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Free Vibrations of Antisymmetric Angle-Ply Laminated Thin Square Composite Plates

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Abstract

Numerical results are presented for vibration frequencies of antisymmetric angle-ply laminated thin square composite plates having different boundary conditions. Boundary conditions are chosen as 2 adjacent free edges and the remaining edges are simply supported, clamped, or free. The Ritz method, along with the displacement assumed in the form of simple polynomials, is applied to solve the problems. Convergence studies are presented to demonstrate the accuracy of the results. The effects of various parameters such as fibre orientation, number of layers, and boundary conditions upon the natural frequencies are studied.

Key words: Antisymmetric angle-ply plates, Vibration, Ritz method.

Introduction

Since composite plates are used widely in many structural applications, the vibration behaviour of composite laminated plates has been studied by many researchers. There are many publications about symmetrically laminated plate vibration. Qatu (1991) studied the vibration of symmetrically laminated composite plates with different boundary conditions. Narita and Leissa (1992, 1990) presented solutions for symmetrically laminated thin cantilevered composite plates. Qatu and Leissa (1991) studied the vibration of thin completely free composite laminated plates. Narita (2000) investigated the free vibration of composite laminated thin plates with general edge conditions. All of the authors mentioned above used classical plate theory in their analyses. Chen et al. (1997) and Lee et al. (1991) studied the vibration of symmetrically laminated composite plates by the higher order shear deformation theories. However, in the literature, solutions for the free vibration problem of antisymmetric laminated composite plates are rare (Moita et al., 1999). Soldatos and Messina (2001) and Messina and Soldatos (1999) investigated the vibration of antisymmetrically laminated composite plates by using a unified shear deformation theory for some boundary conditions. Although there are some studies about antisymmetric angle ply plates, they are restricted to a limited number of boundary conditions.

The objective of this paper is to present a simple, approximate solution for the problem, and to study the vibration behaviour of laminated plates with fibrous composite layers.

Analysis

Figure 1 shows a composite plate made of N layers having dimensions a and b and thickness h. The x-y plane is the mid-plane of the plate and the z axis is normal to the plate. The stress-strain relationships can be written in terms of a global coordinate system in the following form, according to Vinson and Sierakoski (1986) and Jones (1975):

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{1}$$

Here $\bar{Q}_{ij}$ are the components of transformed lamina stiffness matrix. Displacement field components with Kirchoff assumptions can be expressed as

$$\begin{aligned} U(x,y,z,t) &= u(x,y,t) - zw,_x\,, \\ V(x,y,z,t) &= v(x,y,t) - zw,_y\,, \\ W(x,y,z,t) &= w(x,y,t). \end{aligned} \tag{2}$$

where u and v are tangential displacement of the middle surface along the x and y directions, respectively, and w is the transverse displacement of an arbitrary point in the plate. U, V, and W are the displacements of a typical point of the plate again along the x, y, and z directions, respectively.

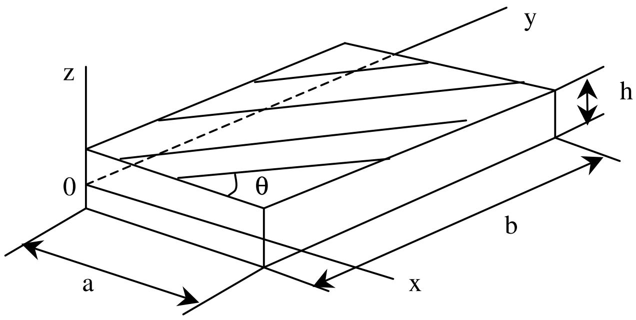

Figure 1. Composite plate geometry and coordinate system.

For small displacements, the strains are defined in terms of displacement components:

$$\varepsilon_x = \frac{\partial U}{\partial x}, \varepsilon_y = \frac{\partial V}{\partial y}, \gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \tag{3}$$

where ε_x and ε_y are the strains along the x and y directions, respectively, and γ_{xy} is the in-plane shear strain. The strain energy of deformation for a thin plate is

$$V_s = \tfrac{1}{2} \iint\limits_A (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}) dxdy \tag{4}$$

