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$I(\chi)$-Reduced representations of generalized Jacobson-Witt algebras $W(m;\mathbf{n})$

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Abstract: This paper studied the representations of graded Cartan type Lie algebra $W(m;\mathbf{n})$ in characteristic $p>2$, by generalizing the arguments of Shu's for restricted Lie algebras $W(m;\mathbf{1})$. Especially, the rank-reducing method exploited in the restricted case was extended to non-restricted case. The simple modules for $W(m;\mathbf{n})$ with generalized p-character χ were described by reduction when χ was regular semisimple.

Key words: graded Lie algebra; Cartan type; regular semisimple; loop algebra

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广义 Jacobson-Witt 代数 $W(m;\mathbf{n})$ 的 $I(\chi)$-约化表示

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摘要: 推广了对限制李代数 $W(m;\mathbf{1})$ 的研究方法, 研究了当特征 $p>2$ 时的阶化 Cartan 型李代数 $W(m;\mathbf{n})$ 的表示. 特别地, 把对限制型李代数所用的降秩的方法推广到了非限制的情形. 描述了当 χ 正则半单时 $W(m;\mathbf{n})$ 的不可约广义 χ-约化表示.

关键词: 阶化李代数; Cartan 型; 正则半单; loop 代数

0 Introduction

So far, except for a few examples of Cartan type Lie algebras in prime characteristic with low ranks, we are far away from understanding their simple modules. Generally speaking, the lower-rank case is easier than the higher-rank case in the study of their representations. For rank-one Cartan type Lie algebras of type W, there is a complete determination for the restricted case[1,2] and for the nonrestricted case[3]. For the study of representations of $W(m;\mathbf{1})$,

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the second author introduced a rank-reducing method and effectively studied simple modules when the corresponding p-characters are so-called regular semisimple.

This paper generalizes Shu's arguments[4] . For this, we have to adopt the machinery of so-called generalized restricted Lie algebras. In general, the graded Cartan type Lie algebras $L = W(m;\mathbf{n})$(generalized Jacobson-Witt algebras) are not restricted unless $\mathbf{n} = \mathbf{1}$. So, the Kac-Weisfeiler's method of classifying the irreducible modules through p-character functions doesn't work. To elude such inconvenience the second author introduced the notion of the generalized restricted Lie algebras and related representation methods[5] . And generalized restricted χ-reduced module category coincides with the $\overline{\chi}$-reduced module category of its primitive p-envelope, where $\overline{\chi}$ is the trivial extension of χ in the primitive p-envelope[3,6] .

Thus, the significance of I associated with an arbitrary χ can be recovered, where I is an index subset of $\{1, 2, \cdots, m\}$ (in this note we will denote it by $I(\chi)$). Especially, associated with I, $L = W(m;\mathbf{n})$ admits a grading structure $L = \sum\limits_{q\geqslant -1} L_{[q],I}$ and thereby all generalized χ-reduced representations admit an I-gradation.

The first main result in this paper is: when χ is regular semisimple(to see Definition 3.1), the irreducible generalized χ-reduced modules of $L = W(m;\mathbf{n})$ are all determined by those of the grade-0 component $L_{\bar{0}} = L_{[0],I}$ associated with their I-gradations. And any irreducible $W(m;\mathbf{n})$-module in this way is isomorphic to the induced module of an irreducible generalized χ-reduced $L_{[0],I}$-module. The conclusion of Guang-yu Shen is generalized (see Section 2). Note that in aid of language of "height" of generalized p-character, Guangyu Shen's graded module category is within the χ-reduced representation category with the height of χ smaller than 1. In the present paper, our work is on the χ-reduced category with the generalized p-character χ be regular semi-simple whose height may be much bigger than 1, for example when $I = \{m\}$, and then $\hat{I} = \{1, 2, \cdots, m-1\}$,the corresponding semi-simple character χ is of the height $p^{\sum_{i=1}^{m-1} n_i} + 1$.

