

Gradient evolution for potential vorticity flows

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Received: 20 June 2000 – Revised: 8 January 2001 – Accepted: 10 January 2001

Abstract. Two-dimensional unsteady incompressible flows in which the potential vorticity (PV) plays a key role are examined in this study, through the development of the evolution equation for the PV gradient. For the case where the PV is conserved, precise statements concerning topology-conservation are presented. While establishing some intuitively well-known results (the numbers of eddies and saddles is conserved), other less obvious consequences (PV patches cannot be generated, some types of Lagrangian and Eulerian entities are equivalent) are obtained. This approach enables an improvement on an integrability result for PV conserving flows (if there were no PV patches at time zero, the flow would be integrable). The evolution of the PV gradient is also determined for the nonconservative case, and a plausible experiment for estimating eddy diffusivity is suggested. The theory is applied to an analytical diffusive Rossby wave example.

1 Introduction

Incompressible unsteady two-dimensional flow is often used to model mesoscale oceanic dynamics (Pedlosky, 1987; Flierl et al., 1987; Pratt et al., 1995; Pierrehumbert, 1991; del Castillo-Negrete and Morrison, 1993; Haller and Poje, 1997; Miller et al., 1996; Rogerson et al., 1999; Miller et al., 1997; Weiss and Knobloch, 1989; Jayne and Hogg, 1999; Brown and Samelson, 1994; Balasuriya et al., 1998). A seemingly key physical consideration in many such models is the conservation, or near conservation, of a scalar quantity called the potential vorticity (PV) following the flow (Pedlosky, 1987). In terms of Ertel's general result (1942), the conservation of the potential vorticity q is given by

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0,$$

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where \mathbf{v} is the unsteady velocity field. If x and y are the local eastward and northwards coordinates, respectively, imposing incompressibility and ignoring stratification leads to the presence of a streamfunction $\psi(x, y, t)$ which relates to the velocity field through $\mathbf{v} = (-\psi_y, \psi_x)$. Under these conditions, the conservation of PV can be written in the form

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad (1)$$

where $q = q(x, y, t)$, and the Jacobian is defined through $J(f, g) = f_x g_y - f_y g_x$. In approximate or balanced models, q and ψ are linked through $q = \mathcal{L}\psi$ for some appropriate operator \mathcal{L} , for example

$$q = \frac{1}{H} \nabla \cdot \left(\frac{1}{H} \nabla \psi \right), \quad q = \nabla^2 \psi - F\psi + \beta y, \quad \text{etc.}$$

where $H(x, y)$ is the depth of the fluid, β the Coriolis parameter, and F measures the size of the horizontal length scale in comparison with the Rossby deformation radius (see Pedlosky, 1987; Hoskins et al., 1985, for more details). Equation (1) would, therefore, be nonlinear, rendering its solution difficult. Somewhat more realistic in oceanographic applications is the case where the PV conservation is broken through

$$\frac{\partial q}{\partial t} + J(\psi, q) = g, \quad (2)$$

for some (small) function g which may model eddy diffusivity, wind-forcing, etc. This paper addresses both cases (1) and (2), and develops in Sect. 2 the evolution equation for the PV gradient following the flow.

The PV gradient evolution equation has many properties from which nice theoretical results can be derived. For the particular case of the PV-conserving flow, precise statements concerning topology-conservation can be derived. These statements strengthen the intuitively well-accepted ideas in the oceanographic community, while also providing some less obvious consequences. Some of the facts shown (and fairly carefully stated) in Sect. 3 are that (i) (Eulerian definitions of) eddies and saddles travel exactly with the flow; i.e.

the Eulerian and Lagrangian objects are equivalent, (ii) the number of (Eulerian) eddies and saddles is each conserved by the flow, and (iii) no PV patches can be generated by the flow.

Section 4 focuses on the integrability and existence of PV-conserving flows. The development of Sect. 3 permits a strengthening of an extant result (Brown and Samelson, 1994) on the integrability of such flows. It is shown that if there were no PV patches to begin with, and if the PV remains a conserved quantity for all time, then the Lagrangian particle trajectories are integrable. Since integrability has been a much debated issue among the oceanographic community, a discussion is provided on its consequences, with comparisons to some available numerical results.

The more general, nonconservative, flow (2) is examined in Sect. 5. Not surprisingly, only limited qualitative results are obtainable from the PV gradient evolution equation. The effects of wind-forcing, bottom-friction and eddy diffusivity are each considered for g in (2). Strong results (akin to those of Sect. 3) are shown to exist in some specialised instances. A simple experiment which can be used to approximate the size of the conservation-breaking function g is presented.

Both the strength and the weakness of the ideas in this paper is the dearth of known analytical solutions to (1) and (2). While being unable to give many rigid examples, it is still instructive that nice qualitative statements concerning solutions are possible. One example, a Rossby wave, which satisfies (1) exactly, is presented in Sect. 6, and is seen to satisfy the topological constraints somewhat trivially. However, this example is used to construct an explicit solution to (2), in which $g = \mathcal{D}\nabla^2 q$ (and models eddy diffusivity). The experiment suggested in Sect. 5 is quantified for this diffusive Rossby wave example, suggesting a quick method of estimating the horizontal eddy diffusivity in the ocean.

2 Evolution of PV gradient

The flow shall be assumed two-dimensional, unsteady and incompressible, in which case, the (Lagrangian) fluid parcel trajectories are given by the solutions to

$$\begin{aligned}\dot{x} &= -\frac{\partial\psi}{\partial y}(x, y, t), \\ \dot{y} &= \frac{\partial\psi}{\partial x}(x, y, t).\end{aligned}\quad (3)$$

In Sects. 2, 3 and 4, the flow (3) is assumed to conserve $q(x, y, t)$, as expressed mathematically through (1). Though referred to as the potential vorticity, q , in this study may, in fact, be any scalar field (active or passive) advected according to (1).

Define the PV gradient function $\lambda(x, y, t)$ by

$$\lambda(x, y, t) := \nabla q(x, y, t) := \left(\frac{\partial q}{\partial x}(x, y, t), \frac{\partial q}{\partial y}(x, y, t) \right),$$

where the ∇ operator refers only to the gradient in the (x, y) variables. Regions in which the PV gradient has large magnitude are associated with regions in which cross-gradient

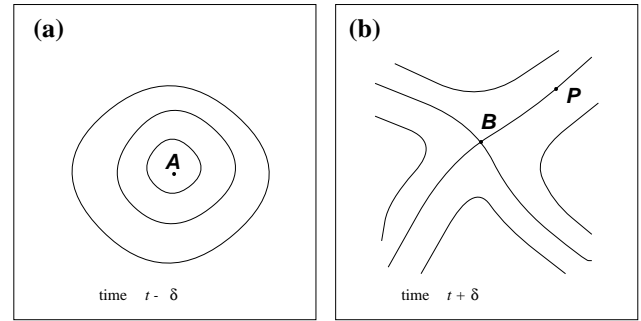


Fig. 1. Qualitative picture of PV contours and associated critical points: (a) an eddy, and (b) a saddle.

transport is suppressed. Additionally, key Eulerian entities can be defined with reference to λ , as is described below.

