On bi-closed subbands of a band and free idempotent semirings

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Abstract: The bi-closed subbands of a band are introduced and its some properties and characterizations are given. By using these results, the free object in the variety of idempotent semirings whose additive reducts are semilattices is established.

Key words: semigroup; idempotent semiring; band; bi-closed subband.

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A semiring $(S, +, \cdot)$ is an algebra with two binary operations '$+$' and '$\cdot$' such that $(S, +)$ and $(S, \cdot)$ are semigroup sconnected by ring-like distributive laws. A semiring is said to be idempotent if its additive reduct and multiplicative reduct are idempotent semigroups(i.e., bands). Idempotent semirings can be regarded as a generalization of distributive lattices. There are a series papers in the literature cosidering idempotent semirings, for example[1~5]. It is easily seen that the Green's relation $\overset{+}{\mathscr{D}}$ on the additive reduct of an idempotent semiring S is always a congruence on semiring S and the quotient semiring $S/\overset{+}{\mathscr{D}}$ is an idempotent semiring whose additive reduct is a semilattice. This is a reason why we study the variety of idempotent semirings whose additive reducts are semilattices. In order to characterize of free object in this variety, we first introduce the bi-closed subbands of a band and we establish the some properties and a characterizations of Bi-closed subbands of a band in section 1. In section 2, we will prove that the set CSB (B) of Bi-closed subbands of a band B is an idempotent semiring whose additivereduct is a semilattice under the following two operations '$+$' and '$\cdot$'

$$C + D = \overline{C \cup D},$$
$$C \cdot D = \overline{CD},$$

for any $C, D \in$ CSB (B), where $\overline{C \cup D}$ and $\overline{CD}$ denote the bi-closed subband of B generated by $C \cup D$ and CD, respectively. Finally, we show that the set FCSB (B_X) of finite bi-closed subbands of free band B_X generated by X forms a subsemiring of semiring CSB (B_X) and is the free object on set X in the variety of idempotent semirings whose additive reducts are semilattices.

1 Bi-closed subbands of a band

Definition *A* subband C of a band B is said to be bi-closed if it has the property

$$sa, sb \in C \Rightarrow sab \in C,$$

and

$$as, bs \in C \Rightarrow abs \in C,$$

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for any $a,b,s\in B$.

A singleton element set of a band is bi-closed. In fact, Let B be a band and $C=\{c\}$ a singleton element set of B. Then, for any $a,b,s\in B, sa,sb\in C$ implies $sa=sb=c$. So we have

$$sab=(sa)b=cb=(sb)b=sb=c.$$

That is to say, $sab=c$, Similarly, we can show that $as,bs\in C$ implies $abs\in C$. It is also clear that C is a subband of B. Now we have shown that C is a bi-closed subband of B.

Obviously, every ideal of a band B is a bi-closed subband of B. Recall that a band B is said to be normal if it satisfies the additional identity

$$uxyv=uyxv.$$

For a normal band B, every subband of B is bi-closed.

For a nonempty subset X of a band B the intersection of all the bi-closed subbands of B containing X is a bi-closed subband of B containing X. It is called the bi-closed subband of B generated by X and written as $\overline{X}$. To characterize $\overline{X}$, we inductively define the following some subbands of B.

$X^{(1)}$ is the subband of B generated by X;

$X^{(i+1)}$ is the subband of B generated by the union set of $\{sab\mid a,b,s\in B,\ sa,sa\in X^{(i)}\}$ and $\{abs\mid a,b,s\in B, as,bs\in X^{(i)}\}$.

Lemma 1 For a nonempty subset X of a band B, the set $\bigcup_{i=1}^{\infty}X^{(i)}$ is a bi-closed subband of B containing X.

Theorem 1 By using Lemma 1, we need only prove that a bi-closed subband C of a band B containing X must contain $\bigcup_{i=1}^{\infty}X^{(i)}$. it is clear that $X^{(1)}\subseteq C$. If $X^{(n)}\subseteq C$ for some $n\in N$, then the union set of the sets

$$\{sab\mid a,b,s\in B,\ sa,sb\in X^{(n)}\}$$

and

$$\{abs\mid a,b,s\in B,\ as,bs\in X^{(n)}\},$$

which is the set of generators of $X^{(n+1)}$, is contained in C and so $X^{(n+1)}$ is contained C since C is a bi-closed subband of B. Hence, $\bigcup_{i=1}^{\infty}X^{(i)}\subseteq C$, as required. □

Lemma 2 Let B be a band and Sub (B) the set of all nonempty subsets of B. Then we have that

$$\overline{X}=\overline{Y}\Rightarrow\overline{ZX}=\overline{ZY},$$

and

$$\overline{XZ}=\overline{YZ},$$

for any $X,Y,Z\in$ Sub (B).

