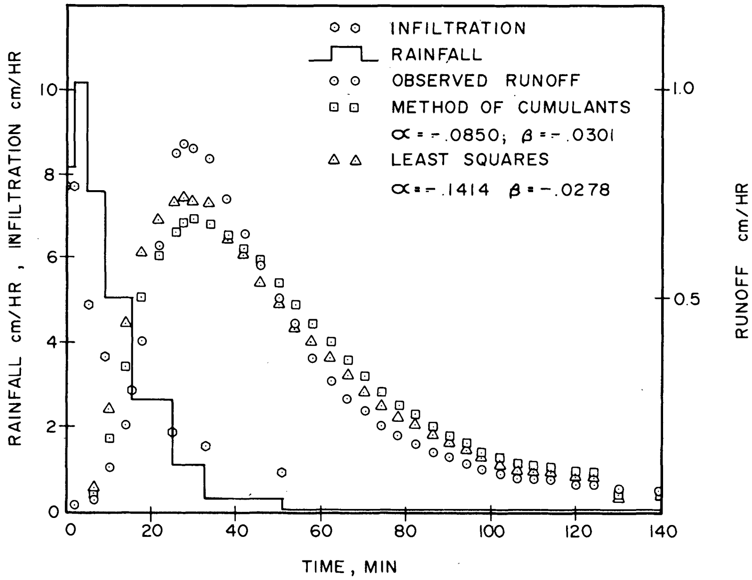

Fig. 2. Comparison of observed surface runoff hydrograph with hydrograph computed by method of moments and least squares method for rainfall-runoff event of 7-16-1961 on watershed SW-17, Riesel (Waco), Texas.

Conclusions

The following conclusions are drawn from this study:

1) The method of cumulants yields values of the model coefficients comparable to those of the least squares method for a natural watershed studied here. The method of cumulants is simple and free of derivatives of rainfall excess and runoff. The least squares method may be unreliable whenever the derivatives of runoff are not estimated accurately.

2) Based on the method of cumulants a rational criterion is developed to determine M in the GHS model. The observations made in this study support the values of M reached empirically by Kulandaiswamy (1964) and Chow and Kulandaiswamy (1971).

Acknowledgements

This study was supported in part by funds provided by the National Science Foundation under the project, »A Hydrodynamic Study of Surface Runoff«, No. NSF-ENG 79-05660.

References

Chaudhary, F. H., Simoes, M. A., and Ferreira Filho, W. M. (1976) »Chow's General Hydrologic System Model,« *J. Hydraul. Div., Proc. ASCE, 102 (HY9),* 1387-1390.

Chaudhary, F. H. (1976) »General Linear Hydrologic Response Models,« Proc. Third Annual Symp. Waterways, Harbors and Coastal Engrg. Div., *ASCE, 1, 367-376,* Colo. State Univ., Ft. Collins, Colo.

Chow, V. T. (1964) *Runoff,* Section 14 in Handbook of Applied Hydrology, edited by V. T. Chow, McGraw-Hill Book Co., New York.

Chow, V. T., and Kulandaiswamy, V. C. (1971) General Hydrologic System Model, *J. Hydraul. Div., Proc. ASCE 97 (HY6),* 791-804.

Conte, S. D. (1965) *Elementary Numerical Analysis:* An Algorithmic Approach, pp. 108-138, McGraw-Hill Book Company, New York.

Dooge, J. C. I. (1973) Linear Theory of Hydrologic Systems, Tech. Bull. No. 1468, Agric, Res. Serv., U.S. Dept. of Agric., Wash., D. C.

Kulandaiswamy, V. C. (1964) A Basic Study of the Rainfall Excess-Surface Runoff Relationship in a Basin System, Unpub. Ph. D. dissertation, Univ. of Illinois, Urbana, Ill.

Kulandaiswamy, V. C., Krishnaswamy, M., and Ramalingam, T. N. (1967) Flood Routing Through Channels, *J. Hydrol., 5,* 279-285.

Kulandaiswamy, V. C., and Subramanian, C. V. (1967) A Nonlinear Approach to Runoff Studies, Proc. Int'l Hydrol. Symp., pp. 72-85, Colo. State Univ., Ft. Collins, Colo.

