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广义经典完整非保守力学系统 Lie 对称性及其守恒量

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摘要:研究了广义完整非保守力学系统的 Lie 对称性及其守恒量. 建立了系统的运动微分方程,给出其确定方程、结构方程和守恒量,得到了系统的 Lie 对称性定理和逆定理,最后举例说明结果的应用.

关键词:广义力学系统; Lie 对称性; 守恒量
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Lie symmetries and conserved quantities of holonomic non-conservative dynamical systems in generalized classical mechanics

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Abstract: Lie symmetries and conserved quantities of holonomic non-conservative dynamical systems in generalized classical mechanics were studied. First, the determining equations of Lie symmetries are established by using Lie's invariance method of infinitesimal transformations of differential equations. Second, the structure equations and the conserved quantities were obtained. Third, the inverse Lie symmetries system problems were discussed. Finally, an example to illustrate the application of the results was given.

Key words: generalized mechanics system; Lie symmetries; conserved quantities

0 引言

描述动力学系统的 Lagrange 量含广义坐标对时间高阶微商情形(广义力学系统)的研究,在数学、力学和物理学等诸多领域广为应用. Ostrogradsky 和 Jacobi 最早对其进行了开创性研究[1],文献[2]用现代数学方法描述了广义经典力学系统和场论. 近年来,对于此类系统的研究更加受到人们的关注,并取得许多研究成果[3-11].

动力学系统的守恒量在力学研究中具有十分重要的作用,它不仅能使复杂的运动微分方程易于求解,而且反映着深刻的物理本质. 对称性和守恒量之间的联系在经典理论中通常由 Noether 定理与 Lie 对称性定理给出[12-17],文献[18]研究了广义经典完整非保守力学系统的 Noether 定理,本文研究系统的 Lie 对称性,建立系统逆变代数形式的运动方程,给出系统的确定方程、结构方程和守恒量,得到 Lie 对称性定理及其逆定理,最后举例说明结果的应用.

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1 系统的运动微分方程

研究广义完整非保守力学系统. 设系统的位形由 n 个广义坐标 $q_s(s=1,\cdots n)$确定,其 Lagrange 函数 L 是时间 t,广义坐标 $q_s(s=1,\cdots n)$及其一阶到 k 阶导数的函数,系统受广义非势力 $Q_s=Q_s(q_s,\dot{q}_s,\cdots\overset{(k)}{q}_s,t)$,则广义 Euler-Lagrange 方程为

$$\sum_{j=0}^{k}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial\overset{(j)}{q}_s}+Q_s(q_s,\dot{q}_s,\cdots\overset{(k)}{q}_s,t)=0 \tag{1}$$

引入广义动量和哈米顿函数

$$p_{s/m}=\sum_{j=0}^{k-m}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial\overset{(j+m)}{q}_s},\ (s=1,\cdots n;\ m=1,\cdots k) \tag{2}$$

$$H(\overset{(m-1)}{q}_s,p_{s/m},t)=\sum_{s=1}^{n}\sum_{m=1}^{k}p_{s/m}\overset{(m)}{q}_s-L, \tag{3}$$

则方程(1)可写成正则形式

$$\frac{\mathrm{d}}{\mathrm{d}t}\overset{(m-1)}{q}_s=\frac{\partial H}{\partial p_{s/m}},\ \frac{\mathrm{d}}{\mathrm{d}t}p_{s/m}=-\frac{\partial H}{\partial\overset{(m-1)}{q}_s}+\widetilde{Q}_s(\overset{(m-1)}{q}_s,p_{s/m},t).\ (s=1,\cdots n;\ m=1,\cdots k) \tag{4}$$

式中

$$\widetilde{Q}_s(\overset{(m-1)}{q}_s,p_{s/m},t)=Q_s(\overset{(m-1)}{q}_s,\overset{(m)}{q}_s(\overset{(m-1)}{q}_s,p_{s/m},t),t).\ (s=1,\cdots n;\ m=1,\cdots k) \tag{5}$$

