Nordic Hydrology, 8, 1977, 129-140

Correlation Structure and Time Scale of Simple Hydrologic Systems

Paper presented at the Nordic Hydrological Conference
(Reykjavik, Iceland, Aug-Sep, 1976)

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Correlation structure of river runoff is a complicated set of different persistence phenomena in the watershed itself and in the meteorological input to the watershed.

Correlation functions and time scale of isolated processes in a watershed (groundwater level and river runoff) are derived analytically from the linearized equations of motion for these processes. Nonlinear effects on the correlation functions are shown for river runoff and for the watershed as a whole.

Introduction

Natural hydrologic systems are stochastic and nonlinear. Nevertheless, deterministic and linear models of hydrologic systems can be well adapted in many situations. The choice what kind of model should be used in a certain situation is actually not free. Whether the model to be used is deterministic or stochastic depends mainly on the character of the specific practical problem, that should be tackled, and whether a linear or nonlinear model is appropriate depends on the studied phenomena.

It is usually so that deterministic models are non-linear while stochastic models are linear, although the same type of phenomena should be described. Further, deterministic models are to a greater extent based on physical understanding of hydrologic processes while stochastic models, almost without exceptions, are totally black box models. There is a controversy among hydrologists which kind of mathematical models should be applied, stochastic or deterministic. It is our opinion that there exists some sort of confusement in this discussion and that the actual controversies are between a physically based and statistical approach for the construction of a model. As it was commented above, stochastic models are of the black box kind and the level of abstraction from the natural hydrological system is high. Structure and parameters of the models are determined by purely statistical methods. Deterministic models, on the other hand, are usually more directly formulated from what we know about the processes in the watershed. Many of them are of the black box kind but less abstract, for instance, - the unit hydrograph. Statistical methods are, however, often used to determine parameters and also make judgements about the structure. Why are stochastic models not based physically then? There is, of course, no opposition between the concepts stochastic and physical. It is, may be, so that it is more difficult to formulate a physical model in stochastic terms. The hydrologist in this case also suffers from the fact that he uses tools of statisticians, always reasoning from the information that can be gained from observations. Hydrologic data usually are poor and that is why we end with a model that is quite simple in its structure, say an antoregressiv or moving average model.

The language of the hydrologist in case of stochastic models abandons natural watersheds, when using terms like correlation and spectral functions etc., which can be hardly found in a watershed, even with the uppermost imagination of the spectator-hydrologist. Applying deterministic models, the hydrologist is speaking in terms of boxes, each box, linear or non-linear, describing in simple terms some part of the watershed like the unsaturated zone of the soil, the groundwater zone, lakes, river reaches etc. and it is already much easier to imagine the actual watershed. Nevertheless, the two languages though different in terms reflect the same phenomena, different reservoir mechanisms in the watershed.

The correlation function is fundamental and is a measure of the reservoir mechanisms in the watershed mentioned above. In this paper we shall try to develop this subject. We start with the equations of motion for ground water and river runoff. We shall examplify effects of different time scales on the structure of the correlation function and also discuss non-linear mechanisms. Finally, we shall discuss stochastic models in general, their structure and the question of stationarity.

Response Function of a Linear System

A linear system, as illustrated in Fig.1, is described by a homogeneous partial differential equation of the form:

$$L\{y(x,t)\} = 0 \tag{1}$$

Fig. 1. Linear system.

where L is a linear partial differential operator, x space coordinate and t time coordinate. Boundary condition is given at $x = 0 : y(0, t) = y_0(t)$. We write down the solution to Eq. (1):

$$y(x,t) = \int_0^t U(x,t-\tau)\, y(0,\tau)\, d\tau \tag{2}$$

where $U(x, t-\tau)$ is the impulse response function. We assume stationarity and that the system is at rest at $t = 0$. If we, in particular, set the input at $x = 0$ equal to Dirac delta function $y(0, t) = \delta(t)$, the solution to Eq. (2) will be given by:

$$y^{\delta}(x,t) = U(x,t) \tag{3}$$

The expression for the response function of a hydrologic system can, thus, be found by solving the linearized equations of motion, that describe best the system, for a Dirac delta input.