Proof: Proof Suppose that $X,Y,Z\in$ Sub(B) and $\overline{X}=\overline{Y}$. Then

$$ZX\in\overline{ZX}=Z\overline{Y},\tag{1}$$

To show $ZX\subseteq\overline{ZY}$, we first prove that

$$(\forall\ n\in N)ZY^{(n)}\subseteq\overline{ZY},\tag{2}$$

Suppose that $z\in Z$ and $y\in Y^{(1)}$. By the definition of $Y^{(1)}$, we can take $y_1,y_2,\cdots,y_m\in Y$ such that $y=y_1y_2\cdots y_m$. hence,

$$zy=zy_1y_2\cdots y_m\in\overline{ZY}$$

since $zy_1, zy_2,\cdots,zy_m\in\overline{ZY}$ and $\overline{ZY}$ is a bi-closed subband of B. This leads to $ZY^{(1)}\subseteq\overline{ZY}$. If $ZY^{(i)}\subseteq\overline{ZY}$, then for any $z\in Z$ and any generator y of $Y^{(i+1)}$, say without generaliztion, $y=sab$ and $sa,sb\in Y^{(i)}$, we have $zy=zsab\subseteq\overline{ZY}$ since $zsa,zsb\subseteq ZY^{(i)}\subseteq\overline{ZY}$ and $\overline{ZY}$ is bi-closed. Hence, $ZY^{(i+1)}\subseteq\overline{ZY}$ again since $\overline{ZY}$ is bi-closed. We have now shown formula (2). Moreover, by using formulas (1) and (2), we have that$Z\overline{Y}=Z\bigcup_{i=1}^{\infty}Y^{(i)}\subseteq\overline{ZY}$ and $ZX\subseteq Z\overline{Y}$. We can dually show that $ZY\subseteq Z\overline{X}$. This leads to $\overline{ZX}=\overline{ZY}$. Analogically, we have $\overline{XZ}=\overline{YZ}$, as required.

Theorem 2 Let B be a band and define a binary relation ρ on the power semigroup Sub (B) of B by

$$X\rho Y\Leftrightarrow\overline{X}=\overline{Y},\tag{3}$$

for any $X,Y\in$ Sub (B), Then ρ is a band congruence on Sub (B).

Corollary Let B be a band and define a binary operation $\circ$ on the set CSB (B) of all bi-closed subband of B as follows

$$X\circ Y=\overline{XY},$$

for any $X,Y\in$ CSB (B). Then (CSB(B), $\circ$) is a band.

Theorem 3 Let X be a countable infinite set and B_X the free band generated by X. Then we have

1) The bi-closed subband $\overline{X_1}$ of B_X generated by a nonempty subset X_1 of X equals to the subband $X^{(1)}$ of B_X generated by X_1.

2) The bi-closed subband $\overline{A}$ of B_X genreated by a nonempty finite subset A of B_X is finite.

Proof Let X be a countable infinite set, B_X the free band generated by X , and X^+ the free semigroup generated by X. For every words w in X^+, we define the content $C(w)$ to be the (necessarily finite) set of elements of appearing in w.

1) Suppose that X_1 is a nonempty subset of X and $X_1^{(1)}$ thesubband of B_X generated by X_1. Then it is clear that $X_1^{(1)}$ consists of all elements of B_X whose contents are contained in X_1. Hence, for any $a,b,s\in B_X$, if $sa,sb\in X_1^{(1)}$, then $C(sa)$, $C(sb)\subseteq X_1$ and so $C(a)$, $C(b)$, $C(s)\subseteq X_1$. This leads tothat $a,b,s\in X_1^{(1)}$, and so $sab\in X_1^{(1)}$. Similarly, we can show that if $as,bs\in X_1^{(1)}$, then $abs\in X_1^{(1)}$. That is to say, $X_1^{(1)}$ is a bi-closed subband of B_X generated by X_1, as required.

2) Suppose that A is a nonempty finite subset of B_X. Recall that $\overline{A}$ denotes the bi-closed subband of B_X genreated by A. For $A_1=\bigcup_{a\in A}C(a)$, we can see that $A^{(1)}\subseteq A_1^{(1)}$. By using 1), this leads to $\overline{A}=\overline{A^{(1)}}\subseteq\overline{A_1^{(1)}}=A_1^{(1)}$. Hence, since the free band on a finite set is finite (Theorem 4. 5. 3 in [6]) and A_1 is finite, we have that both $A_1^{(1)}$ and $\overline{A}$ are finite, as required.