Consider the contours of the PV field drawn at any fixed time t . This is an Eulerian picture, since a fixed time is considered, and no immediate relationship to Lagrangian trajectories is indicated. However, should closed contours exist around a point, one would expect the flow to rotate about that point (since the flow satisfies the constraint of q being preserved), and thus, be associated with an (Eulerian) eddy. The existence of such closed contours implies the presence of a local maximum or minimum at their centre, as pictured as point A in Fig. 1a. Such Eulerian snapshots are often used to identify eddies experimentally (see, for example, Richardson, 1983, in which sea surface height / temperature data from remote sensing is illustrated), or numerically (such as the pictures in Dewar and Gaillard, 1994; Rogerson et al., 1999; Miller et al., 1996; Miller et al., 1997; Poje and Haller, 1999; Constantin et al., 1994; Bush et al., 1996; van Heijst and Clercx, 1998; Flierl et al., 1987). Notice that the centre-point of such an eddy (which, with an abuse of language, shall also be referred to as an eddy) is a local extremum of $q(x, y, t)$, at which $\lambda = \mathbf{0}$.

Also important in transport analyses are saddle points of the Eulerian $q(x, y, t)$ field, which have the qualitative structure of point B in Fig. 1b. Such points appear on the boundaries of cats-eyes or eddies, and have a pivotal role in the analysis of transport across such separatrices (see Bower, 1991; Pierrehumbert, 1991; del Castillo-Negrete and Morrison, 1993; Weiss and Knobloch, 1989; Rogerson et al., 1999; Miller et al., 1996; Miller et al., 1997; Balasuriya et al., 1998). From a dynamical systems viewpoint, saddle points are the endpoints of homo/hetero-clinic trajectories, whose destruction leads to chaotic transport. The behaviour of a saddle point also governs eddy detachment events from oceanic jets; the saddle point defining the “endpoint” of the eddy splits off from the jet boundary (see Fig. 4 in Poje and Haller, 1999). (Notice that $\lambda = \mathbf{0}$ at saddle points as well).

For each fixed time t , isolated points at which $\lambda = \mathbf{0}$ shall be defined as critical points. Eddies and saddles are both included in this definition (analogous definitions of these entities are also considered in the kinematical analyses of Haller

and Poje, 1997, 1998, though their scalar field of interest in the streamfunction, rather than the PV). Connected regions in which $\lambda = \mathbf{0}$ have piecewise constant PV, and shall be defined as PV patches. Saddles, eddies and PV patches are all Eulerian objects, defined through zeros of λ in time snapshots. The Lagrangian behaviour of these Eulerian entities can be assessed through the development of the evolution of λ .

Let $(x(t), y(t))$ be a trajectory of the flow, i.e. the functions $x(t)$ and $y(t)$ are solutions to the differential equations (3), and describe how the position of a fluid parcel (or float) evolves with time. Now, since (1) is the statement of conservation of q along a trajectory, it means that

$$\frac{\partial q}{\partial t}(x(t), y(t), t) + J(\psi(x(t), y(t), t), q(x(t), y(t), t)) = 0.$$

The idea now is to determine an evolution equation for $\lambda(x(t), y(t), t)$; to describe how the PV gradient vector evolves along a fluid trajectory. Taking the x -derivative (partial) of (1),

$$\frac{\partial q_x}{\partial t} + J(\psi_x, q) + J(\psi, q_x) = 0.$$

Since the time-derivative operator on a function f following the flow is given by

$$\frac{D}{Dt} f = \frac{\partial}{\partial t} f + J(\psi, f),$$

this implies that

$$\frac{D}{Dt} q_x = -J(\psi_x, q) = -\psi_{xx}q_y + \psi_{xy}q_x.$$

Similarly taking the y -derivative of (1) gives

$$\frac{D}{Dt} q_y = -J(\psi_y, q) = -\psi_{xy}q_y + \psi_{yy}q_x.$$

These can be combined to form

$$\frac{D}{Dt} \begin{pmatrix} q_x \\ q_y \end{pmatrix} = - \begin{pmatrix} -\psi_{xy} & \psi_{xx} \\ -\psi_{yy} & \psi_{yx} \end{pmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix}. \quad (4)$$

Now note from (3) that the fluid velocity \mathbf{v} is given by

$$\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\psi_y \\ \psi_x \end{pmatrix},$$

and hence, its (matrix) gradient (the stress deformation matrix \mathbf{S}) is

$$\mathbf{S} := \nabla \mathbf{v} = \begin{pmatrix} -\psi_{yx} & -\psi_{yy} \\ \psi_{xx} & \psi_{xy} \end{pmatrix}. \quad (5)$$

Therefore, if $*$ denotes the transpose of a matrix,

$$\mathbf{S}^* := (\nabla \mathbf{v})^* = \begin{pmatrix} -\psi_{yx} & \psi_{xx} \\ -\psi_{yy} & \psi_{xy} \end{pmatrix}. \quad (6)$$

With these definitions, (4) can be represented as follows.

Statement 1 (PV gradient evolution): *As long as the PV is conserved through (1), its gradient $\lambda = \nabla q$ satisfies*

$$\dot{\lambda} = -\mathbf{S}^* \lambda. \quad (7)$$

It must be emphasised that the time derivative denoted by the dot is a time derivative following the flow of (3); when substituting (x, y, t) into the arguments of the above, one is restricted to $(x(t), y(t), t)$, where $(x(t), y(t))$ is a fluid parcel trajectory. (As an aside, it must be stated that (7) may also be instantly derived by invoking the adjoint equation of variations from dynamical systems theory (see, for example, Fiedler and Scheurle, 1996). However, the above development is more transparent.)

3 Topological constraints

Flows in which PV is conserved are well-known in the oceanographic community to “preserve the topology of the PV field.” There are, however, few instances in the literature in which the specifics of this are described, let alone justified. It is possible to utilise the PV gradient evolution equation derived in the previous section to make some precise statements concerning this preservation of topology. Though some of the results of this section are not surprising, it is felt that stating them carefully would avoid misconceptions. The power of using the current approach is that some less obvious consequences can also be stated.