This potential energy of the plate can be written in terms of the displacements of the mid-plane surface of the plate using the relations

$$\begin{aligned} V_s = \tfrac{1}{2} \int\limits_A \{ & A_{11}u_{,x}^2 + 2A_{12}v_{,y}u_{,x} + A_{22}v_{,y}^2 \\ & +2A_{16}\,(u_{,x}u_{,y} + u_{,x}v_{,x}) \\ & +2A_{26}(v_{,y}u_{,y} + v_{,y}v_{,x}) + A_{66}(u_{,y} + v_{,x})^2 \\ & -2B_{11}u_{,x}w_{,xx} - 2B_{16}(u_{,y}w_{,xx} + v_{,x}w_{,xx} \\ & +2u_{,x}w_{,xy}) - 2B_{26}(u_{,y}w_{,yy} + v_{,x}w_{,yy} + 2v_{,y}w_{,xy}) \\ & +2B_{12}(v_{,y}w_{,xx} + u_{,x}w_{,yy}) - 2B_{22}v_{,y}w_{,yy} \\ & -4B_{66}(u_{,y}w_{,xy} + v_{,x}w_{,xy}) + D_{11}w_{,xx}^2 + 2D_{12}w_{,yy}w_{,xx} \\ & +D_{22}w_{,yy}^2 + 4D_{16}w_{,xy}w_{,xx} + 4D_{26}w_{,yy}w_{,xy} \\ & +\, 4D_{66}w_{,xy}^2 \}\, dA \end{aligned} \tag{5}$$

where the A_{ij}, B_{ij} and D_{ij} are conventional laminate stiffness coefficients (Jones, 1975).

The kinetic energy of the composite plate is

$$T = \frac{1}{2}\rho \iint\limits_A (u_{,t}^2 + v_{,t}^2 + w_{,t}^2) dxdy \tag{6}$$

where ρ is the mass per unit area of plate.

For the small-amplitude (linear) free vibrations of the plate, displacements are assumed as

$$\begin{aligned} u(x,y,t) &= \alpha(x,y)e^{i\omega t} \\ v(x,y,t) &= \beta(x,y)e^{i\omega t} \\ w(x,y,t) &= \gamma(x,y)e^{i\omega t}. \end{aligned} \tag{7}$$

Position dependent displacement functions α, β, and γ can be written as double series forms of simple polynomials in terms of the non-dimensional coordinates ξ and η as

$$\begin{aligned} \alpha(\xi,\eta) &= \sum_{i=i_0}^{I} \sum_{j=j_0}^{J} P_{ij}\xi^i\eta^j \\ \beta(\xi,\eta) &= \sum_{k=k_0}^{K} \sum_{l=l_0}^{L} Q_{kl}\xi^k\eta^l \\ \gamma(\xi,\eta) &= \sum_{m=m_0}^{M} \sum_{n=n_0}^{N} R_{mn}\xi^m\eta^n \end{aligned} \tag{8}$$

where ξ = x/a − ξ_o and η = y/b − η_o. Now consider clamped boundary conditions. Equation (8) exactly satisfies the conditions given in Table 1, because of the terms of i = 0, k = 0, m = 0, and m = 1 are neglected for zero displacement and zero slope. Details of these coordinate transformations can be found in the work by Qatu (1991). In the present study, boundary conditions of the plate were chosen in 6 different forms. Two adjacent edges of plates are free (F) and other 2 edges are free, simply supported (S), or clamped (C). In the Ritz method, the geometric boundary conditions (i.e. zero displacement and/or normal slope) must be at least satisfied. Geometric boundary conditions are given in Table 1. The boundary conditions used in this study correspond to classical boundary conditions given by Baharlou and Leissa (1987) (i.e. simply supported, free, and clamped correspond to S_2, F_4, and C_1, respectively). In order to satisfy the boundary conditions, the starting terms of the series given in Eq. (8) can be chosen as given in Table 2.

Table 1. Geometric boundary conditions.

Boundary condition type	at ξ = constant	at η = constant
Free (F)	u $\neq$ 0, v $\neq$ 0, w $\neq$ 0 (no constraints)	u $\neq$ 0, v $\neq$ 0, w $\neq$ 0 (no constraints)
Simply supported	v = w = 0	u = w = 0
Clamped	u = v = $w_{,\xi}$ = 0	u = v = $w_{,\xi}$ = 0

Table 2. Starting indices of the series.

