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带有双障碍的反射倒向随机微分方程的逆比较定理

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摘要:讨论了带有双障碍的反射倒向随机微分方程的逆比较问题,在适当的条件下建立了几个关于其生成元的逆比较定理.

关键词:反射倒向随机微分方程;生成元;比较定理;逆比较定理
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Converse comparison theorems for reflected BSDEs with double obstacles

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Abstract: The converse comparison problem of reflected backward stochastic differential equations (RBSDEs) with double obstacles was explored, and some converse comparison theorems for the generators under some suitable conditions were established.

Key words: reflected backward stochastic differential equation; generator; comparison theorem; converse comparison theorem

0 引言

比较定理是倒向随机微分方程理论的一个重要成就,通过比较两个实值倒向随机微分方程(简记为 BSDE)的生成元和终端条件,可以来比较其解的大小. 该理论在许多有关 BSDE 理论的研究中起到了关键的作用.文[1-4]先后研究了 BSDE 的逆比较问题,即:是否可以通过比较 BSDE 的解来比较其生成元的大小? 并建立了有关 BSDE 的逆比较定理.文[5]首先证明了带有一个障碍的反射倒向随机微分方程(简记为 RBSDE)解的存在惟一性,并且获得了其解的比较定理.此后,文[6]证明了带有两个障碍的 RBSDE 解的存在惟一性,其解的比较定理是由文[7]获得的. 最近,文[8]研究了有关带有一个障碍的 RBSDE 的逆比较问题. 因此,本文讨论带有双障碍的 RBSDE 的逆比较问题是顺其自然的. 本文主要是在一定的假设条件下将逆比较定理推广到了带有双障碍的 RBSDE 情形,其主要证明思想是受文[3,8]的启发而产生的.

1 基本假设,定义,引理

设(Ω, F, P)是一个完备的概率空间,$(W_t)_{t\geqslant 0}$是定义在此概率空间上的一个 d 维的标准布朗运动,并且 $W_0=0$.设$(F_t)_{t\geqslant 0}$是由此布朗运动产生的自然 σ 代数流:$F_t=\sigma\{\sigma\{W_s; s\leqslant t\}\cup N\}$, $\forall t\in[0,T]$,其中 N 表示所有零测集组成的集合.对于任意正整数 n 和 $x\in\mathbf{R}^n$, $|x|$表示其 Euclid 范数.设 $T>0$ 是一个给定的实数,下面定义三个常用的空间:

$L^2(F_t):=\{\xi;\mathbf{R}$ 值 F_t 可测的随机变量且 $E[|\xi|^2]<+\infty\}$, $\forall t\in[0,T]$;

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$S^2(\mathbf{R}):=\{\Psi;\mathbf{R}$ 值连续的可料过程且 $E[\sup_{0\leqslant t\leqslant T}|\Psi_t|^2]<+\infty\}$；

$H^2(\mathbf{R}^n):=\{\Psi;\mathbf{R}^n$ 值可料过程且 $E[\int_0^T|\Psi_t|^2\mathrm{d}t]<+\infty\}$.

接下来定义函数 $g:\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d\to\mathbf{R}$,对 P-a.s., $\forall(x,y)\in\mathbf{R}\times\mathbf{R}^n$,$(g(t,y,z))_{0\leqslant t\leqslant T}$ 为循序可测过程,本文将对 g 做如下假设:

(H1) 存在一个常数 $\mu>0$,使得 $\forall t\in[0,T]$,$\forall(y_i,z_i)\in\mathbf{R}\times\mathbf{R}^n$,$i=1,2$,有:

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|\leqslant\mu|(y_1-y_2)-(z_1-z_2)|.$$

(H2) 过程$(g(t,0,0))_{t\in[0,T]}\in H^2(\mathbf{R})$.

(H3) P-a.s., $\forall(t,y)\in[0,T]\times\mathbf{R}$,$g(t,y,0)=0$.

(H4) P-a.s., $\forall(y,z)\in\mathbf{R}\times\mathbf{R}^n$,$g(t,y,z)$ 关于 t 连续.

注 1　显然假设(H3)可以推出假设(H2).

引理 1.1[9]　假设生成元 g 满足(H1),(H2),则 $\forall\xi\in L^2(F_T)$,如下的 BSDE(g,T,ξ):

$$y_t=\xi+\int_t^T g(s,y_s,z_s)\mathrm{d}s-\int_t^T z_s\mathrm{d}W_s\quad t\in[0,T],$$

存在惟一解$(y_t,z_t)_{t\in[0,T]}\in S^2(\mathbf{R})\times H^2(\mathbf{R}^d)$.本文记 $y_t=E_{g,T}[\xi|F_t]$,$\forall t\in[0,T]$,特别地 $y_0=E_{g,T}[\xi]$.