Notice that an absence of PV patches at some time t_0 can be expressed by the statement: $\lambda(x, y, t_0)$ is zero at most, at finitely many isolated points. With this definition in mind, the following can be shown:

Statement 2 (PV patch prohibition): *Suppose there were no PV patches at time zero. Then, as long as q remains a genuine conserved quantity for the flow (3), no PV patches can appear.*

As long as conservation persists, no PV patches can be generated. This automatically enforces regular motion, since piecewise constant areas of PV, in which particles may roam freely while still satisfying PV conservation, are not formed (piecewise constant PV by itself does not mean nonregular motion; see the PV conserving cusped jet model in Pratt et al., 1995, for example). Statement 2 is proven by showing that for any t , $\lambda(x, y, t) = \mathbf{0}$ at most at isolated points. This is facilitated by the observation from (7) that, if $\lambda = \mathbf{0}$ at some point on a trajectory, λ must be zero at every point on the trajectory. A straightforward intuitive interpretation of this observation yields the desired result; the technicalities associated with the proof are given in Appendix A.

Statement 2 does not preclude the possibility of the PV field gradually flattening over time, such that PV patches are approached as time goes to infinity. On the other hand, the generation of PV patches in finite time is an indication that PV conservation is being violated in some way.

Critical points (where $\lambda(x, y, t) = \mathbf{0}$) identify (Eulerian definitions of) eddies and saddles, whose Lagrangian behaviour is of interest. Do critical points move with the flow, i.e. if a

dyed fluid parcel is at a saddle/eddy at a given instance in time, will it remain at a saddle/eddy? Can an eddy flip into a saddle, or vice versa? Can new saddles/eddies be created? Can they be destroyed? Is the number of eddies in a flow constant? In answering these questions, the PV gradient evolution equation proves invaluable. The structure of (7) shows that, if $\lambda = 0$ at some point on a trajectory, then λ must be zero at every point on a trajectory, since λ is a fixed-point of the evolution equation (7). As long as PV is conserved, this shows that critical points would travel with the flow. In other words, a float placed at a critical point will always remain at a critical point. It is also not possible for a critical point to suddenly appear in a PV conserving flow. If it did, a zero of λ would have emerged from nowhere (which is impossible by (7), since λ must be zero in backwards time along that trajectory). Similarly, critical points cannot disappear, leading to the following result.

Statement 3 (Critical points are Lagrangian and immortal): *As long as PV conservation is satisfied, critical points (i) travel with the flow (3), and (ii) cannot be born or destroyed.*

It is intuitively pleasing that the Eulerian description of critical points (which were defined in terms of a fixed-time scalar field) maintains a strong connection with a Lagrangian description; the Eulerian entity is identified precisely with a Lagrangian particle. A float positioned at a critical point at time zero, would remain exactly at a critical point forever! This is a special feature of PV conserving flows; there is no necessity for such correspondence if PV is not conserved (except in certain special cases, which shall be described in Sect. 5).

Notice that Statement 3 does not, by itself, preclude the possibility of an eddy becoming a saddle, or vice versa, while preserving PV conservation. Critical points remain critical points, but there is no guarantee that eddies remain eddies. In a recent numerical experiment by Constantin et al. (1999), using a surface-geostrophic relationship between q and ψ , (1) was numerically solved to simulate the behaviour near a saddle point. In their Fig. 4, Constantin et al. (1999) noticed that the saddle angle gradually closes with time, thereby getting closer to a front. Nevertheless, the front was never actually achieved; the saddle existed for all finite times. The indications are then that a saddle maintains its structure within a PV conserving flow.

It is indeed possible to prove using (7) that a saddle remains a saddle (and cannot flip into an eddy), under PV conservation. Similarly eddies are prohibited from flipping into saddles. The proof of this result (stated below) is relegated to Appendix A.

Statement 4 (Eddy/Saddle flip): *In the presence of a PV conserving flow, an eddy cannot flip into a saddle (or vice versa) at any instance in time.*

Statement 5 (Eddy/Saddle conservation): *Suppose that the only critical points in the flow (3) are eddies and saddles. If there were e eddies and s saddles initially, there will continue to be e eddies and s saddles as long as PV conservation is satisfied.*

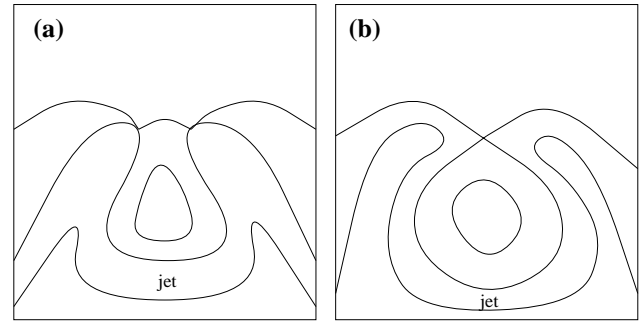


Fig. 2. Merging of two saddles: a cats-eye becomes an eddy ready to detach.

Statement 5 is an immediate consequence of Statement 4, and states that the number of eddies (resp. saddles) in the flow would be conserved as long as PV is conserved. Pathological types of critical points are debarred from the flow in making this statement; only eddies and saddles are permitted (this is an effective constraint on the smoothness of q).

Statements 2, 3, 4 and 5 together give specific instructions on how the potential vorticity field $q(x, y, t)$ must maintain its topological structure. Topological changes would result, for example, in the merging of two saddles, or in the detachment of an eddy from the main jet. Two saddles coalescing can be used to model the creation of an eddy (in readiness for detachment) from a cats-eye structure (see Fig. 2). In this figure, the temporal evolution of PV contours is presented. In Fig. 2a, a cats-eye is shown, and the main jet flows towards its south. The two saddle points, which define the cats-eye, have approached each other, and are preparing to coalesce. By Fig. 2b, these saddles have merged to form just one saddle; the eddy is in a preparatory stage for detachment from the main jet. Thus, a critical point has disappeared, thereby contradicting Statement 3. This scenario is often used as a “thought-experiment” on how mesoscale eddies (rings) may detach from the Gulf Stream. Given the topological change, the time evolution presented in Fig. 2 is an indication of non-conservation of PV.

Formation of an eddy in the form of Fig. 2 can be thought of as a precursor to an eddy detachment event. During the detachment process, some numerical studies show that eddies may maintain a long, thin, attachment to the main stream for some time (see Fig. 4 ($t = 40$) of Rogerson et al., 1999; Fig. 4 ($t = 33.2, 39.8, 49.8$) of Flierl et al., 1987). This can be interpreted as a reluctance to detach, since that would topologically change the PV field further (the saddle at the connection point would disappear). In other words, the presence of the thin attachment would support the fact that the flow is attempting to maintain conservation of PV. In the numerical studies mentioned, the eddies do eventually detach, despite apparently attempting not to do so. Small diffusivity is present in these models; the conjecture from the present results is that nonzero diffusivity can cause eventual topological change, but, if sufficiently small, will display resistance to it. A further discussion on diffusive issues appears

in Sect. 5.3, with a specific example also examined in Sect. 6.

