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Nonlinear interaction between acoustic gravity waves in a rotating atmosphere

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Abstract. The influence of the Earth's rotation on the resonant interaction of atmospheric waves is investigated. The explicit expressions for the coupling coefficients are presented. They are derived by means of two different techniques; first, by a direct expansion derivation from a set of reduced equations, and second, by a Hamiltonian method.

1 Introduction

The nonlinear interaction between the acoustic-gravity waves in the atmosphere of the Earth (Gill, 1982) leads to energy transfer, resonantly as well as non-resonantly, between the high- and low-frequency modes, or from the gravity modes to the vortical modes (Dong and Yeh, 1991) with subsequent gravity wave saturation. In the linearly unstable regions, a turbulent state consisting of acoustic-gravity vortices (Stenflo and Stepanyants, 1995) may be the final result. The diffusion properties of the atmosphere will then be significantly changed (Pavlenko and Stenflo, 1992). It is however also possible that a chaotic development (Lorenz, 1963) takes place. Such situations have been investigated by Stenflo (1996) and Yu and Yang (1996) for atmospheric disturbances where the rotation of the Earth plays an essential role.

In a previous paper (Axelsson et al., 1996) we considered the resonant interaction between three acoustic-gravity waves $(\omega_j, \mathbf{k}_j)$ for arbitrary directions of the wave vectors $\mathbf{k}_j$. The resonance conditions were written as $\sum_{j=1}^{3}\omega_j = 0$ and $\sum_{j=1}^{3}\mathbf{k}_j = 0$. Particular attention was paid to the low-frequency limit. An explicit and symmetric three-dimensional coupling coefficient was derived. In the present paper we are going to generalize that result in order to take the rotation of the Earth into account. Also in this case it will turn out that we can derive the explicit formulas for the coupling coefficients. Due to the complexity of the calculations we shall use two different approaches. Firstly, in section 2, we start from a set of reduced equations for the low-frequency, short-wavelength acoustic-gravity perturbations in the atmosphere, adopting a direct perturbation method to the derive the result. Secondly, in section 3, we use results obtained by generalizing a Hamiltonian method (Larsson, 1996) and then evaluate them in the low-frequency limit. After much algebra, it turns out that the results of section 2 and 3 are identical. This confirms the correctness of both methods.

2 Direct expansion derivation from a set of reduced equations

Let us here consider low-frequency, short-wavelength oscillations in an atmosphere with exponentially decreasing equilibrium density $\rho_0 \sim \exp(-z/H)$ where z represents the vertical direction and H is the density scale height. We thus assume that the frequencies ω and the wavelengths $2\pi/|\mathbf{k}|$ are much smaller than the Brunt-Väisälä frequency ω_g and H, respectively. The evolution of the wave amplitudes can then be described by a reduced set of equations that was recently deduced by Stenflo (1991). It is written as

$$-\partial_t \nabla^2 v_z + (\nabla \times \nabla \times \mathbf{v} \cdot \nabla \mathbf{v})_z = \nabla_\perp^2 \chi + R \qquad (1)$$

$$(\partial_t + \mathbf{v} \cdot \nabla)\chi = \omega_g^2 v_z \qquad (2)$$

and

$$\partial_t R = 4(\Omega\cdot\nabla)^2 v_z - 2\Omega\cdot\nabla(\nabla\times\mathbf{v}\cdot\nabla\mathbf{v})_z \qquad (3)$$

where $\mathbf{v}$ is the fluid velocity, χ represents the density perturbation ρ_1 (normalized by the gravitational acceleration divided by ρ_0), Ω is the angular velocity of the Earth, and R is a new variabel (Stenflo, 1991) that has been introduced in order to account for the rotation of the Earth. For simplicity we shall here consider the high-latitude atmosphere where $\mathbf{\Omega} \approx \Omega\hat{\mathbf{z}}$, assuming that Ω is constant. For $\rho_1/\rho_0 << 1$, i.e. for not too large amplitude disturbances, the fluid motion is almost incompresssible ($\nabla\cdot\mathbf{v}\approx 0$) and according to Eq. (2) in the paper by Stenflo (1991) we therefore replace $\mathbf{v}$ in the nonlinear terms by

$$\mathbf{v} \approx v_z\hat{\mathbf{z}} - \nabla_\perp^{-2}\partial_z(\nabla_\perp + 2\Omega\partial_t^{-1}\hat{\mathbf{z}}\times\nabla_\perp)v_z \qquad (4)$$

In the small amplitude limit the linearised system of equations (1), (2) and (3) yields

$$\omega^2 = \omega_g^2 k_\perp^2/k^2 + 4\Omega^2 k_z^2/k^2 \qquad (5)$$

where $k_\perp^2 = k_x^2 + k_y^2$ and $k^2 = k_\perp^2 + k_z^2$. The equations can also have large amplitude modon solutions (Stenflo, 1991; Stenflo and Stepanyants, 1995).