引理 1.2[10,11]　假设生成元 g,g_1,g_2 满足(H1),(H2).

(ⅰ) 若 $\xi_1,\xi_2\in L^2(F_T)$,$g_1\geqslant g_2$,$\xi_1\geqslant\xi_2$,a.s.,则 $E_{g_1,T}[\xi_1|F_t]\geqslant E_{g_2,T}[\xi_2|F_t]$,a.s., $\forall t\in[0,T]$. 进一步,若 $P(\xi_1>\xi_2)>0$,则 $P(\{E_{g_1,T}[\xi_1|F_t]>E_{g_2,T}[\xi_2|F_t]\})>0$,特别地 $E_{g_1,T}[\xi_1]>E_{g_2,T}[\xi_2]$.

(ⅱ) 对任意停时 $\tau\leqslant T$,$\forall\xi\in L^2(F_\tau)$,有 $E_{g,\tau}[E_{g,T}[\xi|F_\tau]]=F_{g,T}[\xi]$.

(ⅲ) 对任意停时 $\tau\leqslant T$,$\bar{g}=1_{[0,\tau]}g$,$\forall\xi\in L^2(F_\tau)$,有 $E_{g,\tau}[\xi|F_t]=E_{\bar{g},T}[\xi|F_t]$,a.s., $\forall t\in[0,\tau]$.

(ⅳ) 若 g 满足(H3),则 P-a.s.,$E_{g,T}[c|F_t]=c$,$\forall t\in[0,T]$.

定义 1.1[6]　带有双障碍的RBSDE是指具有一个终端条件 $\xi\in L^2(F_T)$ 一个生成元 g,一个低障碍 $L_t\in S^2(\mathbf{R})$ 和一个高障碍 $U_t\in S^2(\mathbf{R})$,并且 $L_t\leqslant U_t$,a.s.我们简记其为 RBSDE(g,ξ,L,U).其解为一个取值于 $\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}_+\times\mathbf{R}_+$ 的四元组(Y,Z,K^+,K^-),并且满足:

(ⅰ) $Z\in H^2(\mathbf{R}^d)$,$Y,K^+,K^-\in S^2(\mathbf{R})$,

(ⅱ) $Y_t=\xi+\int_t^T g(s,Y_s,Z_s)\mathrm{d}s+K_T^+-K_t^+-K_T^-+K_t^--\int_t^T Z_s\mathrm{d}W_s$,$\forall t\in[0,T]$,

(ⅲ) P-a.s.,$L_t\leqslant Y_t\leqslant U_t$,$\forall t\in[0,T]$ 且$\int_0^T(Y_t-L_t)\mathrm{d}K_t^+=\int_0^T(U_t-Y_t)\mathrm{d}K_t^-=0$,

(ⅳ) K^+,K^- 是连续递增的过程且 $K_0^+=K_0^-=0$.

下面给出如下假设:

(H5)[7]:存在两个非负的右连左极的上鞅$\{\eta_t\}_{0\leqslant t\leqslant T}$ 和$\{\theta_t\}_{0\leqslant t\leqslant T}$ 使得 $\forall t\in[0,T)$,$L_t\leqslant\eta_t-\theta_t\leqslant U_t$,并且

$$E[\sup_{0\leqslant t\leqslant T}(|\eta_t|^2+|\theta_t|^2)]<\infty.$$

引理 1.3[6,7]　假设生成元 g 满足(H1),(H2),且(H5)成立,则 RBSDE(g,ξ,L,U) 存在惟一解(Y,Z,K^+,K^-).本文记 $Y_t=E_{g,T}^{L,U}[\xi|F_t]$ $\forall t\in[0,T]$,特别地 $Y_0=E_{g,T}^{L,U}[\xi]$.

2　主要结果

命题 2.1　假设生成元 g 满足(H1),(H3),且存在两个常数 C_1,C_2,使得 $L_t\leqslant C_1<C_2\leqslant U_t$,a.s.,$\tau$ 为任意停时且满足 $0\leqslant\tau\leqslant T$,则 $\forall t\in[0,T]$,$\forall\xi\in L^2(F_\tau)$ 且 $C_1\leqslant\xi\leqslant C_2$,有

$$E_{g,T}^{L,U}[\xi|F_t]=E_{g,T}[\xi|F_t]=\begin{cases}E_{g,\tau}[\xi|F_t], & t\in[0,\tau],\\ \xi, & t\in(\tau,T].\end{cases}$$

证明　令(y_t,z_t)表示 BSDE(g,T,ξ)的解,容易验证 $y_t = E_{g,T}[\xi \mid F_t] = \begin{cases} E_{g,\tau}[\xi \mid F_t], & t \in [0,\tau], \\ \xi, & t \in (\tau,T], \end{cases}$
又由引理 1.2 得　$P\text{-a.s.},\ C_1 = E_{g,T}[C_1 \mid F_t] \leqslant E_{g,T}[\xi \mid F_t] \leqslant E_{g,T}[C_2 \mid F_t] = C_2, \forall t \in [0,T]$.
显然$(y_t,z_t,0,0)$是 RBSDE(g,ξ,L,U)在区间$[0,T]$上的解,容易看出结论成立,证毕.

