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Adjacent-vertex distinguishing total chromatic number on $W_s \vee K_{m,n}$

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Abstract: A conjecture about the concept of the adjacent-vertex distinguishing total colorings (AVDTC) on graphs is stated as this: For any simple graph G, then $\chi_{at}(G) \leq \Delta(G)+3$. The AVDTC-chromatic number of a join graph $W_s \vee K_{m,n}$ is determined in the form $\Delta(W_s \vee K_{m,n})+1 \leq \chi_{at}(W_s \vee K_{m,n}) \leq \Delta(W_s \vee K_{m,n})+2$.

Key words: graph; total coloring; adjacent-vertex-distinguishing total coloring; adjacent-vertex-distinguishing total chromatic number

联图 $W_s \vee K_{m,n}$ 的邻点可区别全色数

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摘要：图的邻点可区别全染色(AVDTC)数为 $\chi_{at}(G)$，有猜想：$\chi_{at}(G) \leq \Delta(G)+3$. 联图 $W_s \vee K_{m,n}$ 的邻点可区别全色数被确定为 $\chi_{at}(W_s \vee K_{m,n}) = \Delta(W_s \vee K_{m,n})+1$ 或 $\Delta(W_s \vee K_{m,n})+2$.

关键词：图；全染色；邻点可区别全染色；邻点可区别全色数
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0 Introduction

The concept of the vertex-distinguishing proper edge-coloring (VDPEC) was introduced and discussed by Burris and Schelp in reference [1], and there are many results have been obtained[2,3]. Although there are many results on the total coloring of graphs[4], the famous total coloring conjecture $\chi_T(G) \leq \Delta(G)+2$ is open. Zhang Zhongfu et al. first introduced the concepts of the adjacent-vertex distinguishing proper edge-colorings (AVDPEC) and the adjacent-vertex distinguishing total colorings (AVDTC) on graphs, and investigated some particular graphs such as paths, cycles, complete graphs, complete bipartite graphs, stars, fans and wheels about the AVDPEC-chromatic index and AVDTC-chromatic numbers in references of [5] and [6]. In 2005, Zhang Zhongfu et al. proposed a conjecture: For any simple graph G, then $\chi_{at}(G) \leq \Delta(G)+3$ in reference[7]. Unfortunately, there is little works on graphs AVDTC nowadays. In this paper, the AVDTC-chromatic number of a join graph $W_s \vee K_{m,n}$ is determined, that is,

$$\Delta(W_s \vee K_{m,n})+1 \leq \chi_{at}(W_s \vee K_{m,n}) \leq \Delta(W_s \vee K_{m,n})+2.$$

For the sake of simplicity, we use the symbol $[k]$ is used instead of the set $\{1, 2, \cdots k\}$, where k is a natural number and $[k]$ is often called a color set. Equivalently, the color set $[m]^0 = \{0,1,2,\cdots m\}$ is also used. Let f be a

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proper total coloring of a graph G from $V(G)\cup E(G)$ to $[k]$, then the color set $C(v)$ of a vertex v of G as the form $C(v)=\{f(v)\}\cup\{f(vw)\mid w\in V(G),\ vw\in E(G)\}$, and $\bar{C}(v)=[k]\setminus C(v)$ is called the color complement set of $C(v)$.

Definition 1[7] Let f be a proper total coloring of a graph G from $V(G)\cup E(G)$ to $[k]$. If f satisfies that

(ⅰ) $\forall\ u,v\in V(G),\ uv\in E(G),\ f(u)\neq f(v),\ f(u)\neq f(uv)\neq f(v)$;

(ⅱ) $\forall\ uv,\ vw\in E(G),\ f(uv)\neq f(vw)$;

(ⅲ) $\forall\ uv\in E(G),\ C(u)\neq C(v)$.

Then f is called a k-AVDTC of G, and the smallest number of k over all k-AVDTCs of G, $\chi_{at}(G)$, is called the AVDTC-chromatic number of G.

Definition 2[8] The join graph $G\vee H$ of two graphs G and H has the vertex set $V(G\vee H)=V(G)\cup V(H)$ and edge set $E(G\vee H)=E(G)\cup E(H)\cup\{uv\mid u\in V(G),\ v\in E(H)\}$.

Some notations and terminologies not mentioned are cited from [8].