Correlation Function of a Linear System

The autocovariance function $C_y(\tau)$ of $y(s,t)$ is calculated as:

$$C_y(x,\tau) = E\,\{y(x,t)\; y(x,t+\tau)\} =$$

$$= \iint_0^{\infty} U(x,u)\; U(x,v)\; E\,\{y(0,t-u)\; y(0,t+\tau-v)\}\; du\; dv \tag{4}$$

Here we have taken advantage of the fact that the mean of the process $y(x, t)$ is equal to zero i.e. $\mu_y = E\{y(x, t)\} = 0$. Let the input at $x = 0$ be defined by:

$$C_y(0,\tau) = E\,\{y(0,t)\; y(0,t+\tau)\} = \delta(\tau) \tag{5}$$

where $\delta(\tau)$ is the Dirac delta function. We thus assume that the input $y(0, t)$ is an independent in time process. Making use of the specific properties of the delta function, the covariance function of $y(x, t)$ for this case is given by the integral expression:

$$C_y(x,\tau) = \int_0^{\infty} U(x,u)\; U(x,u+\tau)\; du \tag{6}$$

The correlation function of $y(x, t)$ is expressed by definition as:

$$\rho_y(x,\tau) = \frac{C_y(x,\tau)}{C_y(x,0)} \tag{7}$$

Time Scale of a Linear System

A simple measure of the persistence or memory of a process is the time scale T defined as the integral of the correlation function over time $\tau > 0$, i.e.:

$$T_y = \int_0^\infty \rho_y(x,\tau)\, d\tau \tag{8}$$

Equation of Motion for Groundwater Level

The linear differential equation for the operation of a groundwater aquifer in terms of groundwater piezometric height $h(x, t)$ is written down as:

$$\frac{\partial h}{\partial t} = \frac{H^\times k}{m} \frac{\partial^2 h}{\partial x^2} \tag{9}$$

where

k is the permeability coefficient,
$H^\times$- thickness of aquifer and
m - the active soil porosity

Eq. (9) is applicable to one dimensional confined flow in a homogeneous isotropic aquifer, and it is a good approximation for unconfined flow if changes in the water level $h(x, t)$ are small compared to the aquifer thickness $H^\times$.
The impulse response function corresponding to the linear partial differential equation is written down (Venetis 1970):

$$h^\delta(x,t) = U(x,t) = (t^\times)^{-3/2} \exp\{-1/t^\times\} \frac{mx^2}{4H^\times k} \tag{10}$$

where $\quad t^\times = \dfrac{4H^\times k \cdot t}{m\, x^2}$

Correlation Function of Groundwater Movement

Inserting the expression Eq. (10) for the impulse response function of groundwater level to Eq. (6) we derive an expression for the autocovariance function of an groundwater aquifer for independent input, namely:

$$C_y(x,\tau) = \left(\frac{2k\,m}{x^2 H^\times}\right)^2 \int_0^\infty \{t^\times(t^\times+\tau)\}^{-3/2} \exp\left\{-\frac{2t^\times+\tau}{t^\times(t^\times+\tau)}\right\} dt^\times \tag{11}$$

For $\tau = 0$ we find that $C_y(x,0) = \left(\frac{km}{x^2 H^\times}\right)^2$ The correlation function is thus expressed as:

$$\rho_y(s,\tau) = 4\int_0^{\infty}\{t^{\times}(t^{\times}+\tau)\}^{-3/2}\exp\{-\frac{2t^{\times}+\tau}{t^{\times}(t^{\times}+\tau)}\}\,dt^{\times} \qquad (12)$$

By integrating Eq. (12) over time τ we derive the time scale as:

$$T_y = \frac{\pi m\, x^2}{2H^{\times} k}$$

In Fig. 2 the correlation function Eq. (12) is drawn. We can notice a very slow descend of the function for large lags. For comparison the ordinary exponential decay function is also drawn for equal time scale.