We shall now use equations (1), (2) and (3) to consider the resonant interaction between three weakly nonlinear waves $(\omega_1,\mathbf{k}_1)$, $(\omega_2,\mathbf{k}_2)$ and $(\omega_3,\mathbf{k}_3)$ where each wave satisfies Eq. (5) and where the resonance conditions $\omega_1+\omega_2+\omega_3=0$ and $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$ are supposed to be satisfied. Keeping only resonant nonlinear terms of second order we then study the excitation of wave "3" due to the interaction of waves "1" and "2". We thus directly obtain from (1), (2) and (3)

$$\begin{aligned}(\partial_t^2\nabla^2 + \omega_g^2\nabla_\perp^2 + 4\Omega^2\partial_z^2)v_{z3} = \\ \partial_t\{\nabla_\perp\cdot[\partial_z(\mathbf{v}_1\cdot\nabla\mathbf{v}_{\perp 2}) - \nabla_\perp(\mathbf{v}_1\cdot\nabla v_{z2}]\} \\ -\left[\partial_t^2\nabla^2 + 4(\Omega\cdot\nabla)^2\right](\mathbf{v}_1\cdot\nabla\textstyle\int v_{z2}\,dt) \\ + 2\Omega\cdot\nabla\left(\nabla\times\mathbf{v}_1\cdot\nabla\mathbf{v}_2\right)_z' + (1\leftrightarrow 2)\end{aligned} \qquad (6)$$

where $(1\leftrightarrow 2)$ means permutation of indices 1 and 2. According to (4) we then insert

$$\mathbf{v}_j = [\hat{\mathbf{z}} - (k_{zj}/k_{\perp j}^2)(\mathbf{k}_{\perp j} + 2i\Omega(\mathbf{k}_j\times\hat{\mathbf{z}})/\omega_j)]\,v_{zj} \quad , \qquad (7)$$

where j =1 and 2, in the right hand side of Eq. (6) and write $\mathbf{v}_j = \mathbf{v}_{jA}\exp(-i\omega_j t + i\mathbf{k}_j\cdot\mathbf{r})$ with slowly varying amplitudes $\mathbf{v}_{jA}$. Dropping for simplicity the subscript A, and passing the straightforward algebra, we write Eq. (6) in the form

$$\partial_t v_{z3} = C v_{z1} v_{z2} \qquad (8)$$

where

$$C = C_0 + C_1 + C_2 + C_3 \qquad (9)$$

with

$$\begin{aligned}C_0 = \frac{i}{2k_3^2 k_{\perp 1}^2}\left[k_{z2}k_{z3}\frac{1}{k_{\perp 2}^2}\mathbf{k}_{\perp 2}\cdot\mathbf{k}_{\perp 3} + k_{\perp 3}^2 - \frac{\omega_3}{\omega_2}k_3^2\right] \\ \mathbf{k}_{\perp 1}\cdot(\mathbf{k}_{\perp 1}k_{z2} - \mathbf{k}_{\perp 2}k_{z1}) + (1\leftrightarrow 2)\end{aligned} \qquad (10)$$

$$\begin{aligned}C_1 = \frac{\Omega}{\omega_1}\frac{k_{z1}}{k_3^2 k_{\perp 1}^2}\left[\frac{\omega_3}{\omega_2}k_3^2 - k_{z2}k_{z3}\frac{1}{k_{\perp 2}^2}\mathbf{k}_{\perp 2}\cdot\mathbf{k}_{\perp 3}\right. \\ \left. - k_{\perp 3}^2\right](\mathbf{k}_{\perp 1}\times\mathbf{k}_{\perp 2})_z \\ +\Omega\left(\frac{1}{\omega_2} - \frac{1}{\omega_3}\right)\frac{k_{z2}k_{z3}}{k_3^2}\frac{1}{k_{\perp 1}^2 k_{\perp 2}^2}(\mathbf{k}_{\perp 2}\times\mathbf{k}_{\perp 3})_z \\ \mathbf{k}_{\perp 1}\cdot(\mathbf{k}_{\perp 1}k_{z2} - \mathbf{k}_{\perp 2}k_{z1}) + (1\leftrightarrow 2)\end{aligned} \qquad (11)$$