4 Integrability and existence of solutions

The paper by Brown and Samelson (1994), which states that particle motion in a PV-conserving incompressible two-dimensional flow is integrable, has led to much discussion among the oceanographic community. In retrospect, this result is not surprising, since it effectively claims that a flow in 2-D, possessing an integral of motion, is integrable. The integral of motion for the trajectory equation (3), in this case, is the PV field $q(x, y, t)$. If, by adopting the dynamical systems viewpoint, one imagines the Lagrangian motion in the three-dimensional phase space (x, y, t) , the presence of $q(x, y, t)$ as a constant of motion implies that the flow of (3) is confined to surfaces $q(x, y, t) = \text{constant}$. If $q(x, y, t)$ possesses the necessary smoothness, and is not degenerate, these surfaces will demarcate the (x, y, t) phase space smoothly; the flow is, therefore, integrable.

It turns out that Brown and Samelson's hypotheses (1994) for integrability can be weakened, using Statement 2. To show how this is achieved, their result is stated first:

Integrability statement of Brown and Samelson (1994): *Assume a PV conserving flow satisfying (1) for all time, in which $\lambda \neq 0$ for all (x, y, t) . Then, the flow (3) is integrable.*

The assumption of interest in the above is that $\lambda \neq 0$ for all (x, y, t) (critical points are prohibited from the flow for all time). This is clearly restrictive, and Brown and Samelson (1994) attempted to weaken this condition by considering certain types of invariant submanifolds in the domain. However, the results of Sect. 2, in fact, can be used to improve this result in a clearer fashion. First, it can be noted that Brown and Samelson (1994) were unnecessarily restrictive in their assumption that $\lambda \neq 0$ at all points (x, y, t) . It is, in fact, sufficient for $\lambda \neq 0$ except possibly at a finite number of isolated points for each fixed t . This is the first improvement that can be made; a justification appears in Appendix A.

To enable an additional weakening of the assumption, the result of Statement 2 (no PV patches can be generated if the PV is genuinely conserved) can be used. The necessary assumption for integrability is that for any time t , there can only be a finite number of isolated points at which $\lambda(x, y, t) = 0$. This can be guaranteed from Statement 2 if $\lambda = 0$ at most at a finite number of isolated points, at time zero. Should there be no PV patches in the initial (time zero) PV field, this condition is satisfied. Thus, the assumption of Brown and Samelson (1994) on the nondegeneracy of q for all time, can be reduced to a nondegeneracy requirement only at time zero, and even at this time, a finite number of isolated critical points may exist. This permits the following stronger version of integrability.

Statement 6 (Integrability): *Assume a PV conserving flow satisfying (1) for all time, in which there were no PV patches at time zero. Then, the flow (3) is integrable.*

Therefore, an incompressible, two-dimensional flow

which possesses a conserved quantity q for all time, and which is not piecewise constant at time zero, produces particle trajectories that are integrable. It must be noted that this is a statement on the integrability of the ordinary differential equations (3), and not on the partial differential equation (1), per se.

There is an innocuously powerful assumption in both Statement 6 and in the original statement in Brown and Samelson (1994): that the flow is PV conserving for all time. Thus, the function $q(x, y, t)$ exists as a smooth function forever. However, since q and ψ are typically interrelated, the conservation equation (1) (when expressed purely in terms of the streamfunction) is nonlinear. Such equations, in general, can be ill-posed, and may possess solutions which blow up in finite time. In other words, it is not clear that smooth solutions $q(x, y, t)$ exist to the conservation equation (1) for all times. Should such solutions not exist, then the integrability result of Statement 6 does not apply. On the other hand, for any flow which genuinely satisfies PV conservation for all time, integrability holds.

It is of interest, then, to address the existence of solutions to (1) for times approaching infinity. Under some restrictive assumptions which are nevertheless sometimes used in modelling, such infinite time solvability of (1) is justifiable. For example, if one looks for solutions which are “steady in a moving frame,” the temporal evolution of the PV can be eliminated, and thus, solutions will automatically exist for all time. Such models are often used in addressing the Gulf Stream (“steady in a frame moving eastward”) or detached eddies (“steady in a frame moving westward”). No complicated behaviour is possible in these simple models. Another simplifying case is that of time-periodicity, an often used assumption in dynamical systems theory (the motion of the Gulf Stream appears “close to” periodic, lending some support to this assumption). In this case, existence of smooth solutions for the time duration of the period immediately gives infinite time existence. In both of these cases, chaotic particle motion cannot occur, since the surfaces $q(x, y, t) = \text{constant}$ (when drawn in the (x, y, t) -space) provide a smooth, infinite-time demarcation of the phase-space.

It is also possible that infinite-time solutions to (1) exist, yet have no “nice” limit as $t \rightarrow \infty$. For example, contours of PV may gradually contort and approach one another as time proceeds, causing some level of mixing in the flow. However, these contours cannot contort arbitrarily, or cross, since topological preservation is required from Sect. 3.

The alternative scenario is that solutions to (1) cease to exist in finite time. The precise nature of the PV model that is in use may have profound implications on this “blow-up in finite time” issue. Depending upon the relationship between q and ψ , different types of nonlinear partial differential equations result. If using a barotropic version, such as $q = \nabla^2 \psi - F\psi + \beta y$, the resulting nonlinear equation (1) is in fact fairly similar to the (non-oceanographic) two-dimensional incompressible inviscid vorticity conservation equation, obtained by simply taking the curl of the Euler momentum equation (Chorin and Marsden, 1993). There

would be additional terms which result from the geophysical considerations, but these would be linear and of lower order. Given the fact that existence and uniqueness of such planar Euler flows for all time are well-known (Yudovich, 1961, 1962), it is reasonable to anticipate similar results for the PV conservation equation (1), for barotropic models. In fact, using $q = \nabla^2 \psi + \beta y$, it is shown in Proposition 3 of Balasuriya (1997) via a priori estimates that solutions, if they exist, remain smooth for arbitrarily large times.

If the surface-quasigeostrophic model (in which q represents a potential temperature) is used instead, certain results concerning the solution of (1) are available (Constantin and Wu, 1999; Constantin et al., 1999, 1994). Existence issues of genuine (strong) solutions for this case are only known for finite times (Constantin et al., 1994; Constantin and Wu, 1999). However, it is shown in Constantin and Wu (1999) that, should a strong solution exist, it remains smooth for finite time. Numerical results of Constantin et al. (1999) also indicate that finite time singularities probably do not occur. The contours steepen in their numerics, gradually approaching the formation of a front, a curve across which the PV abruptly changes value. Though approaching a front-like structure, the contours never actually achieve such discontinuity. Now, should a front form in finite time, λ would possess a singularity along the front, and hence, frontogenesis is associated with the blow-up in finite time of solutions to (1). The numerical results are suggestive that nonsingular solutions probably exist for all time. These arguments provide evidence that the PV conservation equation (1) possesses genuine solutions for all time, should the initial PV distribution be smooth.