令

$$\widetilde{Q}_s=\Omega_{s/m}\frac{\partial H}{\partial p_{s/m}}, \tag{6}$$

$$a^{\mu}=\begin{cases}\overset{(m-1)}{q}_s,(\mu=(2m-2)n+s;\ m=1,\cdots k;\ s=1,\cdots n)\\ p_{s/m},(\mu=(2m-1)n+s;\ m=1,\cdots k;\ s=1,\cdots n)\end{cases} \tag{7}$$

$$R_{\mu}=\begin{cases}p_{s/m},(\mu=(2m-2)n+s;\ m=1,\cdots k;\ s=1,\cdots n)\\ 0,(\mu=(2m-1)n+s;\ m=1,\cdots k;\ s=1,\cdots n)\end{cases} \tag{8}$$

$$\Lambda_{\mu}=\begin{cases}\Omega_{s/m}\dfrac{\partial H}{\partial p_{s/m}},(\mu=(2m-2)n+s;\ m=1,\cdots k;\ s=1,\cdots n)\\ 0.(\mu=(2m-1)n+s;\ m=1,\cdots k;\ s=1,\cdots n)\end{cases} \tag{9}$$

则方程(4)可进一步表示为

$$\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}-\frac{\partial H}{\partial a^{\mu}}=-\Lambda_{\mu}.\ (\mu,\nu=1,\cdots 2kn) \tag{10}$$

与式(10)相对应的逆变代数方程为[19]

$$\dot{a}^{\mu}-s^{\mu\nu}\frac{\partial H}{\partial a^{\mu}}=0.\ (\mu,\nu=1,\cdots 2kn) \tag{11}$$

此处

$$s^{\mu\nu}=\omega^{\mu\nu}+T^{\mu\nu}. \tag{12}$$

其中

$$\omega^{\mu\nu}=\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}},\ (\omega_{\mu\nu})=(\omega^{\mu\nu})^{-1}, \tag{13}$$

且

$$\begin{cases}(\boldsymbol{\omega}^{\mu\nu})=\begin{bmatrix}M_1 & & \\ & \ddots & \\ & & M_k\end{bmatrix}_{2nk\times 2nk},\\ M_i=\begin{bmatrix}0_{n\times n} & I_{n\times n}\\ -I_{n\times n} & 0_{n\times n}\end{bmatrix}.\ (i=1,\cdots k)\end{cases} \tag{14}$$

$$\left\{\begin{aligned}&(\boldsymbol{T}^{\mu\nu})=\begin{bmatrix}0_{n\times n} &&&&&&\\ & H_1 &&&&&\\ && \ddots &&&&\\ &&& 0_{n\times n} &&&\\ &&&& H_i &&\\ &&&&& \ddots &\\ &&&&&& H_k\end{bmatrix}_{2nk\times 2nk},\\ &\boldsymbol{H}_i=\begin{bmatrix}\Omega_{1/m} &&&\\ & \Omega_{2/m} &&\\ && \ddots &\\ &&& \Omega_{s/m}\end{bmatrix}.\ (m=i;\ i=1,2,\cdots k;\ s=1,2,\cdots n.)\end{aligned}\right. \tag{15}$$

式(14)中,$0_{n\times n}$是 n 阶零矩阵,$\boldsymbol{I}_{n\times n}$是 n 阶单位矩阵.

2　系统的 Lie 对称性与守恒量

取 r 参数变换群的无穷小变换

$$\left\{\begin{aligned}&t^{*}=t+\Delta t,\\ &a^{\mu^{*}}=a^{\mu}+\Delta a^{\mu}.\end{aligned}\right.\quad(\mu=1,\cdots 2kn) \tag{16}$$

以及它的扩展形式

$$\left\{\begin{aligned}&t^{*}=t+\varepsilon_{\alpha}\xi_0^{\alpha}(t,a),\\ &a^{\mu^{*}}=a^{\mu}+\varepsilon_{\alpha}\xi_{\mu}^{\alpha}(t,a).\end{aligned}\right.\quad(\mu=1,\cdots 2kn:\alpha=1,\cdots r) \tag{17}$$

这里 ε_{α} 是无穷小参数,ξ_0^{α},ξ_{μ}^{α} 是无穷小变换生成元. 引入无穷小变换生成元向量

$$X^{(0)}=\xi_0^{\alpha}\frac{\partial}{\partial t}+\sum_{\mu=1}^{2kn}\xi_{\mu}^{\alpha}\frac{\partial}{\partial a^{\mu}}.\ (\alpha=1,\ \cdots r) \tag{18}$$

它的一次扩展

$$X^{(1)}=X^{(0)}+\sum_{\mu=1}^{2kn}(\dot{\xi}_{\mu}^{\alpha}-\dot{a}^{\mu}\dot{\xi}_0^{\alpha})\frac{\partial}{\partial\dot{a}^{\mu}}.\ (\alpha=1,\cdots r) \tag{19}$$