命题 2.2　假设生成元 g 满足(H1),(H2),且(H5)成立,则对任意停时 $\tau \leqslant T, \forall \xi \in L^2(F_\tau)$ 且 $L_\tau \leqslant \xi \leqslant U_\tau$,有 $E^{L,U}_{g,\tau}[\xi \mid F_t] = E^{\bar{L},\bar{U}}_{\bar{g},T}[\xi \mid F_t]$,a.s., $\forall t \in [0,\tau]$.其中 $\bar{g} = 1_{[0,\tau]} g, \bar{L}_t = L_{t\wedge\tau}, \bar{U}_t = U_{t\wedge\tau}$.

证明　令(Y_t,Z_t,K_t^+,K_t^-)表示 RBSDE(g,ξ,L,U)在区间$[0,\tau]$上的解,$(\bar{Y}_t,\bar{Z}_t,\bar{K}_t^+,\bar{K}_t^-)$表示 RBSDE$(\bar{g},\xi,\bar{L},\bar{U})$在区间$[0,T]$上的解. 容易验证

$$\bar{Y}_t = \xi,\ \bar{Z}_t = 0,\ \bar{K}_t^+ = K_\tau^+,\ \bar{K}_t^- = K_\tau^-,\ \forall t \in [\tau,T],$$
$$\bar{Y}_t = Y_t,\ \bar{Z}_t = Z_t,\ \bar{K}_t^+ = K_t^+,\ \bar{K}_t^- = K_t^-,\ \forall t \in [0,\tau].$$

显然 $E^{L,U}_{g,\tau}[\xi \mid F_t] = E^{\bar{L},\bar{U}}_{\bar{g},T}[\xi \mid F_t]$,a.s., $\forall t \in [0,\tau]$.证毕.

定理 2.1　假设生成元 g_1,g_2 满足(H1),(H3),(H4),且存在两个常数 C_1,C_2,使得 $\forall t \in [0,T]$, P-a.s.,$L_t \leqslant C_1 < C_2 \leqslant U_t$.此时若 $\forall \xi \in L^2(F_T)$ 且 $C_1 \leqslant \xi \leqslant C_2$,有 $E^{L,U}_{g_1,T}[\xi] \geqslant E^{L,U}_{g_2,T}[\xi]$,则

$$P\text{-a.s.},\ g_1(t,y,z) \geqslant g_2(t,y,z),\ \forall (t,y,z) \in [0,T] \times [C_1,C_2] \times \mathbf{R}^d.$$

证明　$\forall \delta > 0, \forall (y,z) \in (C_1,C_2) \times \mathbf{R}^d$,定义如下停时:

$$\tau_\delta = \tau_\delta(y,z) = \inf\{t \geqslant 0; g_1(t,y,z) \leqslant g_2(t,y,z) - \delta\} \wedge T,$$

显然,若结论不成立,则存在 $\delta_0 > 0,(y_0,z_0) \in (C_1,C_2) \times \mathbf{R}^d$,使得 $P(\{\tau_{\delta_0} < T\}) > 0$.简记 $\tau_{\delta_0} = \tau_0$.

对(δ_0,y_0,z_0)在区间$[\tau_0,T]$上考虑如下两个正向随机微分方程:

$$\begin{cases} -\mathrm{d}Y_t^i = g_i(t,Y_t^i,z_0)\mathrm{d}t - z_0\mathrm{d}W_t \\ Y_{\tau_0}^i = y_0 \end{cases} \quad i = 1,2. \tag{1}$$

显然对 $i = 1,2$,方程(1)在区间$[\tau_0,T]$上有惟一解 $Y_t^i \in S^2(\mathbf{R})$.