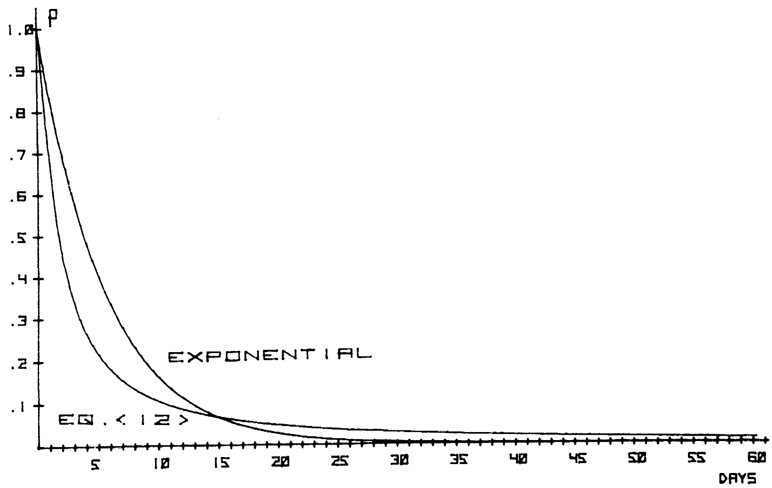

Fig. 2. Correlation function of groundwater level Eq. (12) and exponential function with equal time scales.

Inserting characteristic values for the parameters of the expression for the time scale given above, we find a value T_y equal to eight months (x = 1000 m, m = 0.33, $H^{\times}$= 50 m, k = 5.10-4 m/s). It should be mentioned that groundwater response of precipitation and evaporation input calculated directly from observed data is usually faster than the theoretically derived response from Eq. (9) (Gottschalk and Nordberg 1977).

Linearized Equation of Motion for River Runoff

The linearized differential equation governing the flow in a river reach can be written down in the following way:

$$\left(gH_0 - \left(\frac{Q_0}{F_0}\right)^2\right)\frac{\partial^2 q}{\partial x^2} - \frac{2Q_0}{F_0}\frac{\partial^2 q}{\partial x \partial t} - \frac{\partial^2 q}{\partial t^2} = \frac{3g\,Q_0^2}{F_0^2 C^2 H_0}\frac{\partial q}{\partial x} + \frac{2g\,Q_0}{F_0 C^2 H_0}\frac{\partial q}{\partial t} \tag{13}$$

where

x is the distance along the reach,
t - time,
g - acceleration of gravity,
q - fluctuation of river runoff,
Q_0 - mean river runoff,
H_0 - mean depth,
F_0 - mean section area and
C - chezy coefficient

Considering slow processes we can neglect the inertia terms in Eq. (13) which in this case has the simple form:

$$\frac{\partial^2 q}{\partial x^2} = \frac{3\,Q_0^2}{F_0^2 C^2 H_0^2}\frac{\partial q}{\partial x} + \frac{2\,Q_0}{F_0 C^2 H_0^2}\frac{\partial q}{\partial t} \tag{14}$$

The solution of this latter equation for a delta function input (the impulse response function for the river reach) is:

$$q(x,t) = U(x,t) = \frac{x}{\sqrt{4\pi\frac{F_0 C^2 H_0^2}{2\,Q_0}}}\;\frac{1}{t^{3/2}}\;\exp\left[-\frac{\left(x - \frac{3}{2}\frac{Q_0}{F_0}t\right)^2}{\frac{4\,F_0 C^2 H_0^2}{2\,Q_0}}\right] \tag{15}$$

A complete solution to Eq. (13) for a delta function input can be found also (Lighthill and Whitham 1955) but gives a very complicated expression.

River reaches behave very nearly as linear systems over small runoff ranges. Another condition for good result with linear approximations is that the considered reach is small compared to the wave length of the floodwave. For a larger river the reach should be devided into a set of linear subsystems. The theoretical basis for such division of a river into linear subsystems is given by Kalinin and Miliukov (1958).

Let us consider such a subsystem of a river reach. With a proper choice of the length of this river reach an unambiguous functional relation is valid between volume W and runoff Q:

$$Q = f(W) \tag{16}$$

The derivative dQ/dW is always positive and we can write

$$dW = k \cdot dQ \tag{17}$$

where k is a positive parameter with dimension time. In general case k is not a constant but is dependent on the level of flow. The Glomma river in Norway was devided into