1 Main results and proofs

In order to address the result clearly, some notations are needed. It is known that the wheel W_s is the join graph $K_1\vee C_s$ of an isolated vertex $v_0=K_1$ and a cycle $C_s=\{v_1,\ v_2,\ \cdots v_s\}$, thereby, it has the vertex set $V(W_s)=\{v_0,\ v_1,v_2,\cdots v_s\}$ and the edge set $E(W_s)=\{v_0v_i\mid 1\leq i\leq s\}\cup\{v_iv_{i+1}\mid 1\leq i\leq s \text{ and } v_{s+1}=v_1\}$. For $m\leq n$, let H_m and H_n be non-edge graphs, namely, $E(H_m)=\emptyset$ and $E(H_n)=\emptyset$, and let $V(H_m)=\{u_1,u_2,\cdots u_m\}$ and $V(H_n)=\{w_1,\ w_2,\cdots w_n\}$. A complete bipartite graph $K_{m,n}=H_m\vee H_n$ is defined, write its edge set $E(K_{m,n})=\{u_jw_k\mid 1\leq j\leq m,\ 1\leq k\leq n\}$. Successively, $V(W_s\vee K_{m,n})=V(W_s)\cup V(K_{m,n})$ are obtained and

$$E(W_s\vee K_{m,n})=E(W_s)\cup E(K_{m,n})\cup\{xy\mid x\in V(W_s),\ y\in V(K_{m,n}\},$$

where, in detail,

$$\{xy\mid x\in V(W_s),\ y\in V(K_{m,n})\}=\{v_iu_j\mid 0\leq i\leq s,\ 1\leq j\leq m\}\cup\{v_iw_k\mid 0\leq i\leq s,\ 1\leq k\leq n\}.$$

It is clear that $W_s\vee K_{m,n}=(K_1\vee C_s)\vee(H_m\vee H_n)$. The following Lemma 1 and Lemma 2 are obvious, so the proofs on them are omitted.

Lemma 1 Let $G(V,E)$ be a connected graph. If G has no two adjacent vertices of maximum degree, then $\chi_{at}(G)\geq\Delta(G)+1$, otherwise, have $\chi_{at}(G)\geq\Delta(G)+2$.

Lemma 2 Let f be a k-AVDTC of the join graph $W_s\vee K_{m,n}$, then:

(1) For $1\leq i\leq s$, $|C(v_i)|=|C(v_{i+1})|$, here $v_{s+1}=v_1$.

(2) If $m=n$, for $1\leq j,\ k\leq n$ then $|C(u_j)|=|C(w_k)|$.

(3) For $1\leq j\leq m,\ 1\leq k\leq n$, then $|C(u_j)|=|C(v_i)|$ if $s=m+2$ and $1\leq i\leq s$; and there are $|C(w_k)|=|C(v_i)|$ if $s=n+2$ and $1\leq i\leq s$.

Theorem 1 For $\chi_{at}(W_3\vee K_{m,n})=m+n+5$.

Proof Since there are at least two maximum degree vertices which are adjacent, then $\chi_{at}(W_3\vee K_{m,n})\geq m+n+5$ by Lemma 1. Only a $(m+n+5)$-AVDTC of $W_3\vee K_{m,n}$ is used for completing the proof.

A total coloring g: $V(W_3\vee K_{m,n})\cup E(W_3\vee K_{m,n})\to[m+n+4]^0$ is defined to the join graph $W_3\vee K_{m,n}$ as follows.

Firstly, all vertices and edges in the part W_3 of $W_3\vee K_{m,n}$ are colored. Setting $g(v_0)=0$, $g(v_i)=2i$ and $g(v_0v_i)=i$ for $1\leq i\leq 3$, $g(v_1v_2)=3$, $g(v_2v_3)=5$, $g(v_3v_1)=4$, $g(v_iu_j)=i+j+4\pmod{m+n+5}$ for $0\leq i\leq 3$ and $1\leq j\leq m$, and $g(v_iw_k)=i+k+m+4\pmod{m+n+5}$ for $0\leq i\leq 3$ and $1\leq k\leq n$.

Secondly, colors are assigned to the remaining part of $W_3\vee K_{m,n}$ inthe following two cases.