下面定义停时

$$\tau^i_{C_1} = \inf\{t \geqslant \tau_0; Y_t^i \leqslant C_1\} \wedge T,\ i = 1,2,$$
$$\tau^i_{C_2} = \inf\{t \geqslant \tau_0; Y_t^i \geqslant C_2\} \wedge T,\ i = 1,2,$$
$$\bar{\tau}_0 = \inf\left\{t \geqslant \tau_0; g_1(t,Y^1_{\tau_0},z_0) \geqslant g_2(t,Y^2_{\tau_0},z_0) - \frac{\delta_0}{2}\right\} \wedge T.$$

由 g_i,Y_t^i 的连续性可知　$\{\tau_0 < \tau^i_{C_1}\} = \{\tau_0 < \tau^i_{C_2}\} = \{\tau_0 < \bar{\tau}_0\} = \{\tau_0 < T\},\ i = 1,2$.
记 $\tau = \tau^1_{C_1} \wedge \tau^1_{C_2} \wedge \tau^2_{C_1} \wedge \tau^2_{C_2} \wedge \bar{\tau}_0$,则有 $P(\{\tau_0 < \tau\}) = P(\{\tau_0 < T\}) > 0$.
从而　$C_1 \leqslant E_{g_i,\tau}[Y^i_\tau \mid F_t] = Y^i_t \leqslant C_2,\ \forall t \in [\tau_0,\tau],\ i = 1,2$.
由命题 2.1 可得　$E_{g_iT}[Y^i_\tau \mid F_t] = Y^i_\tau,\ \forall t \in [\tau,T],\ i = 1,2$.
令 $\bar{g}_i = 1_{[0,\tau]} g_i$,由引理 1.2 得

$$C_1 \leqslant E_{g_i,\tau_0}[Y^i_{\tau_0} \mid F_t] = E_{\bar{g}_i,T}[Y^i_{\tau_0} \mid F_t] = E_{\bar{g}_i,T}[y_0 \mid F_t] = y_0 \leqslant C_2,\ \forall t \in [0,\tau_0],\ i = 1,2,$$

此时,令 $y_t^i = E_{g_i,T}[Y^i_\tau \mid F_t], z_t^i = \begin{cases} 0, & t \in (\tau,T] \\ z_0, & t \in [\tau_0,\tau], \\ 0, & t \in [0,\tau_0] \end{cases}$ $k_t^{i,+} = 0, k_t^{i,-} = 0,\ i = 1,2$.
显然$(y_t^i,z_t^i,k_t^{i,+},k_t^{i,-})$为 RBSDE$(g_i,Y^i_\tau,L,U)$在区间$[0,T]$上的解. 从而

$$E^{L,U}_{g_i,T}[Y^i_\tau] = E_{g_i,T}[Y^i_\tau] = y_0, i = 1,2. \tag{2}$$

由命题 2.1 得　$$E^{L,U}_{g_2,T}[Y^1_\tau] = E_{g_2,T}[Y^1_\tau]. \tag{3}$$

因为　$$Y^1_\tau - Y^2_\tau = \int_{\tau_0}^{\tau} [g_2(s,Y^2_s,z_0) - g_1(s,Y^1_s,z_0)]\mathrm{d}s \geqslant \frac{\delta_0}{2}(\tau - \tau_0).$$

由 τ_0,τ 的定义可知　$$Y^1_\tau \geqslant Y^2_\tau \text{且} P(\{Y^1_\tau > Y^2_\tau\}) > 0, \tag{4}$$

从而由引理1.2知 $$E_{g_2,T}[Y_\tau^1] > E_{g_2,T}[Y_\tau^2]. \tag{5}$$

由式(2),(3),(5)与题设条件得

$$y = E_{g_1,T}[Y_\tau^1] = E_{g_1,T}^{L,U}[Y_\tau^1] \geqslant E_{g_2,T}^{L,U}[Y_\tau^1] = E_{g_2,T}[Y_\tau^1] > E_{g_2,T}[Y_\tau^2] = y.$$

推出矛盾,结论成立,证毕.

定理 2.2 假设生成元 g_1,g_2 满足(H1),(H3),(H4),且均与 y 无关,存在常数 C,使得 $\forall t\in[0,T]$, $L_t < C < U_t$, a.s.则以下三个条件等价:

(ⅰ) 对任意停时 $\tau\leqslant T,\forall \xi\in L^2(F_\tau)$ 且 $L_\tau\leqslant\xi\leqslant C$,有 P-a.s., $E_{g_1,\tau}^{L,U}[\xi]\geqslant E_{g_2,\tau}^{L,U}[\xi]$.

(ⅱ) 对任意停时 $\tau\leqslant T,\forall \xi\in L^2(F_\tau)$ 且 $C\leqslant\xi\leqslant U_\tau$,有 P-a.s., $E_{g_1,\tau}^{L,U}[\xi]\geqslant E_{g_2,\tau}^{L,U}[\xi]$.

(ⅲ) P-a.s., $g_1(t,z)\geqslant g_2(t,z),\ \forall(t,z)\in[0,T]\times\mathbf{R}^d$.

证明 由带双障碍的RBSDE的比较定理([7, 定理2.1]),可知(ⅲ)⇒(ⅰ)与(ⅲ)⇒(ⅱ).

(ⅰ)⇒(ⅲ): 若存在常数 $\bar C$ 使得 $L_t\leqslant\bar C < C < U_t$,由定理2.1可知(ⅰ)⇒(ⅲ).