Furthermore, we can generalize the representation theory corresponding to the "loop algebra" constructed by $\mathfrak{gl}(m)$ tensoring divided power algebra to the case of $W(m;\mathbf{n})$[4]. Then we finally reduce the irreducible representations of $L_{\bar{0}}$ to the irreducible representation of L_I.

Thus we obtain the second main result in this paper: when χ is regular semisimple, the irreducible generalized χ-reduced $L_{[0],I}$-modules of $W(m;\mathbf{n})$ coincide with the induced representations of the corresponding irreducible representations of $L_I = S_{\hat{I}} \oplus W(\hat{I}) \oplus \mathfrak{A}(\hat{I}) \otimes \mathfrak{h}(I)$.

It's worthy of mention that Guang-yu Shen extensively and deeply studied irreducible graded modules of Lie algebras of type $L = W(m;\mathbf{n})$[7,8,9] , as well as for types S, and H. Those irreducible modules are determined by the irreducible ones of the grade-0 component $L_{[0]} \cong \mathfrak{gl}(m)$. Especially, when this $L_{[0]}$-irreducible module V_0 is not the exceptional weight restricted module, the graded irreducible module of $L = W(m;\mathbf{n})$ is the mixed products of the divided power algebra $\mathfrak{A}(m;\mathbf{n})$ and V_0. Our results may be regarded as the generalization of Shen's related results.

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1 Notations and Λ_I-gradations of $\mathcal{L}$

Throughout $\mathcal{K}$ will be an algebraically closed field of characteristic $p>2$.

1.1 Notations

Let $J=\{1,2,\cdots,m\}$ be an index set and α be a function on J over $\mathbb{Z}$. Denote $A(m)\triangleq\{\alpha\mid\alpha:J\to\mathbb{Z},0\leqslant\alpha(i)\leqslant p^{n_i}-1,\forall\ i\in J\}$. We can write $(\alpha(1),\alpha(2),\cdots,\alpha(m))\triangleq(\alpha_1,\alpha_2,\cdots,\alpha_m)$ for α. Let I be a subset of J and $I=\{i_1<i_2<\cdots<i_l\}$, then $A(I)\triangleq\{\alpha(I)\mid\alpha\in A(m)\}$, where $\alpha(I)$ means $(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_l})$. In order to emphasize the index subset I of J, we sometimes write α_I for the elements of $A(I)$.

Set $L=W(m;\mathbf{n})=\sum\limits_{i=1}^{m}\mathfrak{A}(m;\mathbf{n})D_i$. Recall that $L=W(m;\mathbf{n})=\sum\limits_{i=1}^{m}\mathcal{K}D_i+\sum\limits_{i=1}^{m}\sum\limits_{|\alpha|>0}\mathcal{K}x^{\alpha}D_i\triangleq L_{[-1]}+L_0$. $(L_0,[p])$ is a restricted subalgebra of L with the standard basis elements like $D\in\{x^{\alpha}D_i\mid\sum\limits_{i=1}^{m}\alpha_i>0,0\leqslant\alpha_i\leqslant p^{n_i}-1,i=1,2,\cdots,m\}$. We have known that $(L,\underline{s},\varphi_{\underline{s}})$ is a generalized restricted Lie algebra with $\underline{s}=(n_1,n_2,\cdots,n_m,1,1,\cdots,1)$ and $\varphi_{\underline{s}}:D_i\mapsto 0,D\mapsto D^{[p]}$[3]. By Schur's Lemma, it's easily shown that an arbitrary given simple L-module (V,ρ) is subject to a linear function χ on L[3,6], which satisfies

$$\begin{aligned}&\rho(D)^p-\rho(D^{[p]})=\chi(D)^p\cdot\mathrm{Id}_V,\quad\forall\ D\in L_0,\\&\rho(D_i)^{p^{n_i}}=\chi(D_i)^{p^{n_i}}\cdot\mathrm{Id}_V,\quad i=1,2,\cdots,m.\end{aligned}\tag{1.1}$$

A representation (resp. module) of L satisfying (1.1) is called a generalized reduced representation (resp. module), or χ-reduced representation (χ-reduced module).