Case 1 $m>1$

$g(u_j w_k) = j + k + m + 7 \pmod{m + n + 5}$ is colored with respect to $1 \le j \le m$ and $1 \le k \le n$,

$$g(u_j) = \begin{cases} 1, & n = 2; \\ m + 7, & n = 3; \\ m + 8, & n > 3. \end{cases} \quad 1 \le j \le m - 1, \quad g(u_m) = \begin{cases} 1, & n = 2, 3; \\ m + 8, & n > 3, \end{cases}$$

as well as $g(w_k) = \begin{cases} 5, & m = 2; \\ m + 4, & m > 2. \end{cases} \quad 1 \le k \le n$.

Case 2 $m = 1$.

$g(u_1 w_n) = 4$, $g(u_1) = 3$, $g(w_k) = 5$ is set with $1 \le k \le n$. If $n = 2$, it must be $g(u_1 w_1) = 2$. If $n > 2$, then $g(u_1 w_k) = k + 9 \pmod{n + 6}$ for $1 \le k \le n - 1$.

This coloring g is shown as satisfying Definition 1. Under the coloring g, then $\bar{C}(v_i) = \{i + 4\} \pmod{m + n + 5}$ for $0 \le i \le 3$. When $m = 1$, if $n > 3$, then $\bar{C}(u_1) = \{9\}$; if $n = 3$, then $\bar{C}(u_1) = \{0\}$; if $n = 2$, then $\bar{C}(u_1) = \{1\}$; if $n = 1$, then $\bar{C}(u_1) = \{2\}$.

If $m = n = 1$, $\{2\} = \bar{C}(u_1) \ne \bar{C}(w_1) = \{3\}$. For $m = n = 2$, $\bar{C}(u_1) = \{0, 4\}$, $\bar{C}(u_2) = \{2, 5\}$, $\bar{C}(w_1) = \{4, 6\}$, $\bar{C}(w_2) = \{6, 7\}$. If $m = n = 3$, $0 \in \bar{C}(u_j)$, but $0 \notin \bar{C}(w_k)$ for $1 \le j, k \le 3$. When $m = n > 3$, $m + 3 \in \bar{C}(w_k)$ with $1 \le k \le n$, but $m + 3 \notin \bar{C}(u_j)$ for $m - 4 \le j \le m - 1$. And then $1 \in \bar{C}(u_m)$ and $m + 4 \in \bar{C}(u_t)$ for $1 \le t \le m - 5$, but these two numbers 1, $m + 4 \notin \bar{C}(w_k)$ when $1 \le k \le n$. Hence, $C(u_j) \ne C(w_k)$ has been verified for $1 \le j, k \le m$.

Gathering up all verifications together, the coloring g is exactly a $(m + n + 5)$-AVDTC of the graph $W_3 \vee K_{m,n}$, which gives out $\chi_{at}(W_3 \vee K_{m,n}) = m + n + 5$, as desired.

Theorem 2 $\chi_{at}(W_4 \vee K_{m,n}) = \begin{cases} n + 7, & m = 1; \\ m + n + 5, & m > 1. \end{cases}$

Proof For $m = 1$, notice that both v_0 and u_1 are two adjacent vertices of maximum degree in $W_4 \vee K_{1,n}$. Thereby, $\chi_{at}(W_4 \vee K_{1,n}) \ge n + 7$ by Lemma 1. A $(n + 7)$-AVDTC f is given to $W_4 \vee K_{1,n}$, namely, f: $V(W_4 \vee K_{1,n}) \cup E(W_4 \vee K_{1,n}) \to [n + 6]^0$.

Let $f(v_0) = 1$, $f(v_1) = f(v_3) = 3$, $f(v_2) = 4$, $f(v_4) = 7$; $f(v_0 v_i) = i + 1$ for $1 \le i \le 4$, $f(v_1 v_2) = 1$, $f(v_2 v_3) = 5$, $f(v_3 v_4) = 6$, $f(v_4 v_1) = 4$, $f(w_k) = 5$ for $1 \le k \le n$.

Set

$$f(v_i u_1) = \begin{cases} 6, & i = 0; \\ 5, & i = 1; \\ i + 5, & i = 2, 3, \end{cases} \quad \text{and } f(v_4 u_1) = \begin{cases} 3, & n = 1; \\ 9 \pmod{n + 7}, & n > 1, \end{cases}$$

$$f(u_1) = \begin{cases} 2, & n = 1, 2; \\ 0, & n = 3; \\ 10, & n > 3, \end{cases} \quad \text{and } f(v_i w_k) = \begin{cases} k + 6, & i = 0; \\ k + 5, & i = 1; \\ k + i + 5 \pmod{n + 7}, & i = 2, 3, 4; \end{cases} \quad 1 \le k \le n.$$

$f(u_1 w_k) = k + 10 \pmod{n + 7}$, for $n \ge 3$ and $1 \le k \le n$.