Statements 1–6 in this paper are all dependent on the conservation equation (1) being satisfied (either for infinite times, or for suitable times). In any genuine PV conserving flow, this should happen by definition. Hence, the results are all valid for genuine PV conserving flows. Should blow-up of solutions occur (in spite of the evidence to the contrary that has been presented), PV conservation is violated, and none of these results hold.

5 Nonconservation of PV

This section returns to the more general flow, as given in (2), in which the PV-conservation is broken through the presence of the function g . However, particle trajectories still satisfy (3). First, the modification of the PV gradient evolution statement is presented.

Statement 7 (PV gradient evolution under nonconservation): *As long as $q(x, y, t)$ satisfies (2), its gradient λ obeys*

$$\dot{\lambda} = -\mathbf{S}^* \lambda + \nabla g. \quad (8)$$

Here, λ and \mathbf{S}^* have the same meanings as in Sect. 2, and λ 's evolution is along a trajectory. The proof of this statement is simple: one follows the argument in Sect. 2 used to prove

Statement 1, and notes that an additional term ∇g appears on the right-hand side.

A crucial property distinguishing (8) from (7) is that if λ is zero at some point on a trajectory, there is no necessity for it to be zero everywhere on that trajectory. Thus, the qualitative equivalence between Eulerian and Lagrangian entities presented in Statement 3 is destroyed when PV conservation breaks. Should g , the conservation breaking function, be “small,” one may expect some approximation to hold.

Such an approximation suggests a “quick and dirty” experiment for estimating the conservation-breaking mechanism’s magnitude for a real flow. Suppose $g = \mathcal{O}(a)$. Imagine that at time zero, a float is placed precisely at a critical point in the ocean (though either an eddy or a saddle may be chosen as this critical point; an eddy would be experimentally better, since the saddle is most likely associated with an unstable direction of flow, rendering the precise positioning of the float at the saddle problematic). Now, think of the PV gradient, measured at the float, as a function of time. Thus, $\lambda(0)$ is zero, and by monitoring the float’s position as time proceeds, and the associated PV values at the float, one may obtain the function $\lambda(t)$. Since $\lambda(0) = \mathbf{0}$, (8) gives that, for small t ,

$$\dot{\lambda}(t) \approx \nabla g(x(t), y(t), t) = \nabla g(0, 0, 0) + \mathcal{O}(t),$$

and thereby

$$\lambda(t) \approx t \nabla g(0, 0, 0) + \mathcal{O}(t^2) = \mathcal{O}(at) + \mathcal{O}(t^2).$$

If $|\lambda(t)|$ were plotted as a function of t , a linear relationship (for small t) is to be expected. Thus, the slope of the graph provides an experimental assessment of the magnitude, a , of the PV conservation-breaking mechanism. The linear relationship is not absolutely accurate, of course, since the quantity ∇g would not remain constant; neither is the argument valid should the float be far from the initial critical point.

The strong eddy/saddle conservation results of Sect. 3 no longer apply to the case where the PV is not conserved. However, a partial result can still be stated, and appears below. Weaknesses in this Statement 8 include (i) identification of eddies/saddles through the sign of the Hessian $H = q_{xx}q_{yy} - (q_{xy})^2$ (critical points where $H = 0$ cannot be handled), and (ii) the result is only valid for short times.

Statement 8 (Critical point persistence): *Assume a flow satisfying (2) for short times, and that the potential vorticity q remains smooth. Moreover, assume that at time zero, there are e critical points where $H > 0$, and s critical points where $H < 0$. Then, for at least a short time beyond zero, there will be e eddies and s saddles in the flow. Moreover, a bifurcation of a critical point is only possible if $H = 0$.*

The proof of this is presented in Appendix A. No information from the dynamical equation (1) was necessary; the only requirement is that q be sufficiently smooth. However, critical points do not travel with the flow; the assertion is merely that eddies/saddles persist as Eulerian objects for short times. Moreover, eddies/saddles are only defined through the sign of H , which provides an incomplete classification of critical points (for example, if $q = x^4 + y^4$, the local minimum

at $(0, 0)$ is not captured through this test). Eddy-to-saddle flips may occur since the value of H , which is positive to begin with, could approach zero at some finite time, and then become negative. Such a bifurcation cannot happen in PV conserving flows (see Statement 4). For related (and more extensive) bifurcation results (though stated with respect to the streamfunction rather than the PV), see Theorem 2.1 in Haller and Poje (1997).

The changes which occur in the other results are now addressed. To do so, different forms of physically applicable g shall be considered as examples.

5.1 Bottom friction

If a flat-bottomed ocean with friction at the ocean bed is hypothesised, the PV would dissipate accordingly. This can be modelled through (2), with the function g chosen to be

$$g(x, y, t) = -a q(x, y, t),$$

where a is a positive constant (Pedlosky, 1987; Jayne and Hogg, 1999). The PV gradient, through substitution in (8), is governed by

$$\dot{\lambda} = -(\mathbf{S}^* + a\mathbf{I})\lambda, \quad (9)$$

where \mathbf{I} is the identity matrix. Equation (9) shares an important property with equation (7): if λ is zero at some point on a trajectory, it shall be zero at all points on that trajectory. Analogous versions of Statements 2, 3, 4 and 5 would, therefore, all hold, even when the PV is dissipated through bottom friction! It is somewhat nonintuitive that, in flows in which PV dissipates through bottom friction, floats placed at saddles remain at saddles. Even integrability (Statement 6) can be proven, since this form of dissipation can be absorbed using the standard integrating factor approach. If (2) (with $g = -aq$) is multiplied through by e^{at} and rearranged, one obtains

$$\frac{\partial}{\partial t} (e^{at} q) + J(\psi, e^{at} q) = 0.$$

Thus, the function $Q(x, y, t) = e^{at} q(x, y, t)$ is conserved by the flow, and can be used to play the role of q in Statement 6. Motion in the (conceptual) three-dimensional (x, y, t) phase-space is confined to the surfaces $Q(x, y, t) = e^{at} q(x, y, t) = \text{constant}$; i.e. the surfaces $q(x, y, t) = C e^{-at}$, where C is a constant. Qualitatively, an exponential spreading of contours of q is, therefore, to be expected (in comparison with a PV conserving flow), thereby [comparatively] reducing the PV gradients exponentially with time. The PV field q approaches a uniform zero value as time goes to infinity; however, if nondegenerate to begin with, it does not become zero at any finite time. It is in this sense that PV dissipation occurs in the presence of bottom friction; however, the parcel trajectories still remain integrable via Statement 6. A bottom-frictional PV jet flow was analysed by Jayne and Hogg (1999) numerically, who then noted that the observed phenomena were well described through a quasi-analytical model they developed. This is consistent with the observation on integrability presented here.