$$\begin{aligned}C_2 = 2i\Omega^2\frac{k_{z3}}{k_{\perp 1}^2 k_{\perp 2}^2 k_3^2}\left[\frac{1}{\omega_2\omega_3}\mathbf{k}_{\perp 1}\cdot(\mathbf{k}_{\perp 1}k_{z2} - \mathbf{k}_{\perp 2}k_{z1})\right. \\ \mathbf{k}_{\perp 2}\cdot(\mathbf{k}_{\perp 1}k_{z2} - \mathbf{k}_{\perp 2}k_{z1}) \\ \left. + \left(\frac{1}{\omega_2} - \frac{1}{\omega_3}\right)\frac{1}{\omega_1}k_{z1}k_{z2}(\mathbf{k}_{\perp 1}\times\mathbf{k}_{\perp 2})_z^2\right] + (1\leftrightarrow 2)\end{aligned} \qquad (12)$$

and

$$\begin{aligned}C_3 = 4\Omega^3\frac{1}{\omega_1\omega_2\omega_3}\frac{1}{k_{\perp 1}^2 k_{\perp 2}^2 k_3^2}\left[k_{z1}k_{z2}k_{z3}(k_{\perp 1}^2 - k_{\perp 2}^2)\right. \\ \left. + k_{z3}^2(k_{z2}k_{\perp 1}^2 - k_{z1}k_{\perp 2}^2)\right](\mathbf{k}_{\perp 1}\times\mathbf{k}_{\perp 2})_z + (1\leftrightarrow 2).\end{aligned} \qquad (13)$$

Using the dispersion relation (5) we then rewrite C_0 as

$$\begin{aligned}C_0 = \frac{i}{2}\omega_g^2\frac{1}{\omega_1^2\omega_2^2\omega_3}\frac{k_{\perp 3}^2}{k_3^2}\left[\omega_1^2(\omega_3 - \omega_2)\frac{\mathbf{k}_{\perp 1}}{k_{\perp 1}^2}\right. \\ \left. + \omega_2^2(\omega_1 - \omega_3)\frac{\mathbf{k}_{\perp 2}}{k_{\perp 2}^2} + \omega_3^2(\omega_2 - \omega_1)\frac{\mathbf{k}_{\perp 3}}{k_{\perp 3}^2}\right] \\ \cdot(\mathbf{k}_{\perp 1}k_{z2} - \mathbf{k}_{\perp 2}k_{z1})\end{aligned}$$

$$+2i\Omega^2 \frac{1}{k_{\perp 1}^2 k_{\perp 2}^2 k_3^2 \omega_1^2 \omega_2^2 \omega_3}$$
$$\Big[\omega_3 \omega_1^2 k_{z2}^2 (k_{\perp 1}^2 \mathbf{k}_{\perp 3} + k_{\perp 3}^2 \mathbf{k}_{\perp 1})$$
$$- \omega_3 \omega_2^2 k_{z1}^2 (k_{\perp 3}^2 \mathbf{k}_{\perp 2} + k_{\perp 2}^2 \mathbf{k}_{\perp 3})$$
$$- \omega_1^2 \omega_2 k_{\perp 2}^2 k_{z3}^2 \mathbf{k}_{\perp 1} + \omega_2^2 \omega_1 k_{\perp 1}^2 k_{z3}^2 \mathbf{k}_{\perp 2}\Big]$$
$$\cdot (\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1}) \ . \tag{14}$$

Next we combine the Ω^2 term in (14) with the terms in (12) to obtain an expression that is symmetric in all wave indices. Similarly it turns out that the terms in (13) can be combined with those of (11), without introducing terms proportional to ω_g^2, using the relations $4\Omega^2 (k_{zi}^2 k_{\perp j}^2 - k_{zj}^2 k_{\perp i}^2) = \omega_i^2 k_i^2 k_{\perp j}^2 - \omega_j^2 k_j^2 k_{\perp i}^2$ (i, j = 1, 2, 3). Thus, it is possible to rewrite C in the obviously symmetric form