对于一般情况:我们定义停时 $\tau_n^1 = \inf\left\{t\geqslant 0;\ L_t\geqslant C-\frac{1}{n}\right\}\wedge T$.

显然 $0\leqslant\tau_n^1\leqslant T$,并且存在 n_0,使得 $\forall n\geqslant n_0$,有 $\tau_n^1>0$.

$\forall n\geqslant n_0$,我们定义 $\bar g_i(t,z) = 1_{[0,\tau_n^1]}g_i(t,z)$, $\bar L_t = L_{t\wedge\tau_n^1}$, $\bar U_t = U_{t\wedge\tau_n^1}$, $\forall(t,z)\in[0,T]\times\mathbf{R}^d$, $i=1,2$.

从而我们有 $$P\text{-a.s.},\ \bar L_t\leqslant C-\frac{1}{n} < C\leqslant\bar U_t,\ \forall t\in[0,T].$$

此时,显然若对任意停时 $\tau\leqslant T,\forall\xi\in L^2(F_\tau)$ 且 $\bar L_\tau\leqslant\xi\leqslant C$,有 $E_{\bar g_1,\tau}^{\bar L,\bar U}[\xi]\geqslant E_{\bar g_2,\tau}^{\bar L,\bar U}[\xi]$,

则由定理2.1可得 $$P\text{-a.s.},\ \bar g_1(t,z)\geqslant\bar g_2(t,z),\ \forall(t,z)\in[0,T]\times\mathbf{R}^d.$$

即 $$P\text{-a.s.},\ g_1(t,z)\geqslant g_2(t,z),\ \forall(t,z)\in[0,\tau_n^1]\times\mathbf{R}^d.$$

令 $n\to\infty$,显然 $\tau_n^1\to T$,从而我们有 P-a.s., $g_1(t,z)\geqslant g_2(t,z)$ $\forall(t,z)\in[0,T]\times\mathbf{R}^d$.

因此,下面我们只需证明 $E_{\bar g_1,\tau}^{\bar L,\bar U}[\xi]\geqslant E_{\bar g_2,\tau}^{\bar L,\bar U}[\xi]$.

定义 $\tilde g_i(t,z) = 1_{[0,\tau]}\bar g_i(t,z)$, $\tilde L_t = L_{t\wedge\tau_n\wedge\tau}$, $\tilde U_t = U_{t\wedge\tau_n\wedge\tau}$,由命题2.2知 $E_{\bar g_i,\tau}^{\bar L,\bar U}[\xi] = E_{\tilde g_i,T}^{\tilde L,\tilde U}[\xi]$, $i=1,2$.

由引理1.2与 $\tilde g_i,\tilde L_t,\tilde U_t$ 的定义知 $E_{\tilde g_i,T}^{\tilde L,\tilde U}[\xi] = E_{\tilde g_i,\tau_n\wedge\tau}^{\tilde L,\tilde U}[E_{\tilde g_i,T}^{\tilde L,\tilde U}[\xi\mid F_{\tau_n\wedge\tau}]] = E_{g_i,\tau_n\wedge\tau}^{L,U}[E_{\tilde g_i,T}^{\tilde L,\tilde U}[\xi\mid F_{\tau_n\wedge\tau}]]$, $i=1,2$.

同时由 $\tilde g_i,\tilde L_t,\tilde U_t$ 的定义也可得 $E_{\tilde g_1,T}^{\tilde L,\tilde U}[\xi\mid F_{\tau_n\wedge\tau}] = E_{\tilde g_2,T}^{\tilde L,\tilde U}[\xi\mid F_{\tau_n\wedge\tau}]$.

令 $\eta = E_{\tilde g_1,T}^{\tilde L,\tilde U}[\xi\mid F_{\tau_n\wedge\tau}]$,显然 $\eta\in L^2(F_{\tau\wedge\tau_n})$ 且 $L_{\tau_n\wedge\tau}\leqslant\eta\leqslant U_{\tau_n\wedge\tau}$,由以上讨论与题设知

$$E_{\bar g_1,\tau}^{\bar L,\bar U}[\xi] = E_{g_1,\tau_n\wedge\tau}^{L,U}[\eta]\geqslant E_{g_2,\tau_n\wedge\tau}^{L,U}[\eta] = E_{\bar g_2,\tau}^{\bar L,\bar U}[\xi].$$

结论成立.

(ⅱ)⇒(ⅲ):若存在常数 $\bar C$ 使得 $L_i < C < \bar C\leqslant U_t$,由定理2.1知(ⅱ)⇒(ⅲ).

对于一般情况:定义停时 $\tau_n^2 = \inf\left\{t\geqslant 0;\ U_t\leqslant C+\frac{1}{n}\right\}\wedge T$.

