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A Unified Approach to Watershed Modelling

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This paper summarizes the main points of a theoretical study of the general watershed behaviour, on the basis of the known deterministic physics-based approaches for overland flow, underground flow and unsaturated medium physics. A parameter $z(\theta)$ which describes the soil water movement in the unsaturated zone and accounts for evaporation is introduced. Thus, a differential equation which allows a state-space formulation of the processes in the watershed is obtained, and the solution of the state equation can be given the form of the Volterra Integral series. The obtained results are compared with those of previous studies.

Introduction

Watershed response to rainfall has been studied extensively in recent years. Models, both linear and non-linear, have been proposed to simulate the watershed behaviour. Though the components of the rainfall-runoff transformation process are well described, serious difficulties arise whenever the models must combine all the components involved.

This paper describes the general behaviour of the watershed by combining a deterministic physics-based model for overland flow, underground flow, and for unsaturated flow. Since hydrological practices are concerned mostly with lumped parameter systems, a lumped formulation of the continuity equation for the watershed process is derived. This formulation introduces as a natural consequence the concept of a state-space approach to watershed modelling.

The Simplified Conceptual Watershed Model

A large number of schemes have been proposed for the representation of the Hydrologic processes in the land phase of the hydrological cycle, but they are generally too complex for the parametrical modelling of the rainfall-runoff process. A description of these models has been made by Dooge (1977). This study is based on the simplified model shown in Fig. 1.

This model makes a distinction between direct storm response and baseflow and also between the movement of soil moisture in the unsaturated zone and the flow of ground water in the saturated zone.

This simplified catchment model has thus two main components: that is, the component of overland flow (direct storm response) and the component of underground flow which is in itself a component of several processes. A detailed description of inter-actions between the two components is given elsewhere (Afouda 1974) and it can be summarized as follows: the unsaturated zone is of main importance in that it separates the total amount of water into two major parts: the overland component and the underground component. The processes in the unsaturated zone also comprise depletion of soil moisture by evaporation and transpiration soil moisture movement. The resulting overland flow can be positive or equal to zero. As a response to the change in the stream stage, due to the overland runoff, the base flow decreases and inflow to the aquifer can occur (i.e. the baseflow can be either positive or negative). These model features are given in the form of block diagram in Fig. 1.

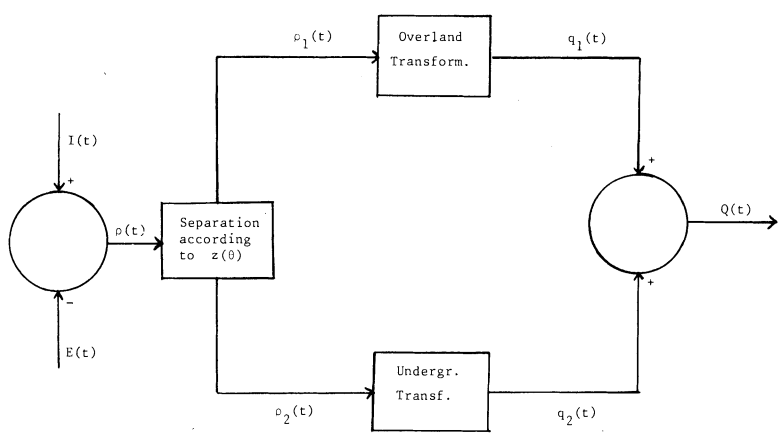

Fig. 1. Simplified catchment model.

Mathematical Description of the Model

Now, let x be the total storage of water not yet discharged from the watershed and be defined by

$$x(t) = \int_{t_0}^{t} (\rho(\tau) - Q(\tau))d\tau$$

where

$$\rho(t) = I(t) - E(t);$$

I(t) is the rainfall intensity and *E(t)* is the evaporation flux in the watershed; *Q(t)* is the discharge at the outlet from the watershed. Let x_1, x_2 be the volume of rainfall not yet discharged from the watershed and thus in surface and subsurface storage.