For $n = 1$, let $f(u_1 w_1) = 4$. If $n = 2$, then $f(u_1 w_1) = 3$, $f(u_1 w_2) = 4$. Obviously, f is a total coloring from $V(W_4 \vee K_{1,n}) \cup E(W_4 \vee K_{1,n})$ to $[n + 6]^0$.

Under this coloring f, then $\bar{C}(v_1) = \{n + 6, 0\}$, $\bar{C}(v_2) = \{2, 6\}$, $\bar{C}(v_3) = \{2, 7\}$. If $n > 1$, then $\bar{C}(v_4) = \{3, 8\}$. When $n = 1$, then $\bar{C}(v_4) = \{1, 8\}$, $\bar{C}(v_0) = \{0\}$, $\bar{C}(u_1) = \{1\}$ and $\bar{C}(w_1) = \{3\}$. When $n = 2$, $\bar{C}(w_1) = \{2, 4\}$ and $\bar{C}(w_2) = \{3, 6\}$.

It is easy to verify that f is a $(n + 7)$-AVDTCof $W_4 \vee K_{1,n}$, that is $\chi_{at}(W_4 \vee K_{1,n}) = n + 7$.

The case of $m > 1$ is dealt with. Since v_0 is the unique vertex of maximum degree in $W_4 \vee K_{m,n}$, $\chi_{at}(W_4 \vee K_{m,n}) \ge m + n + 5$ according to Lemma 1. Therefore it is only needed to prove that there exists a $(m + n + 5)$-AVDTC of $W_4 \vee K_{m,n}$. A total coloring f is constructed from $V(W_4 \vee K_{m,n}) \cup E(W_4 \vee K_{m,n})$ to $[m + n + 4]^0$ as follows. Let $f(v_0)=1$, $f(v_1)=f(v_3)=3$, $f(v_2)=4$, $f(v_0v_i)=i+1$ for $1\le i\le 4$, $f(v_1v_2)=1$, $f(v_3v_4)=6$. The following cases have to be colored.

Case 1 $m=n=2$.

Set $f(v_4)=4$, $f(v_2v_3)=2$, $f(v_4v_1)=0$,

$$f(v_iu_j)=\begin{cases} j+5, & i=0;\\ i+j+2, & i=1,2;\ j=1,2,\ \text{except } i=4 \text{ and } j=2,\\ i+j+3, & i=3,4;\end{cases}$$

$$f(v_iw_k)=\begin{cases} k+7 \pmod 9, & i=0;\\ i+k+4, & i=1,2;\end{cases} k=1,2.$$

Other labels are $f(v_3w_1)=0$, $f(v_3w_2)=1$, $f(v_4w_k)=k+1$ and $f(u_1w_k)=k$ for $k=1,2$, $f(v_4u_2)=1$, $f(u_j)=0$, $j=1,2$, as well as $f(u_2w_k)=k+2$ and $f(w_k)=5$ for $k=1,2$.

It is easy to see that f is a 9-AVDTC, which shows $\chi_{at}(W_4\vee K_{2,2})=9$.

Case 2 $2\le m<n$ and $3\le m\le n$. Let $L=m+n+5$. Set $f(v_4)=7$, $f(v_2v_3)=5$, $f(v_4v_1)=4$, $f(v_2w_n)=6$,

$$f(v_iu_j)=\begin{cases} j+5, & i=0;\\ j+4, & i=1;\\ i+j+4 \pmod L, & i=2,3,4;\end{cases} \quad 1\le j\le m,$$

$$f(v_iw_k)=\begin{cases} k+m+5 \pmod L, & i=0;\\ k+m+4, & i=1;\\ k+i+m+4 \pmod L, & i=2,3,4;\end{cases} \quad 1\le k\le n,\ \text{except } k=n \text{ and } i=2.$$

$f(u_jw_k)=k+j+m+8 \pmod L$ for $1\le j\le m$ and $1\le k\le n$, except the case of $k=n$ and $j=3$.