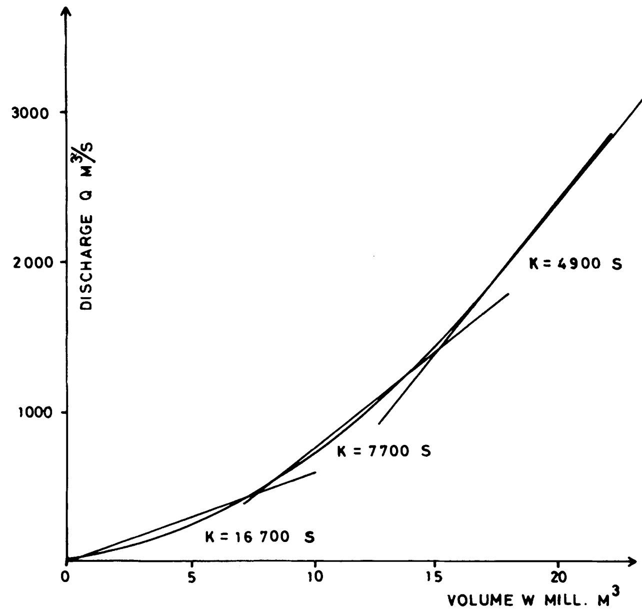

Fig. 3. Volume-discharge curve for a river reach at Skarnes, Glomma river, Norway.

such subsystems for routing purposes (Gottschalk 1975 unpublished). A typical look of the relation (16) for this river is given in Fig. 3, which also illustrates the dependence of k on the level of flow. Linear approximations for different ranges of flow are drawn. For n equal linear subsystems the impulse response function will be

$$U(n,t) = \frac{1}{\Gamma(n)} \left(\frac{t}{k}\right)^{n-1} e^{-t/k} \qquad (18)$$

for a certain range of flow, determined by the parameter k.

Correlation Function and Time Scale of River Runoff

We shall restrict ourselves to the further development of Eq. (18). Inserting this into Eq. (6) we derive the following expression for the covariance function of river runoff

$$C_y(n,\tau) = \int_0^\infty (\Gamma(n)k)^{-2} \left(\frac{u}{k}\right)^{n-1} e^{-u/k} \left(\frac{u+\tau}{k}\right)^{n-1} e^{-(u+\tau)/k} du \qquad (19)$$

Developing $\left(\frac{u+\tau}{k}\right)^{n-1}$ in its binomial series we find after some calculations the expression:

$$C_y(n,\tau) = \frac{1}{\Gamma(n)} e^{-\tau/k} \sum_{r=0}^{n-1} (\frac{\tau}{k})^r \frac{\Gamma(2n-r-1)}{\Gamma(n-r)\Gamma(r+1)} (\frac{1}{2})^{2n-r-1} \qquad (20)$$

For $\tau = 0$ we get:

$$C_y(n,0) = \frac{\Gamma(2n-1)}{\Gamma^2(n)} 2^{-2n+1} \qquad (21)$$

The correlation function we thus write down as:

$$\rho_y(n,\tau) = \frac{C_y(n,\tau)}{C_y(n,0)} = e^{-\tau/k} \sum_{r=0}^{n-1} \binom{n-1}{r} (\frac{\tau}{k})^r \frac{\Gamma(2n-r-1)}{\Gamma(2n-1)} 2^r \qquad (22)$$

Integrating the correlation function Eq.(22) over time $\tau > 0$, we find the time scale to be:

$$T_y = \int_0^\infty \rho_y(n,\tau)\, d\tau = k \cdot \sum_{r=0}^{n-1} \binom{n-1}{r} 2^r \frac{\Gamma(2n-r-1)}{\Gamma(2n-1)} \Gamma(r+1) \qquad (23)$$

Let us return to the routing model for the Glomma river to examplify Eqs. (22) and (23). An average subsystem in this model has the following characteristics: length of river reach 20 km, k= 0.41 when discharge $Q \approx 500$ m³/s, k = 0,21 for $Q \approx 1500$ m³/s and k = 0,17 for ≈2500 m³/s. Fig.4 shows the correlation function Eq. (22) for k = 0,4 and n = 1,..., 10.