Conservation of mass yields:

$$x = x_1 + x_2 \qquad (1)$$

If the action of the unsaturated zone is characterized by a parameter $z(\theta)$ depending on the water content, then x_1 and x_2 are related to the total volume by the following relations.

$$x_1 \equiv [1 - z(\theta)]\, x \qquad (2)$$

$$x_2 \equiv z(\theta)\, x \qquad (3)$$

$z(\theta)$ can vary from 0 to 1 and physically it depends on the rainfall intensity, the antecedent precipitation and evaporation and the soil properties. For any given soil structure however, this parameter will largely be governed by the amount and distribution of moisture in the soil. The soil structure is assumed as constant and $z(\theta)$ is approximated as a function of soil moisture only.

Further, when $\rho(t)$ and *Q(t)* are designating the input and the output of the system, the lumped continuity equation for the system is given by:

$$\frac{dx(t)}{dt} + Q(t) \equiv \rho(t) \qquad (4)$$

and the momentum equation will have the form

$$Q(t) = G[x\,;\, z(\theta)\,;\, \rho(t)\,;\, t] \qquad (5)$$

Eqs. (4) and (5) can be combined to obtain

$$\frac{dx(t)}{dt} \equiv F[x\,;\, z(\theta)\,;\, \rho(t)\,;\, t] \qquad (6)$$

Eq. (6) describes in principle the dynamic behaviour of the hydrologic system in study, and together with Eq. (5) it constitutes the state equation.

Explicit Formulation of the State Equations

In a previous paragraph we have considered the system to be composed of two subsystems of which one produced the overland runoff and the other produced the underground contribution to the watershed response. This decomposition is now used to specify explicitly the state equations of the hydrologic system.

Let us consider the overland component characterized by $x_1(t)$, $\rho_1(t)$; $q_1(t)$ which represent the rainfall volume available for overland transformation and the input and the output of the subsystem respectively. As previously, the lumped continuity equation for the subsystem is given by

$$\frac{dx_1(t)}{dt} + q_1(t) = \rho_1(t) \tag{7}$$

and the dynamic equation takes the form

$$q_1(t) \equiv g_1(x_1 ; \rho_1 ; t) \tag{8}$$

As it will be demonstrated later that hydrological analyses have shown that a simple power relationship can be assumed between discharge at the downstream end of the catchment and the corresponding surface storage over this catchment, q_1 can be assumed to be given by

$$q_1(t) = \varepsilon_2 \; x_1^k(t) \tag{9}$$

If $x_2(t)$, $\rho_2(t)$, $q_2(t)$ are the total amounts of rainfall volume available for underground transformation and the input and output of the underground subsystem respectively, the same argument leads to the following equation

$$\frac{dx_2(t)}{dt} + q_2(t) = \rho_2(t) \tag{10}$$

$$q_2(t) \equiv g_2(x_2 ; \rho_2 ; t) \tag{11}$$

It will be demonstrated later that g_2 $(x; \rho_2; t)$ can have the form

$$q_2(t) = \varepsilon_1 x_2(t) \tag{12}$$

Now Eqs. (7), (9) and (10), (12) can be combined with (1), (2) and (3) to yield the equations of the system as

$$\frac{dx(t)}{dt} + \varepsilon_1 z(\theta)x(t) + \varepsilon_2 [1-z(\theta)]^k [x(t)]^k = \rho(t) \tag{13}$$

$$Q(t) = \varepsilon_1 z(\theta)x(t) + \varepsilon_2 [1-z(\theta)]^k [x(t)]^k \tag{14}$$

where

ε_1 is the coefficient of underground transformation
ε_2 is the coefficient of overland transformation

and the other parameters are defined as previously. Eqs. (13) and (14) are explicit forms of Eqs. (5) and (6).

From Eqs. (13) and (14) it appears that, although the system is described in the lumped form, it remains non-linear.

Non-linearity is introduced by the parameters $z(\theta)$, ε_1, ε_2 which characterize the system and summarize the information about the past history of the system. The basic assumption made is that for all

$$x(t_0) = x\,[\,\rho(t_0)\,;\,z(\theta)|_{t=t_0}\,;\,\varepsilon_1\,;\,\varepsilon_2\,]$$

at t_0 and all $\rho(t)$ for $t>t_0$, knowledge of $x(t_0)$ and $\rho(t)$ uniquely determine $Q(t)$ for the same time interval.