Kulandaiswamy, V. C., and Rao, T. B. (1970) An Investigation of Nonlinearity in Storage-Discharge Relationship for Watersheds, Rept. No. 4, C.S.I.R. Project, Dept. of Hydraul. & Water Resources, Coll. of Engrg., Madras, India.

Kulandaiswamy, V. C., and Rao, T. B. (1971a) Digital Simulation of a Drainage Basin, Proc. Warsaw Symp. Math. Models, *Hydrol., 2,* 443-454, Warsaw, Poland.

Kulandaiswamy, V. C., and Rao, T. B. (1971b) An investigation of Nonlinearity in Runoff Process, Proc. Int'l Symp. Water Resourc., Banglore, India.

Singh, V. P. (1976) Studies on Rainfall-Runoff Modeling: 3. Converging Overland Flow, WRRI Rept. No. 073, 296 p., New Mexico Water Resources Res. Inst., New Mex. State Univ., Las Cruces, New Mex.

Singh, V. P., and McCann, R. C. (1980) A Mathematical Study of the General Hydrologic System Model, Tech. Rep. MSSU – EIRS – CE 80-1, 85 p., Engrg. & Indust. Res. Station, Miss. State Univ., Miss. State, Miss.

U. S. Department of Agriculture (1965) Hydrologic Date for Experimental Agricultural Watersheds in the United States, 1960-1961, Agric. Res. Serv., U. S. Dept. of Agric., Wash., D. C.

First received 28 August, 1979
Revised version received: 2 February, 1980

Address:
Department of Civil Engineering,
Mississippi State University,
Mississippi State, Mississippi 39762,
U.S.A.

Appendix

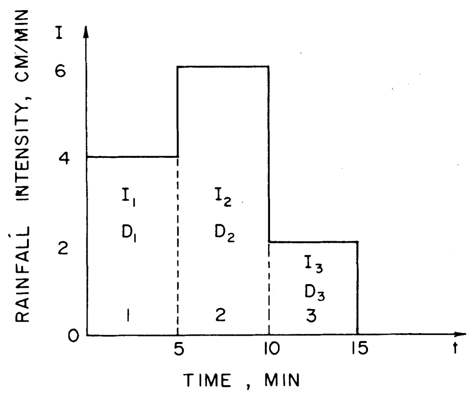

Fig. A. A hypothetical hyetograph of rainfall excess.

Computation of Cumulants

The cumulants can be easily computed for a given set of data by simply recalling their relationship with moments. It is well known in statistics that the first cumulant is equal to the first moment about the origin, the second and third cumulants are respectively equal to the second and third moments about the center of area, the fourth cumulant is equal to the fourth moment about the center of area minus three times the square of the second moment about the center of area, and so on.

For this study we only need the first and second cumulants. To illustrate their determination, consider a hypothetical hyetograph of rainfall excess as shown in Fig. A. Then

$$K_1(I) = \frac{\sum_{i=1}^{3} \bar{t}_i I_i D_i}{\sum_{i=1}^{3} I_i D_i} \tag{A-1}$$

where D_i is the duration of the ith block of rainfall excess whose intensity is I_i and whose center of area is located at a distance of $\bar{t}_i$ from the origin. Then we obtain,

$$K_1(I) \equiv \frac{5 \cdot 4 \cdot 2.5 + 5 \cdot 6 \cdot 7.5 + 5 \cdot 2 \cdot 12.5}{5 \cdot 4 + 5 \cdot 6 + 5 \cdot 2} \equiv 6.667 \text{ min}$$

Similarly, we obtain $K_2(I)$ as

$$K_2(I) = \frac{\sum_{i=1}^{3} (\bar{t}_i - K_1(I))^2 I_i D_i}{\sum_{i=1}^{3} I_i D_i} \tag{A-2}$$

$$= \frac{(2.5 - 6.667)^2 \cdot 20 + (7.5 - 6.667)^2 \cdot 30 + (12.5 - 6.667)^2 \cdot 10}{60} = \frac{708.339}{60} = 11{,}806 \text{ min}^2$$

In like manner the cumulants of higher order can be computed without difficulty.