1.2 Λ_I-Gradations of L

Let $\mathcal{E}=\{$ the standard basis of L_0 presented above $\}\bigcup\{D_1,\cdots,D_m\}$ be the standard basis of L. Define $\deg:\mathcal{E}\to\mathbb{Z}^m$ by

$$\deg x^{\alpha}D_j=(\alpha_1,\cdots,\alpha_j-1,\cdots,\alpha_m).$$

Denote

$$\Lambda(L)=\{\deg(D)\mid D\in\mathcal{E}\},\quad\Lambda_I=\{\alpha(I)\mid\alpha\in\Lambda(L)\},$$

where I is an index subset of $J=\{1,2,\cdots,m\}$. Then there is a Λ_I-gradation of L:

$$L=\bigoplus_{\alpha\in\Lambda_I}L_{\alpha},\quad L_{\alpha}=\mathcal{K}\text{-span}\{D\in\mathcal{E}\mid\deg D(I)=\alpha\}.$$

For Λ_I, we can define an order $\succ_I$. Set

$$L^{+}(I)=\bigoplus\nolimits_{\alpha_I\succ_I0}L_{\alpha_I},\quad L^{-}(I)=\bigoplus\nolimits_{\alpha_I\prec_I0}L_{\alpha_I},\quad L^{0}(I)=L_I,$$

where $L_I\triangleq\mathcal{K}\text{-span}\{x^{\alpha}D_j\mid(\deg x^{\alpha}D_j)(I)=0\}$. Then $L=L^{-}(I)+L^{0}(I)+L^{+}(I)$.

For L_I, we have the following lemma by direct verification.

Lemma 1.1 Set $\hat{I}:=\{1,2,\cdots,m\}\setminus I$, then $L_I=L_I^c\oplus L_I^0$, where $L_I^c=\mathcal{K}\text{-span}\{x^{\alpha_{\hat{I}}+\varepsilon_i}D_i\mid\alpha_{\hat{I}}\in A(\hat{I}),\ i\in I\}$ is an abelian ideal of L_I, and $L_I^0=\mathcal{K}\text{-span}\{x^{\alpha_{\hat{I}}}D_k\mid\alpha_{\hat{I}}\in A(\hat{I}),\ k\in\hat{I}\}$ can be regarded as a simple generalized Witt algebra $W(\hat{I})$ corresponding to $\hat{I}$.

1.3 About $\mathcal{L}$

Now let

$$\mathcal{L} = L + \sum_{i=1}^{m}\sum_{d_i=1}^{n_i-1}\mathcal{K}D_i^{[p]^{d_i}} \subset \operatorname{Der}\mathfrak{A}(m;\mathbf{n}),$$

(i with $n_i > 1$), then $(\mathcal{L},[p])$ is a restricted Lie subalgebra of $\operatorname{Der}\mathfrak{A}(m;\mathbf{n})$. For any simple $\mathcal{L}$-module $(V,\overline{\rho})$ there are following equations:

$$\overline{\rho}(D)^p - \overline{\rho}(D^{[p]}) = \overline{\chi}(D)^p\cdot \mathrm{Id}_V,\quad \forall\ D\in\mathcal{L}. \tag{1.2}$$

Where $\overline{\chi}$ is a character of $\mathcal{L}$ which is a trivial extension of $\chi\in L^*$, i.e, satisfying $\overline{\chi}|L=\chi$ and

$$\begin{aligned}&\overline{\chi}(D)=\chi(D),\quad \forall\ D\in L,\\ &\overline{\chi}(D_i^{[p]^{d_i}})=0,\quad 1\leqslant d_i\leqslant n_i-1.\end{aligned}$$

A representation $(V,\overline{\rho})$ satisfying (1.2) is called a $\overline{\chi}$- reduced representation of $\mathcal{L}$.