那么微分方程(12)在无穷小变换(17)下的不变性归结为确定方程:

$$\dot{\xi}_{\mu}^{\alpha}-\sum_{\nu=1}^{2kn}s^{\mu\nu}\left(\frac{\partial H}{\partial a^{\nu}}+\Lambda_{\nu}\right)\dot{\xi}_0^{\alpha}=X^{(0)}\left[\sum_{\nu=1}^{2kn}s^{\mu\nu}\left(\frac{\partial H}{\partial a^{\nu}}+\Lambda_{\nu}\right)\right].\ (\mu=1,\cdots 2kn) \tag{20}$$

定义 1　如果无穷小变换(17)的生成元 ξ_0^{α}, ξ_{μ}^{α} 满足确定方程(20),那么相应的对称性称为广义完整非保守力学系统的 Lie 对称性.

于是,有下述定理存在

定理　对于满足确定方程(20)的无限小生成元 ξ_0^{α},ξ_{μ}^{α},如果存在规范函数 $G^{\alpha}=G^{\alpha}(t,a)$满足结构方程

$$\left(\sum_{\mu=1}^{2kn}R_{\mu}\dot{a}^{\mu}-H\right)\dot{\xi}_0^{\alpha}+X^{(1)}\left(\sum_{\mu=1}^{2kn}R_{\mu}\dot{a}^{\mu}-H\right)-\sum_{\mu=1}^{2kn}\Lambda_{\mu}(\xi_{\mu}-\dot{a}^{\mu}\xi_0)+\dot{G}^{\alpha}=0,\qquad(\alpha=1,\cdots r) \tag{21}$$

则系统存在如下 Lie 对称性守恒量

$$I=\sum_{\mu=1}^{2kn}R_{\mu}\xi_{\mu}^{\alpha}-H\xi_0^{\alpha}+G^{\alpha}=\mathrm{const}.\ (\alpha=1,\cdots r) \tag{22}$$

证

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&\sum_{\mu=1}^{2kn}\frac{\partial R_{\mu}}{\partial a^{\nu}}\dot{a}^{\nu}\xi_{\mu}^{\alpha}+\sum_{\mu=1}^{2kn}R_{\mu}\dot{\xi}_{\mu}^{\alpha}-\left(\frac{\partial H}{\partial t}+\sum_{\mu=1}^{2kn}\frac{\partial H}{\partial a^{\nu}}\dot{a}^{\nu}\right)\xi_0^{\alpha}-H\dot{\xi}_0^{\alpha}-\left(\sum_{\mu=1}^{2kn}R_{\mu}\dot{a}^{\mu}-H\right)\dot{\xi}_0^{\alpha}-\\ &\sum_{\mu=1}^{2kn}(\dot{\xi}_{\mu}^{\alpha}-\dot{a}^{\mu}\dot{\xi}_0^{\alpha})R_{\mu}-\sum_{\mu=1}^{2kn}\sum_{\nu=1}^{2kn}\frac{\partial R_{\mu}}{\partial a^{\nu}}\xi_{\mu}^{\alpha}\dot{a}^{\nu}+\frac{\partial H}{\partial t}\xi_0^{\alpha}+\sum_{\mu=1}^{2kn}\frac{\partial H}{\partial a^{\mu}}\xi_{\mu}^{\alpha}+\sum_{\mu=1}^{2kn}\Lambda_{\mu}(\xi_{\mu}^{\alpha}-\dot{a}^{\mu}\xi_0^{\alpha})=\\ &\sum_{\mu=1}^{2kn}(\xi_{\mu}^{\alpha}-\dot{a}^{\mu}\xi_0^{\alpha})\left[\frac{\partial H}{\partial a^{\mu}}+\Lambda_{\mu}-\sum_{\nu=1}^{2kn}\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}\right]-\sum_{\mu=1}^{2kn}\sum_{\nu=1}^{2kn}\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\mu}\dot{a}^{\nu}\xi_0^{\alpha}=0.\end{aligned}$$

显然,系统存在守恒量式(22).