5.2 Forcing

The dynamical equation (2) could also model the breaking of PV conservation through the inclusion of wind-forcing. Then, (8) behaves like a linear nonautonomous inhomogeneous equation for λ .

For the extremely restrictive class of spatially-independent forcing, immediate results are available. If $g = g(t)$ alone, this would mean that $\nabla g = \mathbf{0}$. Then, (8) simplifies to (7), the equation whose properties provided Statements 2–5. Thus, if $g = g(t)$, the analogous versions of Statements 2–5 hold even for the equation (2). Potential vorticity breaking through spatially independent forcing provides effectively the same behaviour as PV-conserving flows, at least as far as the aspects addressed in the present work. Integrability can also be shown by defining a function $h(t)$, such that $h'(t) = g(t)$, and rewriting (2) in the form

$$\frac{\partial}{\partial t} [q(x, y, t) - h(t)] + J(\psi(x, y, t), q(x, y, t) - h(t)) = 0.$$

Thus, the new function $Q(x, y, t) = q(x, y, t) - h(t)$ provides a conserved quantity.

Qualitative statements in the spirit of the other results in this paper can no longer be made for a more general forcing function $g(x, y, t)$. Nevertheless, an equation governing the PV gradient vector (the crucial vector which describes the Eulerian objects of critical points and PV patches) has been derived.

5.3 Diffusion

Models, in which eddy diffusivity plays the dissipative role, are commonly used. In such cases, the standard procedure is to set

$$g = \mathcal{D}\nabla^2 q$$

in (2), where \mathcal{D} is the diffusive parameter (or equivalently, the reciprocal Péclet number), which is assumed small but positive (Rogerson et al., 1999; Miller et al., 1997; Babiano et al., 1994). This is a frequently used procedure to model the effect of small scale turbulence in the ocean; the averaged effect of such turbulence may reflect itself in the dynamical equations through a diffusive term of this nature. This results in a PV dissipating flow governed by

$$\frac{\partial q}{\partial t} + J(\psi, q) = \mathcal{D}\nabla^2 q. \quad (10)$$

Even in some numerical models whose intention is to model PV conserving flows, diffusivity is sometimes included in the numerics merely to promote numerical stability (Dewar and Gaillard, 1994; Flierl et al., 1987). Substituting the eddy diffusive version of g in the PV gradient equation (8),

$$\dot{\lambda} = -\mathbf{S}^* \lambda + \mathcal{D}\nabla^2 \lambda.$$

This is a reaction-diffusion equation describing the evolution of the PV gradient along a trajectory. The first term on the

right is nonlinear, in general, since the PV is an active scalar, and thus, \mathbf{S} would depend on λ in some nontrivial fashion. It is clear that none of the Statements 1–6 are automatically applicable for this diffusive case; when λ is zero at a point on a trajectory, there are not any nice implications. A fluid parcel placed at a critical point would distance itself from it with time t , such that $|\lambda(t)|$ approximately goes as $\mathcal{D}t$ for small times. Thus, the magnitude of (horizontal) eddy diffusivity may be roughly approximated by simply releasing a float at a (centre of an) eddy, and observing how its PV gradient deviates from zero, with time. The slope of this graph near time zero would estimate the eddy diffusivity. The usage of this idea for the particular example of a (diffusive) Rossby wave is presented in Sect. 6.

Occasionally, higher-order diffusivities, such as $g = -\mathcal{D}\nabla^4 q$, are also used (Flierl et al., 1987). For such super-diffusivity, the PV gradient would evolve according to

$$\dot{\lambda} = -\mathbf{S}^* \lambda - \mathcal{D}\nabla^4 \lambda.$$

Diffusive models are not expected to satisfy the topology preservation properties outlined in Sect. 3. In fact, it is often seen, both numerically and experimentally, that 2-D dissipative flows tend to self-organise, creating larger eddies (vortices) from smaller ones. Such results were shown by van Heijst and Clercx (1998) experimentally in Figs. 1 and 2, and numerically in Figs. 3 and 4 (using dynamics of the form (10) with the model $q = \nabla^2 \psi$ in their numerics). Pierrehumbert's calculation (1991), which includes numerical diffusion in an attempt to solve PV-conserving flows, also displays this phenomenon (see his Fig. 14). Topology change is also observable in the diffusive calculations of Flierl et al. (1987) (their Figs. 2, 3, 10, 11, 12, 13, for example), in which the number of critical points is seen to change with time. Other aspects of topology change occur during eddy detachment, when a saddle point must disassociate from the jet, or in its preparatory stage when two saddle points merge to form an eddy from a cats-eye (see Fig. 2). Since these phenomena destroy the topology, in a generic sense, PV conservation is violated. Diffusivity could be directly responsible for such events.

6 Rossby wave example

This section develops an analytical example illustrating the use of the theoretical ideas in this paper. The focus is on the barotropic β -plane model, in which q and ψ are linked through

$$q(x, y, t) = \nabla^2 \psi(x, y, t) + \beta y, \quad (11)$$

where β is the Coriolis parameter. Nondimensional variables will be used for convenience. One of the only known nontrivial solutions to the conservative equation (1) with this model is a Rossby wave given by the streamfunction

$$\psi_0(x, y, t) = \sin[k(x - ct)] \sin(ly),$$

where $\beta = -c(k^2 + l^2)$ (Pierrehumbert, 1991; Pedlosky, 1987). (Many authors use either linearised or approximate solutions to (1), given the difficulty in finding exact solutions.) An explicit expression for the PV can be written using (11), and hence, this example is automatically integrable. Moreover, topology conservation is also obvious, since the q_0 field merely shifts at speed c in the eastward direction. Therefore, this example is still somewhat too trivial for illustrating the qualitative statements of Sects. 3 and 4. On the other hand, it is possible to construct a diffusive solution based on this example, as is shown below.

The streamfunction

$$\psi(x, y, t) = \exp\left(\frac{\mathcal{D}\beta t}{c}\right) \psi_0(x, y, t) \quad (12)$$

is a solution to the PV diffusing equation (10), while satisfying the model (11). A (more general) derivation of this appears in Balasuriya et al. (1998); for the current work, this may be verified by straightforward substitution. For short times, the conservative streamfunction ψ_0 and the diffusive streamfunction ψ remain close, since by Taylor expanding the exponential in (12),

$$\psi(x, y, t) - \psi_0(x, y, t) = \mathcal{O}(\mathcal{D}t),$$

where β and c shall be assumed fixed. The corresponding velocity fields, derived from the gradients of the streamfunctions, are, therefore, $\mathcal{O}(\mathcal{D}t)$ -close. The distance between particles which began at the same point at time zero under these two velocity fields would then increase as $\mathcal{O}(\mathcal{D}t^2)$ for small t (since the position is calculated by integrating the velocity field with respect to t).