1.4 Λ_I-Gradations of $\mathcal{L}$

Set

$$\begin{aligned}\mathcal{L}^+(I) &= L^+(I),\\ \mathcal{L}^-(I) &= L^-(I)+S_I \triangleq L^-(I)+\sum_{i\in I}\sum_{d_i=1}^{n_i-1}\mathcal{K}D_i^{[p]^{d_i}},\\ \mathcal{L}_I &= L_I + S_{\hat{I}} \triangleq L_I + \sum_{k\in\hat{I}}\sum_{d_k=1}^{n_k-1}\mathcal{K}D_k^{[p]^{d_k}},\end{aligned}$$

where $\hat{I}=\{1,2,\cdots,m\}\setminus I$. Then $\mathcal{L}=\mathcal{L}^-(I)\oplus\mathcal{L}_I\oplus\mathcal{L}^+(I)$. All three subalgebras are restricted Lie algebras.

The following proposition is clear.

Proposition 1.1 For any $\overline{\chi}\in\mathcal{L}^*$, there exists the smallest index set $I(\chi)$ such that $\overline{\chi}(\mathcal{L}^{\pm}(I))=0$. ($I(\chi)$ will be mostly denoted by I for simplicity whenever the context is clear.)

2 The simplicity of induced modules of $L=W(m;\mathbf{n})$

Fix a $\chi\in L^*$. Then there is the smallest index set I such that $\overline{\chi}(\mathcal{L}^{\pm}(I))=0$. Let

$$\begin{aligned}L_{[0],I} &= L_I + \sum_{\alpha_{\hat{I}}\in A(\hat{I})}\sum_{i(\neq j)\in I}\mathcal{K}x^{\alpha_{\hat{I}}+\varepsilon_i}D_j,\\ L_{[q],I} &= \sum_{\alpha_{\hat{I}}\in A(\hat{I})}\sum_{\substack{\alpha_I\in A(I)\\ |\alpha_I|=q+1}}\sum_{i\in I}\mathcal{K}x^{\alpha_{\hat{I}}+\alpha_I}D_i + \sum_{\alpha_{\hat{I}}\in A(\hat{I})}\sum_{\substack{\alpha_I\in A(I)\\ |\alpha_I|=q}}\sum_{k\in\hat{I}}\mathcal{K}x^{\alpha_{\hat{I}}+\alpha_I}D_k\end{aligned}$$

for $q=-1$ and $q>0$. We have a $\mathbb{Z}$-gradation of $L: L=\oplus_{q\geqslant -1}L_{[q],I}$. Set $L_{q,I}=\sum\limits_{k\geqslant q}L_{[k],I}$ for $q\geqslant -1$. Then $L_{q,I}$ is a filtration of L. Thus we have $L=L_{[-1],I}\oplus L_{[0],I}\oplus L_{1,I}$. Now let

$$\begin{aligned}\mathcal{L}_{1,I} &= L_{1,I},\\ \mathcal{L}_{[0],I} &= L_{[0],I}+S_{\hat{I}} = L_{[0],I}+\sum_{k\in\hat{I}}\sum_{d_k=1}^{n_k-1}D_k^{[p]^{d_k}},\\ \mathcal{L}_{[-1],I} &= L_{[-1],I}+S_I = L_{[-1],I}+\sum_{i\in I}\sum_{d_i=1}^{n_i-1}D_i^{[p]^{d_i}}.\end{aligned}$$

Then $\mathcal{L}=\mathcal{L}_{[-1],I}\oplus\mathcal{L}_{[0],I}\oplus\mathcal{L}_{1,I}$. All three are restricted subalgebras and $\mathcal{L}_{[-1],I}$ is also Abelian.