	B.C. at x = constant			B.C. at y = constant		
	i_0	k_0	m_0	j_0	l_0	n_0
Free	0	0	0	0	0	0
Simply	0	1	1	1	0	1
Clamped	1	1	2	1	1	2

The Ritz method requires the minimisation of the functional (T_{max}-V_{max}) with respect to the coefficients P_{ij}, Q_{kl}, and R_{mn}

$$\begin{aligned} \frac{\partial(T_{\max}-V_{\max})}{P_{ij}} &= 0 i = i_0, ..., I; j = j_0, ..., J \\ \frac{\partial(T_{\max}-V_{\max})}{Q_{kl}} &= 0 k = k_0, ..., K; l = l_0, ..., L \\ \frac{\partial(T_{\max}-V_{\max})}{R_{mn}} &= 0 m = m_0, ..., M; n = n_0, ..., N \end{aligned} \tag{9}$$

If the upper limits of the series are taken equal to each other (I = J = K = L = M = N), Eq. (9) yields a total of 3 × M × N simultaneous, linear, homogeneous equations in unknowns P_{ij}, Q_{kl}, and R_{mn}. Those equations can be described in matrix form as

$$\left([K] - \Omega^2 [M]\right)\{\delta\} = 0 \tag{10}$$

where [K] is the stiffness matrix, [M] is the mass matrix and $\{\delta\}$ is the column vector of unknown coefficients P_{ij}, Q_{kl}, and R_{mn}. For a non-trivial solution, the eigenvalues (Ω) that make the determinant equal to zero correspond to the vibration frequency parameters.

Numerical Results

Since an increasing orthotropy degree results in poor convergence in the Ritz method, the material properties are chosen as E_1/E_2= 40, G_{12}/E_2 = 0.5, and ν_{12} = 0.25. This material has the highest orthotropy degree used in the literature. The nondimensional frequency parameter is taken as $\Omega = \omega a^2(\rho h/D_0)^{1/2}$, where $D_0 = E_2h^3$.

A sample convergence study is presented in Table 3 for 2-layer antisymmetric (30°/–30°) composite plates. The maximum difference between a 6- and 7-term solution is 0.65% for FFFF plates. Based on this study, the following calculations are carried out using 6 × 6 (108 terms).

In Table 4, the numerical results obtained for 2 and 10 layers with different fibre orientation angles are compared to values in the literature in order to establish the validity of the present approach. All of the obtained results are the same as those reported by Soldatos and Messina (2001) for completely free plates. Unfortunately, no other results in the literature for comparison are known by the authors.

Table 3. Convergence of frequency parameters (Ω) for angle-ply square plates (30°/–30°).

M	SSFF	CSFF	FFFF	CFFF	CCFF	SFFF
4	3.975	5.503	7.745	3.307	6.155	7.698
5	3.972	5.499	6.943	3.182	6.134	7.662
6	3.970	5.494	6.943	3.079	6.130	7.586
7	3.970	5.493	6.898	3.073	6.129	7.583

Table 4. Comparison of frequency parameters of completely free plates with previous results.

	$(\theta/{-\theta})$		$(\theta/{-\theta})_5$	
	Present	Soldatos and Messina (2001)	Present	Soldatos and Messina (2001)
0°	0.7453	0.7453	0.7453	0.7453
15°	1.028	1.028	1.036	1.036
30°	1.090	1.090	1.144	1.144
45°	1.153	1.153	1.195	1.195

Table 5 gives the first (fundamental) nondimensional frequencies (Ω) for 4- and 10-layer square (a/b = 1) composite plates, as the fibre orientation angle varies between 0° and 90°. As mentioned by Qatu (1991) for FFFF, SSFF, and CCFF square plates, symmetry about the line $\xi = \eta$ exists. For this reason, the frequencies obtained for plates with fibre orientation angles $\theta = 60°$, 75°, and 90° are the same as those with angles of $\theta = 30°$, 15°, and 0°, respectively. In each column, the maximum value is indicated by an asterisk. It is observed that for all numbers of layers studied for CFFF the maximum fundamental frequency occurs for $\theta = 0°$ plates and for FFFF, SFFF, and SSFF plates for $\theta = 45°$. For 4 and 10 layers, maximum frequencies were obtained for $\theta = 45°$ for CCFF plates and $\theta = 30°$ for CSFF plates. The frequency parameter is insensitive to the number of layers for $\theta = 0°$ and $\theta = 90°$ for all boundary conditions considered.