$$C = \frac{i}{2}\omega_g^2 \frac{1}{\omega_1^2 \omega_2^2 \omega_3} \frac{k_{\perp 3}^2}{k_3^2} \Big[\omega_1^2 (\omega_3 - \omega_2) \frac{\mathbf{k}_{\perp 1}}{k_{\perp 1}^2}$$
$$+ \omega_2^2 (\omega_1 - \omega_3) \frac{\mathbf{k}_{\perp 2}}{k_{\perp 2}^2} + \omega_3^2 (\omega_2 - \omega_1) \frac{\mathbf{k}_{\perp 3}}{k_{\perp 3}^2}\Big]$$
$$\cdot (\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1})$$
$$+ \Omega \frac{1}{\omega_1 \omega_2} \frac{1}{k_{\perp 1}^2 k_{\perp 2}^2 k_3^2} \mathrm{perm}_3 \Big\{\Big[k_{z1} k_{z2} k_{z3} \omega_1 (k_{\perp 2}^2 - k_{\perp 3}^2)$$
$$+ (\omega_3 - \omega_2) k_{z1} k_{\perp 2}^2 k_{\perp 3}^2\Big] (\mathbf{k}_{\perp 1} \times \mathbf{k}_{\perp 2})_z\Big\}$$
$$+ 2i\Omega^2 \frac{1}{\omega_1^2 \omega_2^2 \omega_3} \frac{1}{k_{\perp 1}^2 k_{\perp 2}^2 k_3^2}$$
$$\Big[\mathrm{perm}_6 \big\{\omega_3 \omega_1^2 k_{\perp 3}^2 k_{z2}^2 \mathbf{k}_{\perp 1} \cdot (\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1})\big\}$$
$$- \omega_1 \omega_2 \omega_3 \, \mathrm{perm}_3 \big\{k_{z3} (\mathbf{k}_{\perp 1} \cdot \mathbf{k}_{\perp 2})(\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1})^2\big\}$$
$$+ 6\omega_1 \omega_2 \omega_3 k_{z1} k_{z2} k_{z3} (\mathbf{k}_{\perp 1} \times \mathbf{k}_{\perp 2})_z^2\Big] \tag{15}$$

where perm_3 and perm_6 means the sum of the cyclic permutated respectively totally permutated terms. The coupling coefficient (15) is our main result. It has not been derived previously.

3 The Hamiltonian method

In this section we start from the full set of equations governing the dynamics of a perfect fluid

$$\partial_t \rho + \nabla \cdot (\rho \, \mathbf{v}) = 0 \tag{16}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\mathbf{\Omega} \times \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi \tag{17}$$

and

$$\partial_t S + \mathbf{v} \cdot \nabla S = 0 \ , \tag{18}$$

where Φ represents the potential of some external or inertial forces (e.g. gravitation or centripetal acceleration). The pressure $P = P(\rho, S)$ is a given function of the local values of the mass density ρ and the specific entropy S. The Coriolis acceleration term $2\mathbf{\Omega} \times \mathbf{v}$ in Eq. (17) takes the rotation of the Earth into account. In order for the system of equations (16)-(18) to be Hamiltonian it is only required that $\mathbf{\Omega}(\mathbf{r})$is divergence free.

Consider now a static unperturbed state $(\rho_0, \mathbf{v}_0 = 0, S_0)$, i.e.

$$\frac{\nabla P_0}{\rho_0} + \nabla \Phi = 0 \ . \tag{19}$$

and let

$$(\rho_j, \mathbf{v}_j, S_j) \exp(-i\omega_j t) + \text{comp.conj.} \tag{20}$$

denote three linear waves such that the resonance condition $\omega_1 + \omega_2 + \omega_3 = 0$ is satisfied. The resonant nonlinear interaction of these waves can then be derived from the ansatz

$$(\rho, \mathbf{v}, S) = (\rho_0, \mathbf{0}, S_0) + \sum_{j=1}^{3} \Big[a_j(t)(\rho_j, \mathbf{v}_j, S_j) \exp(-i\omega_j t) + \text{compl.conj.}\Big] \ , \tag{21}$$

where the complex-valued functions $a_j(t)$ are slowly time dependent and satisfy the equations (Axelsson et al., 1996)

$$\frac{d}{dt} a_1^* = c_1 a_2 a_3 , \quad \frac{d}{dt} a_2^* = c_2 a_1 a_3 \quad \text{and} \quad \frac{d}{dt} a_3^* = c_3 a_1 a_2$$

Algebraic expressions for the coupling coefficients c_j have been presented by Axelsson et al. (1996) for the particular case when $\mathbf{\Omega} = \mathbf{0}$. The Hamiltonian structure of the basic equations was then needed for the derivation method. It can however be shown that the system (16)-(18) is Hamiltonian also in the presence of the Coriolis term in (17). The only condition for the Hamiltonian property is that $\nabla \cdot \mathbf{\Omega} = 0$.