A simple $L_{[0],I}$-module V is naturally a simple $\mathcal{L}_{[0],I}$-module via the action: $\overline{\rho}(D_k^{[p]^{d_k}})$ $=\overline{\rho}(D_k)^{p^{d_k}}-\overline{\chi}(D_k)^{p^{d_k}}\cdot \mathrm{Id}_V, k\in\hat{I}$. Notice that $\overline{\chi}(D_k)=0$ for $k\in\hat{I}$, and then $\overline{\rho}(D_k^{[p]^{d_k}})=\overline{\rho}(D_k)^{p^{d_k}}$. If V is a simple $\mathcal{L}_{[0],I}$-module with character $\overline{\chi}$, then V can be extended to a simple $\mathcal{L}_{0,I}$-module via a trivial $\mathcal{L}_{1,I}$- action. Hence we have the induction functor $\mathrm{Ind}_{\mathcal{L}_{0,I}}^{\mathcal{L}}$ from the $u_{\overline{\chi}}(\mathcal{L}_{[0],I})$-module category $u_{\overline{\chi}}(\mathcal{L}_{[0],I})$-**Mod** to the $u_{\overline{\chi}}(\mathcal{L})$-module category $u_{\overline{\chi}}(\mathcal{L})$-**Mod**:

$$\mathrm{Ind}_{\mathcal{L}_{0,I}}^{\mathcal{L}}(V)=u_{\overline{\chi}}(\mathcal{L})\otimes_{u_{\overline{\chi}}(\mathcal{L}_{[0],I})}V.$$

On the other hand, for a $u_{\overline{\chi}}(\mathcal{L})$-module W, set $W^{\mathcal{L}_{1,I}}=\{w\in W\mid \mathcal{L}_{1,I}\cdot w=0\}$. Since $\mathcal{L}_{[0],I}$ normalizes $\mathcal{L}_{1,I}$, $W^{\mathcal{L}_{1,I}}$ is an $\mathcal{L}_{[0],I}$-module. Thus we have the fixed-point functor $\mathfrak{K}=(-)^{\mathcal{L}_{1,I}}$ from $u_{\overline{\chi}}(\mathcal{L})$-**Mod** to $u_{\overline{\chi}}(\mathcal{L}_{[0],I})$-**Mod**.

Proposition 2.1 Set $\pi_{\hat{I}}=\sum\limits_{j\in\hat{I}}(p^{n_j}-1)\varepsilon_j$. Suppose $\chi(G_i)\neq 0$ for a certain $i\in I$ and $G_i=x^{\pi_{\hat{I}}+\varepsilon_i}D_i$. Then the functor $\mathrm{Ind}_{\mathcal{L}_{0,I}}^{\mathcal{L}}$ is one-to-one correspondence between the isomorphism classes of simple $\mathcal{L}_{[0],I}$-modules and simple $\mathcal{L}$-modules with the same character $\overline{\chi}$.

Proof Set $\mathcal{J}=\mathrm{Ind}_{\mathcal{L}_{0,I}}^{\mathcal{L}}$. Parallel to the arguments in restricted cases, we easily know that $\mathcal{J}$ is left adjoint to $\mathfrak{K}$. What we need to do is to prove that both functors $\mathcal{J}$ and $\mathfrak{K}$ send the simple objects to the simple ones. For this, we need to prove that for any $w\in W=\mathcal{J}(V)$, we have $u_{\overline{\chi}}(\mathcal{L})\cdot w=W$, where V is an arbitrary given simple $u_{\overline{\chi}}(\mathcal{L}_{0,I})$-module.

Notice that $W=u_{\overline{\chi}}(\mathcal{L}_{[-1],I})\otimes V$ as vector spaces. For any nonzero vector $w\in W$, we can express it as

$$w=\sum_{\mathbf{a}\in T_1}\sum_{\mathbf{b}\in T_2}F^{\mathbf{a}}E^{\mathbf{b}}\otimes v_{\mathbf{a},\mathbf{b}},$$