显然 $0\leqslant\tau_n^2\leqslant T$,并且存在 n_0,使得 $\forall n\geqslant n_0$,有 $\tau_n^2>0$.

$\forall n\geqslant n_0$,定义 $\bar g_i(t,z) = 1_{[0,\tau_n^2]}g_i(t,z)$, $\bar L_t = L_{t\wedge\tau_n^2}$, $\bar U_t = U_{t\wedge\tau_n^2}$ $\forall(t,z)\in[0,T]\times\mathbf{R}^d$, $i=1,2$.

因此 $$P\text{-a.s.},\ \bar L_t\leqslant C < C+\frac{1}{n}\leqslant\bar U_t,\ \forall t\in[0,T].$$

此时由(ⅰ)⇒(ⅲ)的讨论可知,只需证明对任意停时 $\tau\leqslant T,\forall\xi\in L^2(F_\tau)$ 且 $C\leqslant\xi\leqslant\bar U_\tau$,有

$$E_{\bar g_1,\tau}^{\bar L,\bar U}[\xi]\geqslant E_{\bar g_2,\tau}^{\bar L,\bar U}[\xi].$$

从而对此采用与(ⅰ)⇒(ⅲ)相似的处理方法易证结论成立. 证毕.

定理 2.3 假设生成元 g_1,g_2 满足(H1),(H3),(H4),且存在常数 C,使得 $\forall t\in[0,T]$, $L_t < C < U_t$, a.s.,则以下两个结论成立:

(ⅰ) 若对任意两个停时 $0\leqslant\tau\leqslant\sigma\leqslant T,\forall\xi\in L^2(F_\sigma)$ 且 $L_\sigma\leqslant\xi\leqslant C$,有 $E_{g_1,\sigma}^{L,U}[\xi\mid F_\tau]\geqslant E_{g_2,\sigma}^{L,U}[\xi\mid F_\tau]$, a.s.,则 $\forall X \in S^2(\mathbf{R})$ 且 $L_t \leqslant X_t \leqslant C$,a.s., $\forall t \in [0,T]$,有

$$P\text{-a.s.},\ g_1(t,X_t,z) \geqslant g_2(t,X_t,z),\ \forall (t,z) \in [0,T] \times \mathbf{R}^d.$$

(ⅱ) 若对任意两停时 $0 \leqslant \tau \leqslant \sigma \leqslant T$,$\forall \xi \in L^2(F_\sigma)$ 且 $C \leqslant \xi \leqslant U_\sigma$,有 $E^{L,U}_{g_1,\sigma}[\xi \mid F_\tau] \geqslant E^{L,U}_{g_2,\sigma}[\xi \mid F_\tau]$,a.s.则 $\forall X \in S^2(\mathbf{R})$ 且 $C \leqslant X_t \leqslant U_t$,a.s., $\forall t \in [0,T]$,有

$$P\text{-a.s.},\ g_1(t,X_t,z) \geqslant g_2(t,X_t,z),\ \forall (t,z) \in [0,T] \times \mathbf{R}^d.$$

证明　$\forall X \in S^2(\mathbf{R})$,$\forall \delta > 0$,$\forall z \in \mathbf{R}^d$,定义如下停时:

$$\tau_\delta = \tau_\delta(z) = \inf\{t \geqslant 0; g_1(t,X_t,z) \leqslant g_2(t,X_t,z) - \delta\} \wedge T,$$

(ⅰ):定义空间 $A^2 = \{\Psi \in S^2(\mathbf{R}); L + \varepsilon \leqslant \Psi \leqslant C - \varepsilon, \exists \varepsilon > 0\}$,其中 ε 为常数.从而,由 g_1,g_2 关于 y 的连续性可知,我们只需 $\forall X \in A^2$ 做讨论.

若结论不成立,则存在 $\bar{X} \in A^2$,$\delta_0 > 0$,$z_0 \in \mathbf{R}^d$,使得 $P(\{\tau_{\delta_0} < T\}) > 0$.简记 $\tau_{\delta_0} = \tau_0$.