The derivation of the coupling coefficient is very much the same in this more general case with $\mathbf{\Omega} \neq \mathbf{0}$. Thus we here just present the result:

$$c_j = \frac{i\omega_j V}{2\int \rho_0 \mathbf{v}_j^* \cdot \mathbf{v}_j \, d\mathbf{r}} \ , \tag{22}$$

where the common coefficient (22) is

$$
\begin{aligned}
V = &\, \mathrm{perm}_3 \int \rho_1 \mathbf{v}_2 \cdot \mathbf{v}_3 \, d\mathbf{r} + \int \rho_1 \rho_2 \rho_3 F_{\rho 0} \, d\mathbf{r} \\
&+ 2 \int \boldsymbol{\Omega} \cdot \left[\frac{1}{\omega_1 \omega_2} \mathbf{D}_{231} - \frac{1}{\omega_1 \omega_3} \mathbf{D}_{312} \right] d\mathbf{r} \\
&+ \int (b_0 G_{S0} + \rho_0 d_0) S_1 S_2 S_3 \, d\mathbf{r} \\
&+ \mathrm{perm}_3 \int \left(G_{S0} - \frac{b_0 G_0}{\rho_0} \right) \rho_1 S_2 S_3 \, d\mathbf{r} \\
&+ \mathrm{perm}_6 \int \frac{i G_0}{\omega_2} \mathbf{v}_3 \cdot [\rho_2 \nabla S_1 - b_0 S_1 \nabla S_2] d\mathbf{r} \\
&+ \mathrm{perm}_3 \int \left[F_{S0} - \frac{\omega_3^2}{\omega_1 \omega_2 \rho_0} G_0 \right] \rho_1 \rho_2 S_3 \, d\mathbf{r}
\end{aligned} \tag{23}
$$

and where we have used the notations

$$
F_{\rho 0} = \left[\frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial P}{\partial \rho} \right) \right]_{(\rho_0, S_0)},
$$

$$
F_{S0} = \left[\frac{1}{\rho} \frac{\partial^2 P}{\partial S \partial \rho} \right]_{(\rho_0, S_0)}, \quad G_0 = \left[\frac{1}{\rho} \frac{\partial P}{\partial S} \right]_{(\rho_0, S_0)} \quad \text{and}
$$

$$
G_{S0} = \left[\frac{1}{\rho} \frac{\partial^2 P}{\partial S^2} \right]_{(\rho_0, S_0)}.
$$

The scalar functions $b_0(\mathbf{r})$ and $d_0(\mathbf{r})$ depend only on the equilibrium quantities and are defined by (see Axelsson et al. 1996)

$$
\nabla \rho_0 \equiv -b_0 \nabla S_0 \quad \text{and} \quad d_0 \equiv \frac{2i}{\omega_j} \frac{\mathbf{v}_j \cdot \nabla}{S_j} \left(\frac{b_0 G_0}{\rho_0} \right).
$$

The only new term that explicitly enters the coupling coefficients due to the Coriolis acceleration involves the vector quantity
$\mathbf{D}_{123} \equiv \mathbf{v}_1 \times [\nabla \times (\rho_0 \mathbf{v}_2 \times \mathbf{v}_3)] - \mathbf{v}_2 \times \mathbf{v}_3 \nabla \cdot (\rho_0 \mathbf{v}_1)$. This is also the only term in (23) that is not obviously symmetric with respect to permutations of the subscripts $\{1,2,3\}$. The symmetry of this new term is however easily proved using the vector identity

$$
\nabla (\rho_0 \mathbf{v}_1 \cdot \mathbf{v}_2 \times \mathbf{v}_3) = \mathbf{D}_{123} + \mathbf{D}_{231} + \mathbf{D}_{312}.
$$

In the low-frequency, short-wavelength limit we insert the appropriate linear relations for ρ_j, $\mathbf{v}_j$ and S_j, i.e.