Table 6 gives 4 nondimensional frequencies (Ω) for 2-layer antisymmetric square plates. In each column, the maximum value of the parameters corresponding to each mode for different fibre orientations is indicated by ∗an asterisk. Increasing the fibre angle θ from 0° to 45° increases the lowest 2 nondimensional frequencies for FFFF and SSFF, and decreases fundamental frequency for CSFF and the first 2 frequencies for CFFF plates. The fourth frequencies are maximum for $\theta = 45°$ for all boundary conditions except CFFF plates, where maximum frequency occurs at $\theta = 15°$. Increasing the number of layers increases the frequency parameter and this increase is more pronounced for small numbers of layers.

The effects of material anisotropy on the frequencies of antisymmetric angle-ply square plates with a fibre orientation angle of 45° for CFFF, FFFF, and CSFF edge boundary conditions are demonstrated in Figures 2-4. These results are obtained by keeping the material properties as $G_{12}/E_2 = 0.5$ and $\nu_{12} = 0.25$ constant and changing the E_1/E_2 ratio. As seen from these figures, the frequency parameter increased for all of these boundary conditions as the orthotropy degree increased. These increases are

Table 5. Effects of fibre orientation angles upon frequency parameters (Ω) for the plates with different boundary conditions.

B.C.	θ	4 layers	10 layers
SSFF	0°	1.178	1.178
	15°	2.692	2.730
	30°	4.331	4.415
	45°	4.950∗	5.057∗
CCFF	0°	6.730	6.730
	15°	6.769	7.053
	30°	7.460	7.765
	45°	7.914∗	8.226∗
FFFF	0°	4.714	4.714
	15°	6.542	6.553
	30°	7.186	7.237
	45°	7.531∗	7.559∗
CSFF	0°	6.583	6.583
	15°	6.418	6.751
	30°	6.745∗	7.030∗
	45°	6.507	6.733
	60°	5.224	5.369
	75°	3.257	3.309
	90°	1.750	1.750
SFFF	0°	2.398	2.398
	15°	5.450	5.548
	30°	8.588	8.839
	45°	9.749∗	10.049∗
	60°	7.520	7.708
	75°	4.669	4.678
	90°	2.286	2.286
CFFF	0°	6.424∗	6.424∗
	15°	5.499	5.891
	30°	4.254	4.590
	45°	2.765	2.967
	60°	1.566	1.637
	75°	1.064	1.070
	90°	1.015	1.015

sharper for CFFF and CSFF cases compared to the FFFF case, in which a mild increase in the frequency parameter is observed. Although not shown here, the behaviour of SFFF, CCFF, and SSFF plates is nearly the same as that with a CSFF plate. It is also seen from these figures that when the orthotropy ratio is increased the difference between 6-layer and 2-layer results becomes more pronounced for the CFFF plate.

Table 6. Frequency parameters (Ω) for 2-layer square plates.

B.C.	θ	Ω			
CCFF	0°	6.730*	10.227	20.305	38.076
	15°	5.597	12.557	23.977*	26.071
	30°	6.130	15.547	22.383	32.224
	45°	6.526	19.157*	20.959	41.078*
SSFF	0°	1.178	5.735	15.725	28.382
	15°	2.529	8.988	18.657*	20.194
	30°	3.962	13.337	17.196	27.766
	45°	4.483*	15.575*	16.703	35.089*
FFFF	0°	4.714	6.511	11.810	18.341
	15°	6.549	7.007	16.081	18.509
	30°	6.942	10.193	17.530*	21.156
	45°	7.321*	11.545*	14.960	25.510*
CSFF	0°	6.580*	8.966	17.438	33.853
	15°	5.049	11.027	21.248	25.539
	30°	5.486	14.594	21.762*	28.933
	45°	5.495	16.772*	19.391	36.770*
	60°	4.615	15.179	17.626	31.117
	75°	3.049	10.526	18.376	22.614
	90°	1.732	7.483	18.836	28.449
SFFF	0°	2.398	8.377	20.099	28.165
	15°	5.091	12.646	24.006	27.700
	30°	7.586	14.067*	28.031*	31.267
	45°	8.446*	13.371	25.672	33.858*
	60°	5.375	8.126	25.110	28.129
	75°	4.581	5.277	14.449	16.753
	90°	2.286	4.450	8.404	14.579
CFFF	0°	6.424*	7.070*	10.939	21.389
	15°	3.766	6.312	11.885	23.390*
	30°	2.717	6.661	13.811*	17.614
	45°	1.863	6.743	10.886	17.889
	60°	1.263	5.742	7.636	18.489
	75°	1.036	3.984	6.476	13.207
	90°	1.017	2.829	6.362	10.026