对$(\delta_0,\bar{X}_{\tau_0},z_0)$在区间$[\tau_0,T]$上考虑如下两个正向随机微分方程:

$$\begin{cases} -\mathrm{d}Y^i_t = g_i(t,Y^i_t,z_0)\mathrm{d}t - z_0\mathrm{d}W_t, \\ Y^i_{\tau_0} = \bar{X}_{\tau_0} \end{cases} \quad i = 1,2. \tag{6}$$

显然对 $i = 1,2$,方程(6)在区间$[\tau_0,T]$上有惟一解 $Y^i_t \in S^2(\mathbf{R})$.此时定义停时

$$\tau^i_{L_t} = \inf\{t \geqslant \tau_0; Y^i_t \leqslant L_t\} \wedge T,\ \tau^i_C = \inf\{t \geqslant \tau_0; Y^i_t \geqslant C\} \wedge T,\ i = 1,2.$$

$$\bar{\tau}_0 = \inf\left\{t \geqslant \tau_0; g_1(t,Y^1_{\tau_0},z_0) \geqslant g_2(t,Y^2_{\tau_0},z_0) - \frac{\delta_0}{2}\right\} \wedge T,$$

显然有$\{\tau_0 < \tau^i_{L_t}\} = \{\tau_0 < \tau^i_C\} = \{\tau_0 < \bar{\tau}_0\} = \{\tau_0 < T\}$, $i = 1,2$.

记 $\tau = \tau^1_{L_t} \wedge \tau^1_C \wedge \tau^2_{L_t} \wedge \tau^2_C \wedge \bar{\tau}_0$,则我们有 $P(\{\tau_0 < \tau\}) = P(\{\tau_0 < \tau\}) > 0$.

从而
$$L_t \leqslant Y^i_t \leqslant C,\ \forall t \in [\tau_0,\tau],\ i = 1,2.$$

显然,此时有
$$E^{L,U}_{g_i,\tau}[Y^i_\tau \mid F_t] = E_{g_i,\tau}[Y^i_\tau \mid F_t],\ \forall t \in [\tau_0,\tau].\ i = 1,2. \tag{7}$$

特别地
$$E^{L,U}_{g_i,\tau}[Y^i_\tau \mid F_{\tau_0}] = E_{g_i,\tau}[H^i_\tau \mid F_{\tau_0}] = \bar{X}_{\tau_0},\ i = 1,2. \tag{8}$$

易知此时有与(4)相似的结果, 即 $Y^1_\tau \geqslant Y^2_\tau$ 且 $P(\{Y^1_\tau > Y^2_\tau\}) > 0$.

从而令 $\bar{g}_2 = 1_{[0,\tau]}g_2$,由引理 1.2 可知
$$P(\{E_{\bar{g}_2,T}[Y^1_\tau \mid F_{\tau_0}] > E_{\bar{g}_2,T}[Y^2_\tau \mid F_{\tau_0}]\}) > 0, \tag{9}$$

由(7)与引理 1.2 可知

$$L_t \leqslant E^{L,U}_{g_2,\tau}[Y^2_\tau \mid F_t] = E_{g_2,\tau}[Y^2_\tau \mid F_t] = E_{\bar{g}_2,T}[Y^2_\tau \mid F_t] \leqslant E_{\bar{g}_2,T}[Y^1_\tau \mid F_t] = E_{g_2,\tau}[Y^1_\tau \mid F_t] = E_{\bar{g}_2,T}[Y^1_\tau \mid F_t] \leqslant E_{\bar{g}_2,T}[C \mid F_t] = C,\ \forall t \in [0,\tau].$$

因此有
$$E^{L,U}_{g_2,\tau}[Y^1_\tau \mid F_t] = E_{g_2,\tau}[Y^1_\tau \mid F_t],\ \forall t \in [\tau_0,\tau]. \tag{10}$$

由(7)(9)与题设条件得

$$\bar{X}_{\tau_0} = E^{L,U}_{g_1,\tau}[Y^1_\tau \mid F_{\tau_0}] \geqslant E^{L,U}_{g_2,\tau}[Y^1_\tau \mid F_{\tau_0}] = E_{g_2,\tau}[Y^1_\tau \mid F_{\tau_0}] = E_{\bar{g}_2,T}[Y^1_\tau \mid F_{\tau_0}] \geqslant E_{\bar{g}_2,T}[Y^2_\tau \mid F_{\tau_0}] = E_{g_2,\tau}[Y^2_\tau \mid F_{\tau_0}] = \bar{X}_{\tau_0},$$

再由(9)可知 $P(\{\bar{X}_{\tau_0} > \bar{X}_{\tau_0}\}) > 0$.显然矛盾,结论成立.

(ⅱ):与(ⅰ)的证明相似,定义空间 $B^2 = \{\Psi \in S^2(\mathbf{R}); C + \varepsilon \leqslant \Psi \leqslant U - \varepsilon, \exists \varepsilon > 0\}$,其中 ε 为常数.若结论不成立,则存在 $\tilde{X} \in B^2$,$\delta_0 > 0$,$z_0 \in \mathbf{R}^d$,使得 $P(\{\tau_{\delta_0} < T\}) > 0$.简记 $\tau_{\delta_0} = \tau_0$.