$$
\rho_j = \frac{i(\gamma - 1)}{\gamma H} \frac{\rho_0}{\omega_j} v_{zj}
$$

and

$$
S_j = -i \gamma H \omega_g^2 \frac{1}{\omega_j} \rho_0^{1-\gamma} v_{zj}
$$

together with (7) to directly obtain

$$
\begin{aligned}
V = &\, \rho_0 \omega_g^2 \frac{1}{\omega_1^2 \omega_2^2 \omega_3^2} \mathrm{perm}_3 \\
&\left\{ \omega_3^2 (\omega_1 - \omega_2) \frac{\mathbf{k}_{\perp 3}}{k_{\perp 3}^2} \cdot (\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1}) \right\} v_{z1} v_{z2} v_{z3} \\
&+ 2i \Omega \rho_0 \frac{1}{k_{\perp 1}^2 k_{\perp 2}^2 k_{\perp 3}^2} \mathrm{perm}_3 \\
&\left\{ \left[\frac{\omega_g^2}{\omega_1^2 \omega_2^2 \omega_3^2} k_{z3} k_{\perp 1}^2 k_{\perp 2}^2 \omega_3 (\omega_1 - \omega_2) \right. \right. \\
&\left. \left. + \frac{k_{z2} k_{z3}}{\omega_2 \omega_3} \mathbf{k}_{\perp 1} \cdot (\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1}) \right] (\mathbf{k}_1 \times \mathbf{k}_2)_z \right\} \\
&v_{z1} v_{z2} v_{z3} \\
&+ 4 \Omega^2 \rho_0 \frac{1}{k_{\perp 1}^2 k_{\perp 2}^2 k_{\perp 3}^2} \frac{1}{\omega_1 \omega_2 \omega_3} \\
&\left[\frac{1}{\omega_1 \omega_2 \omega_3} \mathrm{perm}_3 \{ \omega_1^2 (\omega_2 - \omega_3) k_{z2} k_{z3} (\mathbf{k}_{\perp 2} \cdot \mathbf{k}_{\perp 3}) \right. \\
&\left. \mathbf{k}_{\perp 1} \cdot (\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1}) \} - 3 k_{z1} k_{z2} k_{z3} (\mathbf{k}_1 \times \mathbf{k}_2)_z^2 \right] \\
&v_{z1} v_{z2} v_{z3} \\
&+ 8i \Omega^3 \rho_0 \frac{1}{k_{\perp 1}^2 k_{\perp 2}^2 k_{\perp 3}^2} \frac{1}{\omega_1^2 \omega_2^2 \omega_3^2} \\
&\mathrm{perm}_3 \{ [k_{z1} k_{z2} k_{z3} \omega_1 (\omega_2 - \omega_3) \mathbf{k}_{\perp 2} \cdot \mathbf{k}_{\perp 3} \\
&\quad - \omega_1^2 k_{z2} k_{z3} \mathbf{k}_{\perp 1} \cdot (\mathbf{k}_{\perp 1} k_{z2} - \mathbf{k}_{\perp 2} k_{z1})] \\
&\quad (\mathbf{k}_1 \times \mathbf{k}_2)_z \} v_{z1} v_{z2} v_{z3}.
\end{aligned} \tag{24}
$$

After much algebra it is possible to show that (24) can be rewritten as

$$
V = 2i \frac{1}{\omega_3} \frac{k_3^2}{k_{\perp 3}^2} C \, v_{z1} v_{z2} v_{z3},
$$

where C is defined by (15). Thus the results of section 3 are in complete agreement with those of section 2.

4 Conclusions

In this paper we have presented the explicit expression for the common symmetric coupling coefficient for the interaction of three low-frequency, short-wavelength acoustic-gravity waves in a rotating atmosphere. This is a basic formula within the nonlinear theory of atmospheric waves. We have shown that the reduced set of equations (Stenflo, 1991) for low-frequency atmospheric waves is very useful for such calculations. In addition, we extended the Hamiltonian technique (Larsson, 1996; Axelsson et al., 1996), derived results for all frequency ranges, and showed that it is straightforward to apply them to various limiting cases.

Our coupling coefficients agree with those of previous works in the appropriate limits. We have also generalized the findings of other authors in order to shed light on previously hidden symmetry properties. The rotation of the Earth turns out to weaken the threshold conditions and to enhance the nonresonant interaction between nearly vertically propagating acoustic-gravity waves (cf. Dong and Yeh, 1991).

We also expect that the Hamiltonian approach can be useful for other waves, e.g. Rossby waves, that have not been studied in the present work. This will be an obvious subject for forthcoming investigations.

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