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PREDICTION OF THE DIMENSIONLESS UNIT HYDROGRAPH

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Brief Note

Much attention has been given to rationalizing the dimensionless unit hydrograph during the past two decades. Recently, Schultz, Pinkayan & Komsartra (1971) reviewed some of this literature and proposed the relation:[1]

$$\frac{Q}{Q_p} = \left(\frac{t}{t_p}\right)^{n-1} \exp\left(-(n-1)[(\frac{t}{t_p}) - 1]\right) \tag{1}$$

where Q and t are the discharge rate and the time after the occurrence of a sudden storm. The subscript p denotes the coordinates of the peak runoff. The paramter n is given as a free constant in the expression.

In 1964 the author (Lienhard 1964) developed a statistical mechanical prediction of the dimensionless unit hydrograph in which all constants were rationalized. Lienhard & Meyer (1967) subsequently showed that this specific case was but one member of a family of distribution functions, all of which obeyed the following physical model:

1. One moment of the distribution is known:

$$\left[\int_0^\infty t^\beta f(t)dt\right] \Big/ \left[\int_0^\infty f(t)dt\right] \equiv (t_{rm\beta})^\beta \tag{2}$$

where t is the argument of the distribution and β is the degree of the moment.

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[1] The original paper contained a typographical error in equation (1). This has been corrected here.

If $\beta = 1$, then $t_{rm\beta}$ is the mean of t; if $\beta = 2$, then $t_{rm\beta} = t_{rms} =$ the root-mean-square of t; etc.

2. The population of the distribution is fixed.

3. The number of ways that a member of the population can reach a given value of t, is proportional to t^{n-1}. Here n has the physical significance of characterizing the "accessibility" of the distributed event to a given value of t.

The result of this model was the generalized gamma distribution:

$$f(t) = \frac{\beta}{\Gamma(n/\beta)}\left(\frac{n}{\beta}\right)^{n/\beta}\frac{1}{t_{rm\beta}}\left(\frac{t}{t_{rm\beta}}\right)^{n-1}\exp\left[-\frac{n}{\beta}\left(\frac{t}{t_{rm\beta}}\right)^{\beta}\right] \tag{3}$$

Next, Lienhard & Davis (1971) discussed the "thermodynamics" that related the properties of the watershed described by this distribution.

In the first of these papers, the author (1964) developed distribution functions based on a known 2nd moment ($\beta = 2$). It was shown on the basis of physical arguments that the accessibility number, n, tended toward 3 for fan-shaped watersheds and toward 2 for long slender watersheds. These expressions worked well when they were compared with the runoff from small watersheds in Illinois after sudden storms.

Let us now cast equation (3) in the form of the conventional dimensionless unit hydrograph to facilitate comparison both with data and with other hydrograph expressions. To do this we set the derivative of the equation equal to zero, to find t_p in terms of $t_{rm\beta}$. The result is

$$t_{rm\beta} = \left(\frac{n}{n-1}\right)^{1/\beta} t_p \tag{4}$$

Equation (4) is then substituted in equation (3) to give $f(t_p)$. Finally, noting that $Q/Q_p = f(t)/f(t_p)$, we obtain from the generalized gamma distribution:

$$\frac{Q}{Q_p} = \left(\frac{t}{t_p}\right)^{n-1}\exp\left[-\frac{n-1}{\beta}\left(\left[\frac{t}{t_p}\right]^{\beta} - 1\right)\right] \tag{5}$$

The author's (1964) hydrographs then take the form

$$\left.\frac{Q}{Q_p}\right|_{\substack{\text{fan shaped}\\ \text{watershed}}} = \left(\frac{t}{t_p}\right)^{2}\exp\left[-\left(\left[\frac{t}{t_p}\right]^{2} - 1\right)\right] \tag{6}$$

$$\left.\frac{Q}{Q_p}\right|_{\substack{\text{long slender}\\ \text{watershed}}} = \left(\frac{t}{t_p}\right)\exp\left[-\frac{1}{2}\left(\left[\frac{t}{t_p}\right]^{2} - 1\right)\right] \tag{7}$$

The expression given by Schultz et al., on the other hand, corresponds with equation (5) based on a *known first moment,* or mean, time of runoff (i.e. ß = 1). Furthermore they obtained a reasonable correlation of data using n = 3 (or slightly more than 3) in the equation. This suggests that their watersheds were very nearly fan-shaped.

Thus equation (5) subtends a large class of hydrograph equations – some in use, and others not. Its particular, merit lies in the fact that the constants n and β are not arbitrary. They are fixed by the shape of the watershed and by the moment that is assumed to be known for the watershed.

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