3 系统 Lie 对称性的逆定理

假设已知广义完整非保守力学系统(1)有初积分

$$I = I(t, a) = \text{const}. \tag{23}$$

则

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \sum_{\mu=1}^{2kn} \frac{\partial I}{\partial a^{\mu}} \dot{a}^{\mu} = 0. \tag{24}$$

将系统运动方程(10)的两端乘以 $\bar{\xi}_{\mu}^{\alpha} = \xi_{\mu}^{\alpha} - \dot{a}^{\mu}\xi_0^{\alpha}$, 并对 s 求和,得

$$\sum_{\mu=1}^{2kn} \bar{\xi}_{\mu}^{\alpha} \left(\sum_{\mu=1}^{2kn} \omega^{\prime\mu\nu} \dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} + \Lambda_{\mu} \right) = 0.\ (\alpha = 1, \cdots r) \tag{25}$$

将式(24)与(25)相加后,分离出含 $\dot{a}^{\nu}$ 的项,并令其系数为 0,得到

$$\frac{\partial I}{\partial a^{\nu}} + \sum_{\mu=1}^{2kn} \omega_{\mu\nu} \xi_{\mu}^{\alpha} + \left(\frac{\partial H}{\partial a^{\mu}} - \Lambda_{\nu} \right) \xi_0^{\alpha} = 0.\ (\nu = 1, \cdots 2kn,\ \alpha = 1, \cdots r) \tag{26}$$

考虑到式(13),有

$$\xi_{\mu}^{\alpha} = \sum_{\nu=1}^{2kn} \omega^{\prime\mu\nu} \left[\frac{\partial I}{\partial a^{\nu}} + \left(\frac{\partial H}{\partial a^{\nu}} - \Lambda_{\nu} \right) \xi_0^{\alpha} \right].\ (\mu = 1, \cdots 2kn,\ \alpha = 1, \cdots r) \tag{27}$$

其中

$$\sum_{\nu=1}^{2kn} \omega^{\prime\mu\nu} \omega_{\nu\rho} = \delta_{\mu\rho}.\ (\mu, \rho = 1, \cdots 2kn) \tag{28}$$

为使式(17)为 Lie 对称变换,令初积分式(23)等于守恒量式(22),得

$$\sum_{\mu=1}^{2kn} R_{\mu} \xi_{\mu}^{\alpha} - H\xi_0^{\alpha} + G^{\alpha} = I.\ (\alpha = 1, \cdots r) \tag{29}$$

定义 2 如果式(27)和(29)确定的无限小生成元 $\xi_0^{\alpha}, \xi_{\mu}^{\alpha}$ 满足确定方程(20). 则称相应的变换为 Lie 对称变换.

4 举例

设力学系统的 Lagrange 函数为

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}q_1^2\dot{q}_2^2 + \frac{1}{2}\ddot{q}_1^2. \tag{30}$$

系统受有非势广义力为 $Q = -\ddot{q}_1$,试研究系统的对称性与守恒量.

解 首先研究 Lie 对称性正问题

第一步 列写系统的运动微分方程

由式(2)求得

$$P_{1/1} = \frac{\partial L}{\partial \dot{q}_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}_1} = \dot{q}_1 - \dddot{q}_1,$$

$$P_{1/2} = \frac{\partial L}{\partial \ddot{q}_1} = \ddot{q}_1,$$

$$P_{2/1} = \frac{\partial L}{\partial \dot{q}_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}_2} = q_1^2\dot{q}_2,$$

$$P_{2/2} = \frac{\partial L}{\partial \ddot{q}_2} = 0. \tag{31}$$

式(3)给出

$$H = P_{1/1}\dot{q}_1 + P_{1/2}\ddot{q}_1 + P_{2/1}\dot{q}_2 - \frac{1}{2}\dot{q}_1^2 - \frac{1}{2}q_1^2\dot{q}_2^2 - \frac{1}{2}\ddot{q}_1^2, \tag{32}$$

令

$$\begin{aligned} &a^1 = q_1, \quad a^2 = q_2, \quad a^3 = P_{1/1}, \quad a^4 = p_{2/1}, \\ &a^5 = \dot{q}_1, \quad a^6 = \dot{q}_2, \quad a^7 = P_{1/2}, \quad a^8 = 0, \\ &R_1 = P_{1/1}, \quad R_2 = P_{2/1}, \quad R_3 = 0, \quad R_4 = 0, \\ &R_5 = P_{1/2}, \quad R_6 = P_{2/2}, \quad R_7 = 0, \quad R_8 = 0. \end{aligned} \tag{33}$$