$$f(u_3w_n)=\begin{cases} m+11 \pmod L, & n\ge 8;\\ n+m+3, & n\le 7, \text{ but } m+n>8;\\ 1, & n\le 7, \text{ but } m+n\le 8.\end{cases}$$

When $n=3$, $f(u_j)=m+8 \pmod L$ for $1\le j\le m-1$ and $f(u_m)=2$. When $n=6$, $f(u_j)=m+9$ for $1\le j\le m$ except $j=3$, $f(u_3)=m+10$. When $n\ne 3,6$, then $f(u_j)=m+9$ with $1\le j\le m$. When $m=2,3$, $f(w_k)=6$ for $1\le k\le n-1$ as well as $f(w_n)=8$. When $m\ne 2,3$, $f(w_k)=m+4$ for $1\le k\le n$.

Obviously, f is a total coloring from $V(W_4\vee K_{m,n})\cup E(W_4\vee K_{m,n})$ to $[m+n+4]^0$. It is going to verify that the coloring f satisfies Definition 1.

We have $\bar{C}(v_1)=\{0\}$, $\bar{C}(v_2)=\{2\}$, $\bar{C}(v_3)=\{7\}$, $\bar{C}(v_4)=\{8\}$.

When $m=2$, if $n=3$, then $\bar{C}(u_1)=\bar{C}(u_2)=\{1\}$; if $n=4$, then $\bar{C}(u_1)=\{10\}$, $\bar{C}(u_2)=\{1\}$; if $n>4$, then $\bar{C}(u_1)=\{10\}$, $\bar{C}(u_2)=\{12\} \pmod L$.

When $m=n=3$, $7\in\bar{C}(w_k)$, but $7\notin\bar{C}(u_j)$ for $1\le j,\ k\le 3$.

When $m=n\ge 4$, $\bar{C}(w_1)=\{5,6,\cdots m+3\}$, $\bar{C}(w_k)=\{4+k,\ \cdots\ m+3\}\cup\{m+5,\ \cdots\ m+k+3\}$ for $2\le k\le m-1$.

If $n\ge 8$, then $\bar{C}(w_m)=\{1,m+5,m+6,\cdots m+10,m+12,\cdots 2m+3\}$.

If $n\le 7$ and $m+n>8$, then $\bar{C}(w_m)=\{1,m+5,m+6,\cdots 2m+2\}$.

If $n\le 7$ and $m+n\le 8$, then $\bar{C}(w_m)=\{m+5,m+6,\cdots 2m+3\}$. But $\bar{C}(u_1)=\{10,11,\cdots m+8\}$, and $\bar{C}(u_2)=\{1,11,12\}$, $\bar{C}(u_3)=\{2,6,12\}$, $\bar{C}(u_4)=\{1,2,3\}$ when $m=n=4$.

When $m=n\ge 5$, $\bar{C}(u_j)=\{9+j,\cdots m+8\}\cup\{m+10,\cdots m+j+8\} \pmod L$ for $2\le j\le m-1$ and $j\ne 3$, $\bar{C}(u_m)=\{m+10,\cdots 2m+4\}\cup\{0,1,2,3\}$.

If $m=n=5$, then $\bar{C}(u_3)=\{0,1,6,12\}$. If $m=n\ge 6$, then

$$\bar{C}(u_3)=\{6\}\cup\{12,13,\cdots m+8\}\cup\{m+11\} \pmod L.$$

Furthermore, it is shown that f is a $(m+n+5)$-AVDTC of $W_4 \vee K_{m,n}$, namely, $\chi_{at}(W_4 \vee K_{m,n}) = m+n+5$. The proof of this theorem is completed.

Theorem 3 For integer $s \geq 5$, $\chi_{at}(W_s \vee K_{m,n}) = \begin{cases} s+n+3, & m=1; \\ s+m+n+1, & m>1. \end{cases}$

Proof In the case of $m=1$, both v_0 and u_1 are two adjacent vertices of maximum degree of $W_s \vee K_{m,n}$, $\chi_{at}(W_s \vee K_{1,n}) \geq s+n+3$ by Lemma Lemma 1. A definition is needed for a $(s+n+3)$-AVDTC of $W_s \vee K_{1,n}$ from $V(W_s \vee K_{1,n}) \cup E(W_s \vee K_{1,n})$ to $[s+n+2]^0$.