The PV fields q and q_0 , corresponding to the diffusive and conservative Rossby wave solutions, respectively, are related through

$$q(x, y, t) = \exp\left(\frac{\mathcal{D}\beta t}{c}\right) q_0(x, y, t) + \beta y \left[1 - \exp\left(\frac{\mathcal{D}\beta t}{c}\right)\right].$$

This is derivable by applying the Laplacian to (12), and then adding βy . By now taking the gradient of the above, the corresponding PV gradient vectors are seen to obey

$$\lambda(x, y, t) = \exp\left(\frac{\mathcal{D}\beta t}{c}\right) \lambda_0(x, y, t) + \beta \left[1 - \exp\left(\frac{\mathcal{D}\beta t}{c}\right)\right] \hat{y}, \quad (13)$$

where $\lambda = \nabla q$ and $\lambda_0 = \nabla q_0$.

Suppose a critical point is identified in the diffusive flow at time zero, by using the PV gradient field λ . Since $\lambda(x, y, 0) = \lambda_0(x, y, 0)$ from the above expression, this would be a critical point of identical structure with respect to the conservative PV field. Now, suppose a float is placed at this critical point, at time zero, and is permitted to travel according to the diffusive streamfunction ψ . Let this float's

trajectory be given by $(x(t), y(t))$, along which the evolution of the PV gradient is to be analysed.

Now, the float trajectory $(x_0(t), y_0(t))$ associated with the conservative streamfunction ψ_0 , would remain within $\mathcal{O}(\mathcal{D}t^2)$ of $(x(t), y(t))$. Hence,

$$\lambda_0(x(t), y(t), t) = \lambda_0(x_0(t), y_0(t), t) + \mathcal{O}(\mathcal{D}t^2),$$

by Taylor expansions. However, Statement 3 tells us that $(x_0(t), y_0(t))$ remains a critical point for all t , and hence, $\lambda_0(x_0(t), y_0(t), t)$ is zero. Thus, $\lambda_0(x(t), y(t), t)$ is of the order of $\mathcal{O}(\mathcal{D}t^2)$. Substituting in (13),

$$\begin{aligned} \lambda(x(t), y(t), t) &= \exp\left(\frac{\mathcal{D}\beta t}{c}\right) \mathcal{O}(\mathcal{D}t^2) \\ &\quad + \beta \left[1 - \exp\left(\frac{\mathcal{D}\beta t}{c}\right)\right] \hat{y} \\ &= [1 + \mathcal{O}(\mathcal{D}t)] \mathcal{O}(\mathcal{D}t^2) \\ &\quad + \beta \left[-\frac{\mathcal{D}\beta t}{c} + \mathcal{O}(\mathcal{D}t)^2\right] \hat{y} \\ &= -\frac{\beta^2 \mathcal{D}t}{c} \hat{y} + \mathcal{O}(\mathcal{D}t^2) + \mathcal{O}(\mathcal{D}t)^2. \end{aligned}$$

Therefore, for small times, the PV gradient's magnitude behaves like

$$|\lambda(x(t), y(t), t)| \approx \left| \frac{\beta^2 \mathcal{D}t}{c} \right| = \left| \beta (k^2 + l^2) \mathcal{D}t \right|.$$

This quantifies the argument presented in Sect. 5, which described how the PV gradient's evolution could estimate the horizontal eddy diffusivity in the ocean. For the specific Rossby wave model that has been examined here, and for a float placed at a critical point (for example, at the centre of an eddy) at time zero, the following has been established: The PV gradient, measured at the float as it is transported with the diffusive flow, increases in size linearly with time, with the proportionality factor $\beta^2 \mathcal{D}/|c|$. If data were gathered from the float and $|\lambda|$ plotted versus t , the initial slope of the graph could be experimentally calculated, and then multiplied by $|c|/\beta^2$ to give an immediate estimate of the effective horizontal eddy diffusivity parameter. Knowledge of the relevant wavespeed c (or equivalently, the wavenumbers k and l) and the local Coriolis parameter β are necessary in this estimate; the process works if a dominant wavespeed can be identified.

7 Conclusions

The PV gradient vector's evolution along fluid trajectories in two-dimensional incompressible flows has been established in this paper. Both PV conserving and nonconserving flows were considered.

It was shown that, in the presence of PV conserving flows, in which q remains a smooth function, its topological structure must be preserved; more precisely: (i) PV patches cannot be generated, (ii) eddies and saddles will travel exactly

with the flow, and (iii) the numbers of eddies (resp. saddles) remains constant. The equivalence established between Eulerian entities (eddies/saddles, defined through fixed-time PV contours) and their Lagrangian counterparts (float trajectories) is an important observation. Moreover, an integrability result was also presented: flows with no PV patches at time zero, and which conserve PV for all times, have integrable trajectories. This improves an available result, and an attempt was also made to shed additional light on the oceanographic consequences of such integrability, and the closely related issue of the existence of infinite time solutions.

The PV gradient's evolution in the presence of eddy diffusivity, bottom friction, or wind-forcing was also obtained. Some of the qualitative results of PV-conserving flows extended to special cases of such flows. In general, however, such statements would be true only in some approximate sense. This could be taken advantage of, in constructing a simple experiment which could be used to estimate horizontal eddy diffusivity. Through the use of a particular analytical example (of a dissipative Rossby wave), a quantification of such an experiment was presented.

Acknowledgements. The author is grateful to Chris Jones for providing encouragement concerning the issues in this paper, to Björn Sandstede for his construction of the diffusive solution of Sect. 6, and to an anonymous referee for constructive criticism. This research was supported in part by the NSF through grant DMS-97-04906, and by the ONR through grant N-00014-92-J-1481.

Appendix A: Proofs of statements

Proof of Statement 2 (PV patch prohibition):

It is necessary to show that, if $\lambda(x, y, 0) = \mathbf{0}$ at most, at (finitely many) isolated points, and if the PV $q(x, y, t)$ is genuinely conserved by the flow for all time, then at each fixed t , $\lambda(x, y, t) = \mathbf{0}$ at most, at isolated points. This is proven simply by considering its contrapositive statement, i.e. it shall be shown that if for some t , $\lambda(x, y, t) = \mathbf{0}$ at more than at isolated points, then the conditions of the statement are violated.