则有

$$H = a^3 a^5 + \frac{1}{2}(a^7)^2 + a^4 a^6 - \frac{1}{2}(a^5)^2 - \frac{1}{2}(a^1)^2(a^6)^2. \tag{34}$$

再令

$$\tilde{Q}_1 = Q = -\ddot{q}_1, \ \tilde{Q}_2 = 0.$$

由式(6),(9)和(33),得

$$\Lambda_1 = -a^7, \ \Lambda_2 = 0, \ \Lambda_3 = 0, \ \Lambda_4 = 0, \ \Lambda_5 = -a^7, \ \Lambda_6 = 0, \ \Lambda_7 = 0, \ \Lambda_8 = 0. \tag{35}$$

式(12)给出

$$\boldsymbol{S}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -a^7/a^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{36}$$

据式(11),系统的运动微分方程为

$$\begin{aligned} &\dot{a}^1 = a^5, \ \dot{a}^2 = a^6, \ \dot{a}^3 = a^1(a^6)^2 - a^7, \ \dot{a}^4 = 0, \\ &\dot{a}^5 = a^7, \ \dot{a}^6 = 0, \ \dot{a}^7 = a^5 - a^4 - a^7, \ \dot{a}^8 = (a^1)^2 a^6 - a^4. \end{aligned} \tag{37}$$

第二步　建立确定方程并求解

取 $r = 1$,确定方程式(20)给出

$$\begin{cases} \dot{\xi}_1 - a^5 \dot{\xi}_0 = \xi_5, \\ \dot{\xi}_2 - a^6 \dot{\xi}_0 = \xi_6, \\ \dot{\xi}_3 + [a^1(a^6)^2 + a^7]\dot{\xi}_0 = (a^6)^2 \xi_1 + 2a^1 a^6 \xi_6 + \xi_7, \\ \dot{\xi}_4 = 0, \\ \dot{\xi}_5 - a^7 \dot{\xi}_0 = \xi_7, \\ \dot{\xi}_6 = 0, \\ \dot{\xi}_7 + (a^3 - a^5 + a^7)\dot{\xi}_0 = \xi_3 - \xi_5 + \xi_7, \\ \dot{\xi}_8 = \xi_4 - 2a^1 a^6 \xi_1 - (a^1)^2 \xi_6. \end{cases} \tag{38}$$

取无穷小变换生成元

$$\xi_0 = -1, \ \xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = \xi_6 = \xi_6 = \xi_7 = \xi_8 = 0. \tag{39}$$

$$\xi_0 = 0, \ \xi_1 = 0, \ \xi_2 = 1, \ \xi_3 = \xi_4 = \xi_5 = \xi_6 = \xi_7 = \xi_8 = 0. \tag{40}$$

显然,生成元式(39)和(40)满足确定方程(37),因此,它们对应系统的 Lie 对称,对于生成元(39),结构方程(21)给出

$$a^7(a^5 + a^7) + \dot{G} = 0. \tag{41}$$

由此找不到规范函数,对于生成元(40),结构方程(21)给出

$$G = 0. \tag{42}$$

将式(40)和(42)代入式(22),得系统的守恒量

$$I = a^4 = \text{const}. \tag{43}$$

其次,研究 Lie 对称性逆问题. 假设系统有守恒量(43),求与其相应的 Lie 对称性.

方程(27),(29)给出

$$\begin{cases} \xi_1 = a^5 \xi_0, \\ \xi_2 = 1 + a^6 \xi_0, \\ \xi_3 = [a^1 (a^6)^2 - a^7] \xi_0, \\ \xi_4 = 0, \\ \xi_5 = a^7 \xi_0, \\ \xi_6 = 0, \\ \xi_7 = (a^5 - a^3 - a^7) \xi_0, \\ \xi_8 = [(a^1)^2 a^6 - a^4] \xi_0, \\ (a^3 a^5 + a^4 a^6 + a^5 - a^3 - a^7) \xi_0 - H\xi_0 + G = 0. \end{cases} \tag{44}$$

引入规范函数

$$G = 0 \tag{45}$$

则

$$\xi_0 = 0,\ \xi_1 = 0,\ \xi_2 = 1,\ \xi_3 = \xi_4 = \xi_5 = \xi_6 = \xi_7 = \xi_8 = 0. \tag{46}$$

容易验证,生成元(46)满足确定方程(38),因此,这个对称性是系统的 Lie 对称性.

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