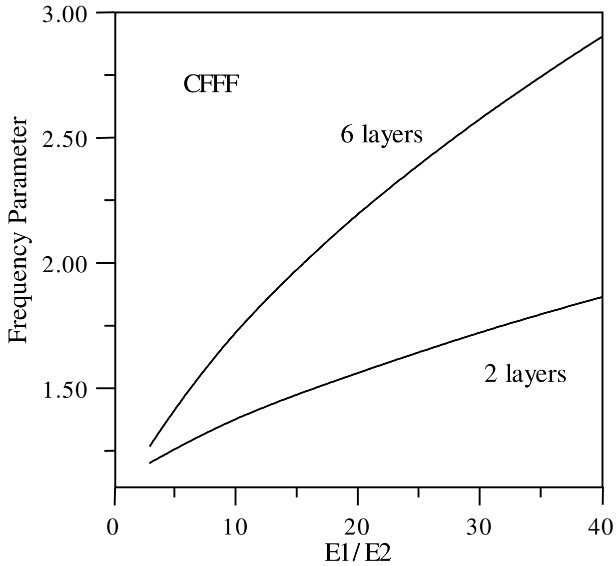

Figure 2. The effect of material anisotropy on the frequency parameters of [45°/–45°/...] antisymmetric angle-ply plates with CFFF boundary condition.

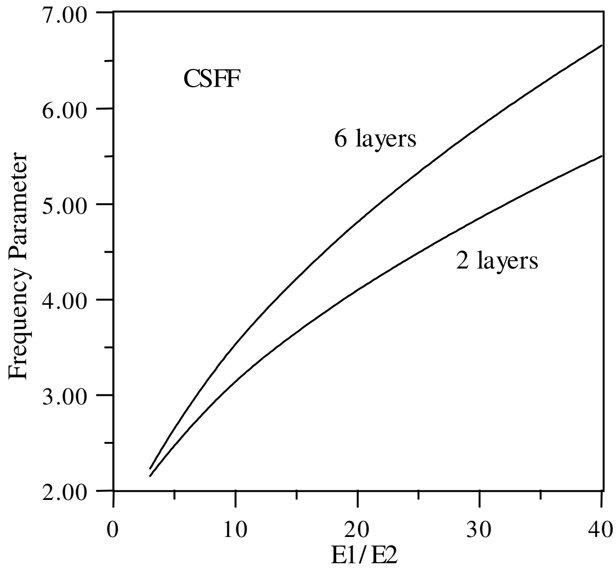

Figure 3. The effect of material anisotropy on the frequency parameters of [45°/–45°/...] antisymmetric angle-ply plates with CSFF boundary condition.

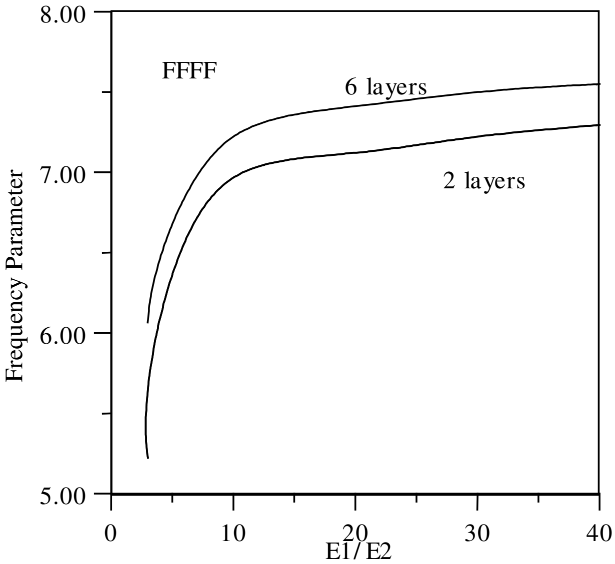

Figure 4. The effect of material anisotropy on the frequency parameters of [45°/–45°/...] antisymmetric angle-ply plates with FFFF boundary condition.

Conclusions

This study dealt with the free vibration analysis of antisymmetric angle-ply laminated thin square composite plates subjected to 6 different types of boundary conditions on the basis of a classical plate theory. The Ritz method was employed to find fundamental frequencies of 2-, 4-, and 10-layered antisymmetric composite plates and the first 4 frequencies of 2-layered antisymmetric thin composite plates.