Pick T such that \mathcal{G}_T is nonempty, contains more than just isolated points, and is defined by

$$\mathcal{G}_T = \{(X, Y) : \lambda(X, Y, T) = \mathbf{0}\}.$$

It is necessary to prove the existence of a subset of $\{(x, y) : \lambda(x, y, 0) = \mathbf{0}\}$ which is nonempty and not a collection of finitely many isolated points. This is constructed by defining \mathcal{G}_0 , which is obtained by letting the set \mathcal{G}_T flow for a time $-T$ with respect to the flow (3). Note from (7) that if $\lambda = \mathbf{0}$ at some point on a trajectory, then $\lambda = \mathbf{0}$ at all points on the trajectory. Therefore, for all $(x, y) \in \mathcal{G}_0$, $\lambda(x, y, 0) = \mathbf{0}$. It remains to be shown that \mathcal{G}_0 is not a collection of isolated points. Since \mathcal{G}_T is not such a collection, there exists a sequence (X_i, Y_i) contained in \mathcal{G}_T , converging to a point $(\bar{X}, \bar{Y}) \in \mathcal{G}_T$, such that $(X_i, Y_i) \neq (X_j, Y_j)$ if $i \neq j$. Now construct (x_i, y_i) by flowing (X_i, Y_i) for time $-T$ by (3),

for each i . By the continuity of the flow operation, (x_i, y_i) must converge to (\bar{x}, \bar{y}) , which is itself obtainable by flowing (\bar{X}, \bar{Y}) by time $-T$. Hence, (\bar{x}, \bar{y}) is in \mathcal{G}_0 . Now consider the set of points $\{(\bar{x}, \bar{y}), (x_1, y_1), (x_2, y_2), (x_3, y_3), \dots\}$ which is contained in \mathcal{G}_0 . If it consists of only finitely many isolated points, the sequence (x_i, y_i) must eventually be identically (\bar{x}, \bar{y}) . But this implies that if (\bar{x}, \bar{y}) follows the flow for time T , it is mapped to infinitely many points, which contradicts the uniqueness of solutions to smooth ordinary differential equations. Hence, \mathcal{G}_0 is nonempty, and not a collection of finitely many isolated points. Thus, the contrapositive statement has been proven.

Proof of Statement 4 (Eddy/Saddle flip)

Suppose an eddy suddenly becomes a saddle (this does not violate the critical point persistence of Statement 3, since both the eddy and the saddle are critical points). More specifically, suppose the picture given in Fig. 1a existed at time $t - \delta$, and that of Fig. 1b occurs by time $t + \delta$, where δ is assumed to be small. The eddy at A (at time $t - \delta$) has become a saddle at B (at time $t + \delta$); this models the flipping of an eddy to a saddle at time t . The quantity δ is assumed to be as small as required, and the contour structures in Fig. 1 should be assumed to be local. Now, since critical points travel with the flow by Statement 3, the fluid parcel at A at time $t - \delta$ has travelled to B by time $t + \delta$. Suppose the PV value at A is q_0 . Since the PV is conserved by the flow, the PV value at B must also be q_0 . This implies that any point P chosen on the contours passing through the saddle B , must also have a PV value of q_0 . Now, consider the fluid parcel which is at P at time $t + \delta$. At time $t - \delta$, this parcel must have been at a point at which the PV value is q_0 , since the PV is assumed conserved by the flow. However, the only point in the local picture at time $t - \delta$ of Fig. 1a at which $q = q_0$ is the point A , since A is a local maximum/minimum which is enclosed by closed contours. Thus, the parcel at P must have originated at A . This is a contradiction, since it is known that the parcel which originated at A is now at B , and not at P . Therefore, an eddy cannot transform to a saddle at some instance in time. It is clear that the reverse process is also impossible.

Proof of Statement 8 (Critical point persistence):

It is assumed that e Eulerian eddies exist in the flow at time zero, defined through the sign of H being positive. Pick one such point, say (x_0, y_0) . Thus, it is known that $\lambda(x_0, y_0, 0) = \nabla q(x_0, y_0, 0) = \mathbf{0}$ and $H(x_0, y_0, 0) = J(q_x, q_y)(x_0, y_0, 0) > 0$. Now, it is required to prove the existence of functions $x(t)$ and $y(t)$ for short times, such that $\lambda(x(t), y(t), t) = \mathbf{0}$ and $J(q_x, q_y)(x(t), y(t), t) > 0$. Here, $(x(t), y(t))$ would be the location of the eddy at time t ; proving these conditions would show that the eddy exists by definition. Now, recall that the implicit function theorem from calculus asserts that the set of equations

$$q_x(x(t), y(t), t) = 0 \quad ; \quad q_y(x(t), y(t), t) = 0$$

can be solved for $(x(t), y(t))$ near (x_0, y_0) if (i) (x_0, y_0) satisfy the equations at $t = 0$, and (ii) $H = J(q_x, q_y) \neq 0$ at $(x_0, y_0, 0)$. These conditions are satisfied since an eddy (a critical point where $H \neq 0$) exists at $t = 0$. Hence, a solution $(x(t), y(t))$, which is a critical point, exists for small enough t . However, by continuity of the derivatives of q , for small enough t , $J(q_x, q_y)$ must be positive at $(x(t), y(t), t)$ as well, and hence, this critical point will continue to be an eddy. This argument can be made for each and every one of the e eddies which exist at time zero, and hence, each of these will persist as eddies for short times. An analogous argument serves to show that each of the s saddles (defined by $H < 0$) also persist for short times, since this satisfies the $H \neq 0$ requirement to apply the implicit function theorem. The only instance when the implicit function theorem does not apply is when $H = 0$, and, therefore, it is only if $H = 0$ that a critical point may change its nature.

First improvement to Brown-Samelson integrability:

Brown and Samelson (1994) showed integrability by transforming the nonautonomous one degree of freedom system (3) to an autonomous two degrees of freedom Hamiltonian system, through defining the Hamiltonian function

$$H(x, y, t, r) = \psi(x, y, t) + r,$$

where r is the (artificially introduced) variable conjugate to t . If it could be shown that H and q were independent in the (x, y, t, r) space, two integrals of motion for the two degree of freedom Hamiltonian system exist, and thus, (3) would be integrable by quadratures (Brown and Samelson, 1994). Independence involves showing that

$$a \text{ grad } H(x, y, t, r) + b \text{ grad } q(x, y, t) = \mathbf{0}$$

has only the solutions $a = b = 0$ for constants a and b . Here, "grad" is the gradient in \mathbb{R}^4 , and thus, the vector equation of interest is

$$a(\psi_x, \psi_y, \psi_t, 1) + b(q_x, q_y, q_t, 0) = 0.$$

Brown and Samelson (1994) argue that $a = 0$ is necessary by considering the final component above, and that $b = 0$ if $\|\lambda(x, y, t)\| > 0$ for all $(x, y, t) \in \mathbb{R}^3$. However, notice that for b to equal zero, it is, in fact, sufficient that q not be degenerate. In other words, as long as $|(q_x, q_y, q_t, 0)|$ is not zero in connected (nonzero measure) areas of \mathbb{R}^4 , $b = 0$ emerges as the only possibility. (Compare this argument with the functions x^2 and x^3 on \mathbb{R} , which are clearly independent functions, but each has a zero gradient at $x = 0$, an isolated point.) Hence, for independence, it is sufficient that, for each t , $|\lambda(x, y, t)|$ have only an isolated number of zeros.

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