In fact this is an approximation as the hyrological system of a watershed is well known to behave in a way that is dynamic, non-linear and stochastic. Nevertheless, for the moment this deterministic approximation of the physical process can be accepted.

Further, non-linearity is introduced by the power relationship between overland flow and the storage. This non-linearity is of a dynamic character and related to the turbulence processes while the first mentioned non-linearity is related to the structural characteristics of the watershed.

Solution of the State Equations

Parameter estimation – Since the parameters of Eqs. (13) and (14), summarize all the information about the past behaviour of the system, it can be expected that available information on the past behaviour of the system allows for their objective estimation.

Estimation of $z(\theta)$ – From Eqs. (2) and (3) it appears that the estimation of $z(\theta)$ involves either the knowledge of x and x_2 or the knowledge of x and x_1. Let us define

$$x_2 = \int (r(\tau) - q_2(\tau))d\tau$$

where $r(t) = \rho_2(t)$ is the infiltration rate. This infiltration rate can be estimate from soil physics or hydrologic models of infiltration and q_2 can be evaluated utilizing traditional hydrograph separation methods. It is not the purpose of this study to present in detail these methods and their validity. Basic information and discussions can be found in classical hydrological textbooks such as Roche (1963), Eagleson (1970), Remenieras (1970), Overton and Meadows (1976), and others.

The mathematical expression for both $z(\theta)$ and other parameters is here presented.

Suppose that $r(t)$ and $q_2\,(t)$ have been adequately evaluated, then $z(\theta)$ can be estimated from the following:

$$z(\theta) = \frac{\int_{t_0}^{t} (r(\tau)-q_2(\tau))d\tau}{\int_{t_0}^{t} (\rho(\tau) - Q(\tau))d\tau} \tag{15}$$

For a given rainfall event $z(\theta)$ is thus the ratio of the volume of rainfall not yet discharged from subsurface storage and the total amount of water not yet discharged from the watershed. When calculated from past events, $z(\theta)$ is a constant parameter for each rainfall event. Hence the mean value can be estimated for a particular watershed by:

$$\bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i\,(\theta) \tag{16}$$

where n is the number of storms considered.

Estimation of ε_1 – From ground water flow theory, it is well known that under appropriate assumptions (Rorabaugh 1969, Dooge 1973), the contribution of groundwater to runoff for a uniform rate of recharge is given by

$$q = 2T\,\frac{ho}{a} \sum_{i=1,3,5,\ldots}^{\infty} \exp\left\{ - i^2\,\frac{\pi^2 T}{4a^2 S}\,t \right\} \tag{17}$$

where q = the horizontal flow per unit width and for one side of the watercourse.

ho = the maximum elevation of water table

T = the transmissivity coefficient

S = the underground storage coefficient

a = the distance to the topographic divide of the ground watershed

It has been shown that as t becomes large and if $(T/a^2S)\,t>0{,}2$ the first term in the infinite series will dominate and the outflow will approximate that from a single reservoir (Dooge 1973).

Taking this result into consideration and bearing in mind the knowledge of $\bar{z}$; then ε_1 can be calculated from

$$\varepsilon_1 = \frac{\pi^2 T}{4a^2 S\bar{z}} \tag{18}$$

Thus ε_1, which is the coefficient of underground transformation is shown to depend on the transmissivity, the underground storage coefficient, the size of the watershed and the initial saturation condition. Hence $1/\varepsilon_1$ which corresponds to the time constant for this linear groundwater reservoir is related to the characteristics of he watershed by a physics based expression.

Estimation of ε_2 – Among the parameters to be determined ε_2 and k are the most familiar to hydrologists. These two parameters can be easily derived from the kinematic wave theory. Henderson (1969), Eagleson (1970), Woolhiser (1977) Overton et al. (1976) have discussed the governing equation of this approximation. Following Henderson's analysis on the order of magnitude and taking into account the kinematic wave results, the momentum equation for overland flow can be written in the form:

$$q_2 = A^{-k} \, C \, \eta \, I_0 \, \frac{k+1}{3} \, x_2^k \tag{19}$$

where

C = the Chezy resistance coefficient or laminar resistance coefficient
η = the wetted area of the section
I_0 = the bed slope
A = the watershed area.