where $F^{\mathbf{a}}=F_1^{\mathbf{a}(1)}F_2^{\mathbf{a}(2)}\cdots F_{l_1}^{\mathbf{a}(l_1)}$, $E^{\mathbf{b}}=E_1^{\mathbf{b}(1)}E_2^{\mathbf{b}(2)}E_{l_2}^{\mathbf{b}(l_2)}$, the F_q's are standard basis elements of $L_{[-1],I}$ like $F_q=x^{\alpha_{\hat{I}}}D_i$, $i\in I$; the E_q's are elements like $D_j^{[p]^{d_j}}$, $j\in I$, $1\leqslant d_j\leqslant n_j-1, n_j>1$. T_1 and T_2 are two subsets of the multiple number set $\{(x_1,x_2,\cdots,x_{l_1})\mid 0\leqslant x_i\leqslant p^{n_i-1}\}$ and $\{(x_1,x_2,\cdots,x_{l_2})\mid 0\leqslant x_i\leqslant p^{n_i-1}\}$ respectively for $l_1=\dim L_{[-1],I}$ and $l_2=\sum\limits_{i\in I}(n_i-1)$.

If $w\in 1\otimes V$, it is naturally true that $w=u_{\overline{\chi}}(\mathcal{L})\cdot w$ because V is simple. So we only need to consider the case $w\notin 1\otimes V$, for which the proof will be divided into two cases as follows.

Case 1 $w\notin 1\otimes V$ and $\mathbf{a}\neq 0$. Let $F_1=x^{\alpha_{\hat{I}}}D_{i_1}$, $\widetilde{F}_1=x^{\pi_{\hat{I}}-\alpha_{\hat{I}}+\varepsilon_{i_1}+\varepsilon_i}D_i\in L_{1,I}$, this is because $[\widetilde{F}_1,D_j^{[p]^{d_j}}]=0$ for all $j\in I$, $1\leqslant d_j\leqslant n_j-1$, $n_j>1$.

Case 2 $w\notin 1\otimes V$ and $\mathbf{a}=0$. Then there is a $\mathbf{b}\neq 0$ and $v_{\mathbf{b}}\neq 0$ such that

$$w=\sum_{\mathbf{b}\in T_2}E^{\mathbf{b}}\otimes v_{\mathbf{b}}. \qquad (*)$$

Without loss of generality we may suppose $E_1=D_{j_1}^{[p]^{d_{j_1}}}$ with $\mathbf{b}(1)\neq 0$ for a certain $\mathbf{b}\in T_2$ such that d_{j_1} is the smallest one among all the d as long as $E_q=D_{j_q}^{[p]^d}$ with $\mathbf{c}_q\neq 0$ for a certain $\mathbf{c}\in T_2$ appearing in the sum expression of w $(*)$, and simultaneously such that $j_1=i$ if $D_i^{[p]^{d_i}}$ appears there, as a factor of a certain summand. Thus we can rewrite w as follows.

$$w=E_1^{r_1}\widehat{E}_{1r_1}+E_1^{r_1-1}\widehat{E}_{1(r_1-1)}+\cdots+E_1\widehat{E}_{11}+\widehat{E}_{10},$$

where

$$\widehat{E}_{1q}=\sum_{\substack{\mathbf{b}\in T_2\\ \mathbf{b}(1)=q}} E^{\mathbf{b}|_2^{l_2}}\otimes v_{\mathbf{b}}, q=0,1,2,\cdots,r_1(<p),$$
$$\mathbf{b}|_2^{l_2}=(\mathbf{b}(2),\mathbf{b}(3),\cdots,\mathbf{b}(l_2)).$$

By the assumption, $\widehat{E}_{1r_1}\neq 0$.