对$(\delta_0,\tilde{X}_{\tau_0},z_0)$在区间$[\tau_0,T]$上考虑如下两个正向随机微分方程:

$$\begin{cases} -\mathrm{d}Y^i_t = g_i(t,Y^i_t,z_0)\mathrm{d}t - z_0\mathrm{d}W_t, \\ Y^i_{\tau_0} = \tilde{X}_{\tau_0}, \end{cases} \quad i = 1,2. \tag{11}$$

显然对 $i = 1,2$,方程(11)在区间$[\tau_0,T]$上有惟一解 $Y^i_t \in S^2(\mathbf{R})$此时定义停时

$$\bar{\tau}^i_C = \inf\{t \geqslant \tau_0; Y^i_t \leqslant C\} \wedge T,\ \tau^i_{U_t} = \inf\{t \geqslant \tau_0;\ Y^i_t \geqslant U_t\} \wedge T,\ i = 1,2.$$

$$\bar{\tau}_0 = \inf\left\{t \geqslant \tau_0;\ g_1(t, Y^1_{\tau_0}, z_0) \geqslant g_2(t, Y^2_{\tau_0}, z_0) - \frac{\delta_0}{2}\right\} \wedge T,$$

显然有$\{\tau_0 < \tau^i_{U_t}\} = \{\tau_0 < \bar{\tau}^i_C\} = \{\tau_0 < \bar{\tau}_0\} = \{\tau_0 < T\}$, $i = 1,2$.

记 $\tau = \tau^1_{U_t} \wedge \bar{\tau}^1_C \wedge \tau^2_{U_t} \wedge \bar{\tau}^2_C \wedge \bar{\tau}_0$,则有 $P(\{\tau_0 < \tau\}) = P(\{\tau_0 < T\}) > 0$.

从而
$$C \leqslant Y^i_t \leqslant U_t,\ \forall t \in [\tau_0, \tau],\ i = 1,2$$
与(8),(9),(10)的证明相似,此时我们也可证得

$$E^{L,U}_{g_i,\tau}[Y^i_\tau \mid F_{\tau_0}] = E_{g_i,\tau}[Y^i_\tau \mid F_{\tau_0}] = \widetilde{X}_{\tau_0},\ i = 1,2, \tag{12}$$

$$P(\{E_{\bar{g}_1,T}[Y^2_\tau \mid F_{\tau_0}] < E_{\bar{g}_1,T}[Y^1_\tau \mid F_{\tau_0}]\}) > 0, \tag{13}$$

$$E^{L,U}_{g_1,\tau}[Y^2_\tau \mid F_t] = E_{g_1,\tau}[Y^2_\tau \mid F_t],\ \forall t \in [\tau_0, \tau]. \tag{14}$$

由(12),(14)与题设条件得

$$\widetilde{X}_{\tau_0} = E^{L,U}_{g_2,\tau}[Y^2_\tau \mid F_{\tau_0}] \leqslant E^{L,U}_{g_1,\tau}[Y^2_\tau \mid F_{\tau_0}] = E_{g_1,\tau}[Y^2_\tau \mid F_{\tau_0}] = E_{\bar{g}_1,T}[Y^2_\tau \mid F_{\tau_0}] \leqslant$$
$$E_{\bar{g}_1,T}[Y^1_\tau \mid F_{\tau_0}] = E_{g_1,\tau}[Y^1_\tau \mid F_{\tau_0}] = \widetilde{X}_{\tau_0}.$$

从而再由(13)可知 $P(\{\widetilde{X}_{\tau_0} < \widetilde{X}_{\tau_0}\}) > 0$.推出矛盾,结论成立. 证毕.

注 2　事实上,定理2.1,定理2.2和定理2.3中的题设条件(H4)减弱为(H6):

(H6):P-a.s.,$\forall (y,z) \in \mathbf{R} \times \mathbf{R}^n$,$g(t,y,z)$关于$t$在区间$[0,T]$上右连续.

注 3　定理2.1和定理2.3均是有关RBSDE生成元的局部的逆比较定理,事实上,下面这个例子(见文[8])表明,在整个空间$\Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d$上的逆比较定理是不一定成立的.

例如,设$c_1 < c_2 < c_3$,$\mu_1 > \mu_2$均为常数,令$g_i(t,y,z) = \mu_i(y - c_i)^- |z|$, $i = 1,2$.显然g_1,g_2满足立(H1),(H3),(H4).令$L = c_2$,$U = c_3$,此时对任意两个停时$0 \leqslant \tau \leqslant \sigma \leqslant T$,$\forall \xi \in L^2(F_\sigma)$且$c_2 \leqslant \xi \leqslant c_3$,有$E^{L,U}_{g_1,\sigma}[\xi \mid F_\tau] = E^{L,U}_{g_2,\sigma}[\xi \mid F_\tau]$,a.s.,但是有$g_1 \geqslant g_2$或者$g_1 \leqslant g_2$.

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