2 Free objects in the variety of idempotent semirings.

It is will known that the set Sub (B) of all nonempty subsets of a band B is a semiring whose addition is the operation of union and multiplication is usual multiplication of subsets of band B. By using Theorem 2, we can see that the binary relation ρ defined by formula (3) is a congruence on semigroup (Sub (B), $\circ$). Moreover, if $X,Y\in$ Sub (B) and $X\rho Y$, *i. e.*, $\overline{X}=\overline{Y}$, then $X\subseteq\overline{X}=\overline{Y}\subseteq\overline{Y\cup U}$ and $Z\subseteq\overline{Y\cup U}$. Hence, $X\cup Z\subseteq\overline{Y\cup Z}$. Dually, we can show that $Y\cup Z\subseteq\overline{X\cup Z}$. Thus, $\overline{X\cup Z}=\overline{Y\cup Z}$. This leads to that ρ is a congruence on semilattice (Sub (B), $\cup$). We now have the following lemma.

Lemma 3 The binary relation ρ defined by formula (3) is a congruence on semiring (Sub (B), $\cup$, $\circ$).

Theorem 4 Let B be a band and define addition and multiplication on the set CSB (B) of all bi-closed subband of B as follows

$$C+D=\overline{C\cup D}\qquad C\circ D=\overline{CD},$$

for any $C,D\in$ CSB(B). Then (CSB(B), $+$, $\circ$) is an idempotent semiring whose additive reduct is a semilattice.

Proof It is easily seen that (CSB(B), $+$, $\circ$) is the quotient semiring of semiring (Sub(B), $\cup$, $\circ$) under congruence ρ and its additive reduct (CSB(B), $+$) is a semilattice . By using the Corollary of Theorem 2, we can see that that its multiplicative reduct (CSB (B), $\circ$) is a band. This leads to that (CSB(B), $+$, $\circ$) is an idempotent semiring whose additive reduct is a semilattice, as required. □

It now is easy to see that the set FCSB (B_X) of all finite bi-closed subband of free band B_X on a set X (finite or countable infinite) is asubsemiring of (CSB(B_X), $+$, $\circ$) whose additive reduct is a semilattice. In fact, for any $C,D\in$ FCSB(B_X), since $C\cup D$ and CD containfinite elements of B_X, by using Theorem 3, we have that $\overline{C\cup D}$ and $\overline{CD}\in$ FCSB(B_X). We will next show that FCSB(B_X) is a free object in the variety of idempotent semirings whose additive reducts are semilattices. For this purpose, the following lemmas are need.

Lemma 4 Let S be an idempotent semiring whose additive reduct is a semilattice. Then we have that

$$u=u+sa=u+sb\Rightarrow u=u+sab;\qquad(4)$$

$$u=u+as=u+bs\Rightarrow u=u+abs.\qquad(5)$$

for any $a,b,s,u\in S$.

Proof Let S be an idempotent semiring whose additive reduct is a semilattice. For any a, $b,s,u\in S$, if $u=u+sa=u+sb$, then we have

$$\begin{aligned}u&=u+sa+sb=u+s(a+b)=\\&u+s(a+b)^2=\\&u+sa+sb+sab+sba=\\&u+sab+sba=\\&u+sab.\end{aligned}$$

This leads to that formula (4) holds. Similarly, we can show that formula (5) holds, as required. □

Recall that for a band B and a subset A of B, we have that $\overline{A}=\bigcup_{i=1}^{\infty}A^{(i)}$ (Lemma 1), where $\overline{A}$ denotes the bi-closed subband of generated by A and $A^{(n)}\subseteq A^{(n+1)}$ for any $n\in N$. Hence, if $\overline{A}$ is finite, then there exists $m\in N$ such that $\overline{A}=\bigcup_{i=1}^{m}A^{(i)}$ and each one of $A^{(n)}$ is finite.

Lemma 5 Let S be an idempotent semiring whose additive reduct is a semilattice, B a band and φ band homomorhpism form B to $(S, \circ)$. For any finite subset A of B, if $\overline{A}$ is finite, then we have that

$$\sum_{a\in A}\varphi(a)=\sum_{a\in A^{(1)}}\varphi(b)=\sum_{a\in \overline{A}}\varphi(c), \qquad (6)$$

where $A^{(1)}$ and $\overline{A}$ are respectively the subband and the bi-closed subband of B generated by A.

Theorem 5 FCSB(B_X) is the free object in the variety of idempotent semirings whose adddi-tive reducts are semilattices.

Corollary Let S be an idempotent semiring whose additivereduect is a semilattice. Then S satisfies the following identity

$$x_1+\cdots+x_m+x_{\lambda(1)}\cdots x_{\lambda(m)}\approx x_1+\cdots+x_m$$

where λ is a transformation on the set $\{1,2,\cdots,m\}$.

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带的双闭子带和自由幂等元半环

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摘要:给出带的双闭子带和它的性质及特征。在此基础上,给出加法半群为半格的幂等元半环簇中的自由对象。

关 键 词:半群;幂等元半环;带;自由对象