The upper curve (n=10) thus reflects the reservoir mechanism for a 200 km long river reach when discharge is about 500 m³/s. The time scale of the process, calculated from Eq. (23), is 2.2 days. Fig. 5 illustrates nonlinearities in this reservoir mechanism where

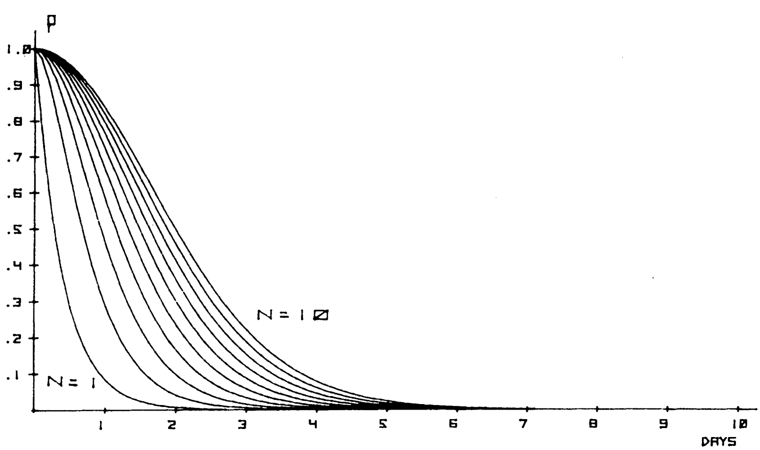

Fig. 4. Correlation function for cascade of n river reservoirs (n = 1, ..., 10) Eq. (22) for K = 0.4.

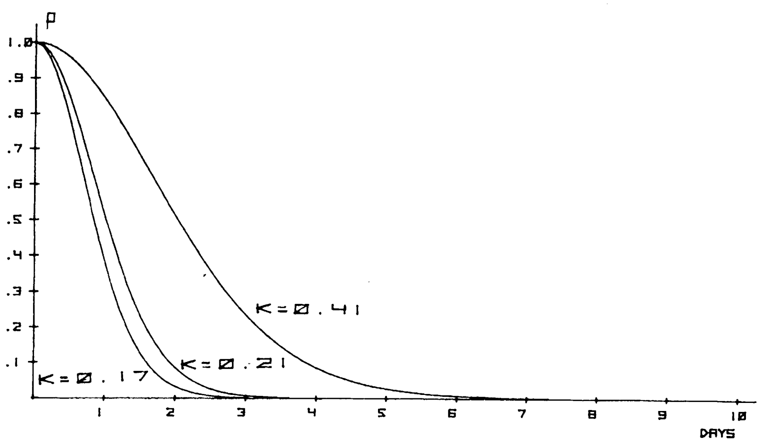

Fig. 5. Correlation function for cascade of 10 river reservoirs Eq. (22) for $K = 0.41$, 0.21 and 0.17.

correlation functions are drawn for $n = 10$ when discharge is around 500 m^3/s, 1500 m^3/s and 2500 m^3/s, respectively. The time scales are 2.2, 1.1 and 0.9 days for respective range of the flow.

Linearized model of watershed

We have analysed the correlation structure for isolated elements in the watershed, and shown that they in general have a complicated form. The correlation function of river runoff reflects composition of these different elements. Each element like river runoff, groundwater and soil water flow, for a certain range, can be approximated by cascades of linear reservoirs, characterized by their respective time scales. The total watershed is thus described by a network of linear reservoir in parallel and in series. A simple example of such network is shown in Fig. 6, where we have a soil water system in parallel with a groundwater system and both are joined to a river system.

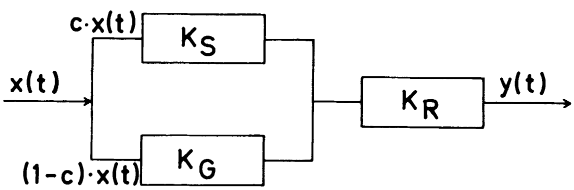

Fig. 6. Linear watershed system.

The correlation function for this system has the expression:

$$\rho(\tau) = \{A_S \cdot e^{-\tau/K_S} + A_G \cdot e^{-\tau/K_G} + A_R \cdot e^{-\tau/K_R}\} / \{A_S + A_G + A_R\} \tag{24}$$

where

$$A_S = C^2\{K_S/2 - K_R K_S/(K_S-K_R)\} / (K_S-K_R)^2 + (1-C)\, C\{K_G K_S/(K_S-K_G) - K_R K_S/(K_S-K_R)\} / (K_G-K_R) / (K_S-K_R)$$

$$A_G = (1-C^2)\{K_G/2 - K_G K_R/(K_G-K_R)\}/(K_G-K_R)^2 + (1-C)\, C\{K_S K_G/(K_G-K_S) - K_R K_G/(K_G-K_R)\} / (K_S-K_R) / (K_G-K_R)$$

$$A_R = C^2\{K_R/2 - K_R K_S/(K_R-K_S)\} / (K_S-K_R)^2 + (1-C^2)\{K_R/2 - K_R K_G/(K_R-K_G)\} / (K_G-K_R)^2 + C(1-C)\{K_R - K_R K_S/(K_R-K_S) - K_R K_G/(K_R-K_G)\} / (K_S-K_R) / (K_G-K_R)$$