Eagleson (1970) has shown that for laminar flow $k = 3$ and for turbulent flow $k = 5/3$. However, for a typical vegetated surface the flow regime may vary between laminar and turbulent. Horton found that for natural surface $k = 2$. As noted by Eagleson this result is supported by the results of other investigators when dealing with surfaces varying from clipped grass to tar and gravel.

The above analysis leads to the following expression for ε_2 and k.

$$\varepsilon_2 = \frac{C \eta \, I_0}{A^2} \tag{20}$$

$$k = 2 \tag{21}$$

Hence, ε_2 depends on the surface characteristics of the watershed i.e. the slope, the Chezy's resistance coefficient which describes the effect of roughness and the degree of vegetated surface, the watershed area and the wetted area of the channel cross-section.

The parameter k gives the indication about the degree of flow turbulence.

Thus each parameter of the model has a precise physical meaning, and together they describe the dynamics of the watershed rainfall and runoff transformation.

Mathematical Derivation of the Solution

The state equations of the system is now given by:

$$\frac{dx}{dt} + \lambda_1 x + \lambda_2 x^2 \equiv \rho(t) \tag{22}$$

$$Q(t) = \lambda_1 x + \lambda_2 x^2 \tag{23}$$

where

$$\lambda_1 = \varepsilon_1 \bar{z} \tag{24}$$

$$\lambda_2 = \varepsilon_2 [1-\bar{z}]^2 \tag{25}$$

Eqs. (22) and (23) can be solved using the local inverse theorem as presented by Halme and Orava (1972) and applied to hydrologic systems by Afouda (1974). When appropriate initial conditions are found, the solution of the state equation is given by:

$$x(t) \equiv \sum_{j=1}^{m} \int_0^t \cdots \int_0^t k_j(\tau_1 \ldots \tau_j) \prod_{\ell=1}^{j} \rho(t-\tau_\ell) d\tau_1 \ldots d\tau_j \tag{26}$$

If we call

$$X_1 \equiv \int_0^t k_1(\tau_1)\rho(t-\tau_1)d\tau_1 \tag{27}$$

the linear part of Eq.(26), then

$$k_1 \equiv \exp\{-\lambda_1 \tau_1\} \tag{27a}$$

The k_j Kernel functions for the non-linear components are derived from the following convolution product

$$X_j \equiv -\lambda_2 (k_1 * (\sum_{\nu=1}^{j} x_\nu x_{j-\nu+1})) \tag{28}$$

The discharge is then given by:

$$Q(t) = \sum_{i=1}^{n} \int_0^t \cdots \int_0^t H_i(\tau_1 \ldots \tau_i) \prod_{\ell=1}^{i} \rho(t-\tau_\ell) d\tau_1 \ldots d\tau_i \tag{29}$$

where the H_i $(t;\ \tau_1 \ldots \tau_i)$ Kernel functions are obtained by appropriate combination of Kernel function k_j in accordance with Eq. (23).

Discussion

The following discussion deals with the theoretical aspects of applicability. Whether the developed model has a good performance in practical applications has yet to be proved.

Comparison with earlier models – Most of the earlier physics-based models are devoted either to overland flow studies (kinematic wave theory) or to the transient saturated-unsaturated flow-studies.

Nevertheless Smith and Woolhiser (1971) have presented a physics-based model that combines the differential equation for overland flow with infiltration from rainfall. The infiltration model developed in the form of Richard's equation was combined with the overland flow equation formulated as a cascading kinematic wave equation. The model presented in this paper allows for the use of the complete infiltration equation when $z(\theta)$ is to be found for a selected rainfall event. Moreover, in Eqs. (13) and (14) the combination of overland flow and underground flow appears clearly through parameter $z(\theta)$ as an internal coupling while in the studies of Smith et al. an external coupling was obtained between overland flow and infiltration. It should also be noted that Smith et al. did not take into account the effect of underground flow and the effect of evaporation.