In applying the Ritz method, the displacement components were assumed as the double series expansions of simple algebraic polynomials. It appears that the Ritz method employing simple algebraic polynomials can yield quite reliable results even for a few initial terms in the series as far as the thin anti-symmetric angle-ply laminates are concerned.

Finally, the effects of orthotropy degree on the frequencies of angle-ply plates of 3 different combinations of edge boundary conditions (FFFF, CSFF, and CFFF) were demonstrated in graphical forms. The study can also be used for the vibration problem of angle-ply composite plates by using higher order shear deformation theory.

Nomenclature

a,b	plate dimensions in x,y directions
E_1, E_2	elastic moduli for a composite layer
G_{12}	shear modulus for a composite layer
ν_{12}	Poisson's ratio
h	plate thickness
$\sigma_x, \sigma_y, \tau_{xy}$	stress components in cartesian coordinates
U,V,W	displacements in x-, y-, and z-directions, respectively
u, v, w	displacement components in the midplane
Q_{ij} (i, j = 1, 2, 6)	reduced stiffnesses
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	strain components
x,y,z	cartesian coordinates
t	time
A_{ij}, B_{ij}, $D_{ij}(i, j = 1, 2, 6)$	stiffness
[K]	stiffness matrix
[M]	mass matrix
$\{\delta\}$	column vector of undetermined coefficients
Ω	nondimensional frequency parameter
V_s	strain energy
T	kinetic energy

References

Baharlou, B. and Leissa, A.W., "Vibration and Buckling of Generally Laminated Composite Plates with Arbitrary Edge Boundary Conditions". International Journal of Mechanical Sciences, 29, 545-555, 1987.

Chen, C.C., Liew, K.M., Lim, C.W. and Kitipornchai, S., "Vibration Analysis of Symmetrically Laminated Thick Rectangular Plates Using the Higher-Order Theory and p-Ritz Method", J. Acoust. Soc. Am., 102, 1600-1611, 1997.

Lee, J.M., Chung, J.H. and Chung, T.Y., "Free Vibration Analysis of Symmetrically Laminated Composite Rectangular Plates", Journal of Sound and Vibration, 199, 71-85, 1991.

Jones, R.M., Mechanics of Composite Materials, Scripta, Washington D.C. 1975.

Messina, A. and Soldatos, K.P., "Vibration of Completely Free Composite Plates and Cylindrical Shell Panels by a Higher Order Theory", International Journal of Mechanical Sciences, 41, 891-918, 1999.

Moita, J.S., Soares, C.M.M. and Soares, C.A.M., "Buckling and Dynamic Behaviour of Laminated Composite Structures Using Discrete Higher-Order Displacement Model", Computers and Structures, 73, 407-423, 1999.

Narita, Y. and Leissa, A.W., "Frequencies and Mode Shapes of Cantilevered Laminated Composite Plates", Journal of Sound and Vibration, 154, 161-172, 1992.

Narita, Y. and Leissa, A.W., "Free Vibration Analysis of Cantilevered Composite Rectangular Plates". Japan Society for Composite Materials, Proceedings of the 5^{th} Japan-U.S. Conference on Composite Materials, Tama-City, Tokyo, p. 679-686, 1990.

Narita, Y., "Combinations for the Free-Vibration Behaviours of Anisotropic Rectangular Plates under General Edge Conditions", Journal of Applied Mechanics Transactions of the ASME 67, 568-573, 2000.

Qatu, M.S., "Free Vibration of Laminated Composite Rectangular Plates", Int. J. Solids Structures, 28, 941-954, 1991.

Qatu, M.S. and Leissa, A.W., "Free Vibrations of Completely Free Doubly Curved Laminated Composite Shallow Shells", Journal of Sound and Vibration, 151, 9-29, 1991.

Soldatos, K.P. and Messina, A., "The Influence of Boundary Conditions and Transverse Shear on the Vibration of Angle-Ply Laminated Plates, Circular Cylinders and Cylindrical Panels", Comput. Methods Appl. Mech. Engrg, 190, 2385-2409, 2001.

Vinson, J.R. and Sierakowski, R.L., The Behaviour of Structures Composed of Composite Materials. Dordrecht: Martinus Nijhoff, 1986.