C is a coefficient distributing input between soil water and groundwater systems and K_S, K_G and K_R are the respective time scales. The time scale corresponding to Eq. (24) is easily found:

$$T = \{A_S K_S + A_G K_G + A_R K_R\} / \{A_S + A_G + A_R\} \tag{25}$$

We shall note that the coefficient C is not a constant but is dependent on the intensity of the flow from the soil water zone. The nonlinear behaviour of the system in Fig. 6 is illustrated by calculating the correlation function Eq. (24) for three different values of C which is shown in Fig. 7. The time scale for the three cases are also given in the figure.

Conclusions

Correlation functions estimated from observed series of river runoff do not have the necessary accuracy to distinguish specific effects of nonlinearities and varying time scales. The empirical correlation function is therefore in many respects a very coarse instrument to describe the complex nature of the river runoff process.

The empirical correlation functions of river runoff are also influenced by correlation structure and time scale of climatic input to the watershed. For Scandinavian conditions this structure is quite complicated. During winter the input to the watershed system is zero. Snowmelt has a relatively high memory as long as the snowcover has

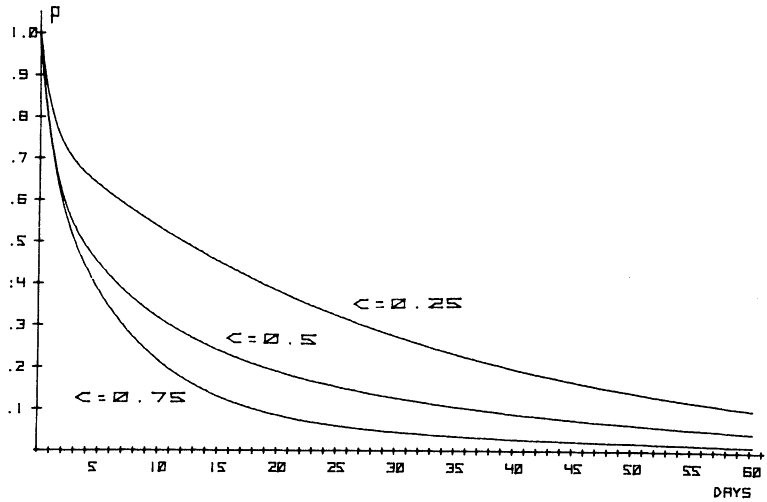

Fig. 7. Correlation function of river runoff from watershed system Eq. (24) with C = 0.25, 0.50, 0.75, K_S = 7 days, K_G = 30 days and K_R = 1 day.

not melted away, but is influenced by abrupt changes in temperature. Summer rain is an intermittent process with short time scale. We thus have a highly nonstationary process. The extent of its influence on the correlation structure of river runoff depends on the time basis, on which the derivation of the correlation function is done.

When using daily values to calculate the correlation function, the reservoir mechanisms of the watershed will be predominant. The time dependence is therefore small. We shall maybe note a larger instability and some lowering in calculated values of the correlation function during summer. This is caused by the intermittent precipitation input and nonlinearities in reservoir behaviour.

Integration of the river runoff process over larger time intervals, results in the fact that the nonstationarity in the climatic input will give dominant influence on the river runoff correlation structure. We can note, however, that the time scale of groundwater aquifers and lakes is quite large and can be of importance. Nonlinearities in the reservoir mechanisms of aquifers and lakes can cause a complicated correlation structure of river runoff in a large time scale.

We shall also stress the fact that runoff as an integrated process over some time interval is dependent on the correlation structure within this time interval. This has been examplified by Gottschalk, 1975, giving relations between statistics of monthly and annual processes.

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Received: 13 December, 1976

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