Another physics-based model that combines three dimensional transient saturated-unsaturated subsurface flow and one dimensional gradually varied unsteady channel flow models has been developed by Freeze (1972). But as reported by Natale et al. (1977), the computer implementation of the model uses programs that are very complex and requires a large amount of cumputer time. Moreover the model was concerned only with base flow generation. The model studied in this paper deals with the major components involved in the complex runoff process that occurs in a watershed.

Parameter variations – The parameters involved in the mathematical formulation have been shown to have precise physical meaning. The accuracy in estimating the watershed behaviour are of course related to the approximations made on the parameters, in particular $z(\theta)$.

If $z(\theta) = 1$; $\rho(t)>0$ the watershed response take the form of underground contribution

If $z(\theta) \equiv 1$; $\rho(t)<0$; that is $I(t)=0$ then $\rho(t) \equiv -E(t)$ and $r(t) = -E(t)$. Assuming $Q(t) = 0$ and $E = \eta_a(\theta_0-\theta_e)$ Eq. (30) is obtained:

$$\frac{dx}{dt} = -\eta_a(\theta_0-\theta_e) \qquad (30)$$

where

η_a is an atmosphere transfer coefficient
θ_0 refers to the soil moisture at the soil surface

and θ_e is the surface moisture content which would be in equilibrium with the vapor pressure existing at elevation h above the soil surface.

If $z(\theta) = 0; \rho(t)>0$, the model corresponds to the well known Horton-Izzard model for urban studies.

If $z(\theta) = \text{const.} \neq 0; \rho(t)>0$, for a given interval we necessarily underestimate and overestimate the volume of infiltration for dry and wetted period respectively. This is the reason why time invariant model linear (linear reservoirs IUH), as well as non-linear (non-linear reservoirs, Volterra series) overestimate the low flow and underestimate the peak. The performance of the model could be improved if the values of $\bar{z}$ are considered for each appropriate subinterval. In summary the parameter $z(\theta)$ together with ε_1 and ε_2 relate the internal dynamics of the system to the input-output and contribute to the answer to the question »What is going on inside the black-box«.

Advantages of the Volterra Series Formulation

The solution of the state equations assuming a time invariant system lead to the Volterra' series representation of the watershed processes. This form of representation has earlier been applied to hydrologic systems by Amorocho and Orlob (1961), but their result did not have widespread application because of the difficulties involved in the identification of the non-linear Kernel function. The problem to be solved was that of the non-linear deconvolution which is a complicated inverse problem. The formulation given in this paper is a well defined problem (in a mathematical sense) which lead to separable Kernel functions which are more easy to handle. A major advantage is that the solution can be used in its analytical form (directly or with the multidimensional Laplace transform), provided the input function is known in its analytical form. An example is given for the Horton-Izzard model when $\rho(t) = \rho_0 = \text{const.}$ Then

$$x(t) = \sum_{j=1}^{m} \rho_0^j \int_0^t \cdots \int_0^t k_j(\tau_1 \ldots \tau_j) \prod_{\ell=1}^{j} u(t-\tau_\ell) d\tau_1 \ldots d\tau_j$$

where $u(t)$ is a unit step function. By simple calculation and algebraic transformation

$$x(t) = \frac{\rho_0^{\frac{1}{2}}}{\varepsilon_2^{\frac{1}{2}}} \tanh[\varepsilon_2^{\frac{1}{2}} \rho_0^{\frac{1}{2}} t] \quad \text{and} \quad Q(t) = \rho_0 \tanh[\varepsilon_2^{\frac{1}{2}} \rho_0^{\frac{1}{2}} t]$$

Conclusion

One of the basic ideas of this study is to consider a watershed as an inseparable environmental unit which consists of the input- the storage – the output. Any combination of the action of the major components (overland, underground) must preserve this physical unity. It has been shown that the interaction of these components through a defined parameter $z(\theta)$ together with a lumped formulation of the continuity equation preserves the environmental unity and satisfies the theoretical physic laws of the system. The obtained mathematical formulation of the system introduces the concept of a state approach to watershed modelling. The parameters of this model are shown to have precise physical meaning and they relate the internal behaviour of the system to its input – output response.

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