Let $M = s+n+3$. Set $f(v_0)=1$, $f(v_0v_i)=i+1$ for $1 \leq i \leq s$, $f(v_sv_1)=4$,

$$f(v_i) = \begin{cases} 3, & i \text{ is odd;} \\ 4, & i \text{ is even.} \end{cases} \quad i=1,2,\cdots s-1, \qquad f(v_s) = \begin{cases} 8, & s=5; \\ s, & s \neq 5, \end{cases}$$

$$f(v_iu_1) = \begin{cases} s+2, & i=0; \\ i+4, & 1 \leq i \leq s-3; \\ i+5 \pmod M, & i=s-2,\ s-1, \end{cases} \qquad f(v_su_1) = \begin{cases} 3, & n=1; \\ s+5 \pmod M, & n>1, \end{cases}$$

$$f(v_iw_k) = \begin{cases} k+s+2, & i=0; \\ k+i+4, & 1 \leq i \leq s-3; \\ k+i+5 \pmod M, & i=s-2,\ s-1,\ s, \end{cases} \quad 1 \leq k \leq n.$$

$f(u_1w_k) = k+s+6 \pmod M$ for $n \geq 3$ and $1 \leq k \leq n$. When $n=1$, $f(u_1w_1)=4$. When $n=2$, $f(u_1w_1)=3$ and $f(u_1w_2)=4$. Then $f(v_iv_{i+1}) = \begin{cases} 1, & i \text{ is odd;} \\ 2, & i \text{ is even.} \end{cases}$ for $i=1,2,\cdots s-l$, and $f(v_tv_{t+1}) = t+3$ with $t=s-l+1, \cdots s-1$ where $l=3$ if s is odd and $l=4$ if s is even. $f(u_1) = \begin{cases} 2, & n=1,2; \\ s+6 \pmod M, & n \neq 1,2. \end{cases}$ and $f(w_k)=5$ for $1 \leq k \leq n$.

Obviously, f is a total coloring from $V(W_s \vee K_{1,n}) \cup E(W_s \vee K_{1,n})$ to $[s+n+2]^0$.

Under this coloring f, $C(v_i) \neq C(v_{i+1})$, $i=0,1,2,\cdots s$ when $i=s$, $v_{i+1}=v_1$.

When $n \leq 2$, $\bar{C}(u_1)=\{1\}$. When $n>2$, $\bar{C}(u_1)=\{4\}$. When $m=n=1$, $\bar{C}(w_1)=\{3\}$. when $s \geq 7$ and $s=n+2$, $5 \in C(w_k)$, $1 \leq k \leq n$, but $5 \notin C(v_l)$, $2 \leq l \leq s$, $l \neq 4$. Furthermore, $\bar{C}(v_1)=\{n+6, n+7, \cdots s+n+2, 0\} \neq \bar{C}(w_k)$, $1 \leq k \leq n$, $\bar{C}(v_4)=\{3,6,7,n+9,\cdots s+n+2,0\} \neq \bar{C}(w_k)$ for $1 \leq k \leq n$. When $s=5,6$, $\{3,8\} \subseteq \bar{C}(v_4)$, but $\{3,8\} \nsubseteq \bar{C}(w_k)$, $10 \in \bar{C}(v_1)$, but $10 \notin \bar{C}(w_k)$ with $1 \leq k \leq n$.

Clearly, this coloring f is a $(s+n+3)$-AVDTC of $W_s \vee K_{1,n}$, thus, $\chi_{at}(W_s \vee K_{1,n}) = s+n+3$.

For the case of $m>1$, $\chi_{at}(W_s \vee K_{m,n}) \geq s+m+n+1$ by means of Lemma 1, because that v_0 is the unique vertex of maximum degree of $W_s \vee K_{m,n}$. The key is to prove the existence of a $(s+m+n+1)$-AVDTC of $W_s \vee K_{m,n}$. A total coloring f is defined from $V(W_s \vee K_{m,n}) \cup E(W_s \vee K_{m,n})$ to $[s+m+n]^0$ in the following.

Let $R = s+m+n+1$. We let $f(v_0)=1$, $f(v_0v_i)=i+1$ for $1 \leq i \leq s$, $f(v_sv_1)=4$,

$$f(v_i) = \begin{cases} 3, & i \text{ is odd;} \\ 4, & i \text{ is even.} \end{cases} \quad i=1,2,\cdots s-1.\ f(v_s)=s;$$

$$f(v_iu_j) = \begin{cases} s+j+1, & i=0; \\ j+i+3, & 1 \leq i \leq s-3;\ 1 \leq j \leq m, \text{ except } i=s \text{ and } j=m. \\ j+i+4 \pmod R, & s-2 \leq i \leq s \end{cases}$$

$$f(v_su_m) = \begin{cases} 1, & m=n=2; \\ s+m+4 \pmod R, & \text{otherwise.} \end{cases}$$

$$f(v_iw_k) = \begin{cases} s+m+k+1 \pmod R, & i=0; \\ i+m+k+3, & 1 \leq i \leq s-3;\ 1 \leq k \leq n. \\ i+m+k+4 \pmod R, & s-2 \leq i \leq s. \end{cases}$$

Again, $f(u_jw_k)=s+m+j+k+4 \pmod R$, $1\le j\le m$, $1\le k\le n$, and

$$f(u_j)=\begin{cases}2, & n=2;\\ s+m+4 \pmod R, & n=3;\\ s+m+5 \pmod R, & \text{otherwise}.\end{cases} \quad 1\le j\le m-1.$$

$$f(u_m)=\begin{cases}2, & n=2,3;\\ s+m+5 \pmod R, & \text{otherwise}.\end{cases}$$

$$f(w_1)=\begin{cases}m+3, & m+4=s;\\ m+4, & \text{otherwise}.\end{cases} \quad f(w_k)=\begin{cases}m+4, & m+4\ne s;\\ m+5, & \text{otherwise}.\end{cases} \quad 2\le k\le n-1.$$

When $s=5$ and $m=n=3$, $f(w_n)=8$, otherwise $f(w_n)=\begin{cases}m+4, & m+4\ne s;\\ m+5, & \text{otherwise}.\end{cases}$

$f(v_iv_{i+1})=\begin{cases}1, & i \text{ is old};\\ 2, & i \text{ is even}.\end{cases}$ $i=1,2,\cdots s-l$, and $f(v_tv_{t+1})=t+3$, $t=s-l+1,\cdots s-1$ where $l=3$ if s is odd and $l=4$ if s is even.

Therefore, f is a proper total coloring from $V(W_s\vee K_{m,n})\cup E(W_s\vee K_{m,n})$ to $[s+m+n]^0$. Under this coloring f, $C(v_i)\ne C(v_{i+1})$ for $0\le i\le s$, here $v_{s+1}=v_1$. And $C(w_k)\ne C(v_i)\ne C(u_j)$ for $1\le j\le m$, $1\le k\le n$ and $1\le i\le s$. Several cases are considered in detail.

Case 1 When $m=n=2$, $\bar{C}(u_1)=\{1\}$ and $\bar{C}(u_2)=\{3\}$.

If $m+4=s$, then $\bar{C}(w_1)=\bar{C}(w_2)=\{6\}$. If $m+4\ne s$, so $\bar{C}(w_1)=\{5\}$ and $\bar{C}(w_2)=\{7\}$.

Case 2 When $m=n=3$, $1\in\bar{C}(u_j)$, but $1\notin\bar{C}(w_k)$ for $1\le j,\ k\le 3$.

Case 3 When $m=n>3$, $\bar{C}(u_1)=\{s+6,\cdots s+m+4\}$, and

$\bar{C}(u_j)=\{s+j+5,\cdots\ s+m+4\}\cup\{s+m+6,\cdots s+m+j+4\} \pmod R$ for $2\le j\le m$.

If $m+4=s$, then $\bar{C}(w_1)=\{5,6,\cdots\ m+2,\ m+4\}$, $\bar{C}(w_2)=\{6,7,\cdots m+4\}$, $\bar{C}(w_k)=\{k+4,\cdots m+4\}\cup\{m+6,\cdots m+k+3\}$ for $3\le k\le n$.

If $m+4\ne s$, then $\bar{C}(w_1)=\{5,6,\cdots m+3\}$, $\bar{C}(w_k)=\{k+4,\cdots m+3\}\cup\{m+5,\cdots m+k+3\}$ for $2\le k\le n-1$, and $\bar{C}(w_n)=\{m+5,\cdots m+n+3\}$.

At last, $C(u_j)\ne C(w_k)$ for $1\le j,k\le m$ is obtained, it means that f is a $(s+m+n+1)$-AVDTC of $W_s\vee K_{m,n}$ as claimed.

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