Set $\widetilde{E}_1=x^{\pi_{\bar{I}}+p^{d_{j_1}}\varepsilon_{j_1}+\varepsilon_i}D_i\in\mathcal{L}_{1,I}$. Then for all $D_{j_q}^{[p]^d}$ appearing in the sum expression $(*)$

$$[\widetilde{E}_1,D_{j_q}^{[p]^d}]=\begin{cases}-G_i, & \text{if } d=d_j, j_q=j_1,\\ 0, & \text{otherwise.}\end{cases}$$

which implies the trivial action of $\widehat{E}_1$ on $\widehat{E}_{1q}$ for $q=0,1,2,\cdots,r_1$. Furthermore,

$$\begin{aligned}\widetilde{E}_1\cdot E_1^k &= \sum_{q=0}^{k}(-1)^q\binom{k}{q}E_1^{k-q}(\mathrm{ad}E_1)^q\widetilde{E}_1\\ &= E_1^k\widetilde{E}_1-kE_1^{k-1}G_i.\end{aligned}$$

Note that $[G_i,E_q]=0$ for $q\geqslant 2$. From the above, we first have

$$\widetilde{E}_1\cdot E_1^{r_1}\widehat{E}_{1r_1}=E_1^{r_1}\widetilde{E}_1\cdot\widehat{E}_{1r_1}-r_1E_1^{r_1-1}G_i\widehat{E}_{1r_1}=-r_1E_1^{r_1-1}G_i\widehat{E}_{1r_1}\neq 0.$$

This is due to the fact $G_i^p\widehat{E}_{1r_1}=\overline{\chi}(G_i)^p\widehat{E}_{1r_1}\neq 0$, along with the fact $G_i\widehat{E}_{1r_1}$ is in $u_{\overline{\chi}}(\oplus_{j=2}^{n_2}\mathcal{K}E_j)\otimes V$. Then we have

$$\widetilde{E}_1\cdot w=E_1^{r_1-1}\widehat{\widehat{E}}_{1(r_1-1)}+E_1^{r_1-2}\widehat{\widehat{E}}_{1(r_1-2)}+\cdots+E_1\widehat{\widehat{E}}_{11}+\widehat{\widehat{E}}_{10},$$

such that $\widehat{\widehat{E}}_{1q},q=0,1,2,\cdots,r_1-1$ are all in $u_{\overline{\chi}}(\sum\limits_{j=2}^{n_2}\mathcal{K}E_j)\otimes V$ and $\widehat{\widehat{E}}_{1(r_1-1)}=-r_1G_i\widehat{E}_{1r_1}\neq 0$. Hence $\widetilde{E}_1\cdot w\neq 0$. The above argument can be repeated. Thus, we have $\widetilde{E}_1^{r_1}\cdot w\neq 0$ and there is no factor E_1 appearing in any nonzero component of the sum expression of $\widetilde{E}_1^{r_1}\cdot w$, similar to $(*)$. Iterating the above process, we will have

$$0\neq\widetilde{E}_s^{r_s}\cdots\widetilde{E}_1^{r_1}\cdot w\in 1\otimes V.$$

where the meaning of $\widetilde{E}_q$ $(q=2,3,\cdots,s)$ is similar to that of $\widetilde{E}_1$. This implies the simplicity of $\mathrm{Ind}_{\mathcal{L}_{0,I}}^{\mathcal{L}}(V)$.

Conversely, suppose W is an arbitrary simple $u_{\overline{\chi}}(\mathcal{L})$-module. If $\mathfrak{K}(W)$ contains a simple submodule W', by the above argument $\mathcal{J}(W')$ is a simple $u_{\overline{\chi}}(\mathcal{L})$-module. Hence $\mathcal{J}(W')\cong W$ and $\mathfrak{K}\circ\mathcal{J}(W')\cong\mathfrak{K}(W)$. Furthermore, the above argument shows that for any $w\in\mathcal{J}(W')\setminus W'$, there is an $\widetilde{F}_1\in\mathcal{L}_{1,I}$ or a $\widetilde{E}_1\in\mathcal{L}_{1,I}$ such that $\widetilde{F}_1\cdot w\neq 0$ or $\widetilde{E}_1\cdot w\neq 0$, Hence $\mathfrak{K}\circ\mathcal{J}(W')=W'$. This implies that W' coincides with $\mathfrak{K}(W)$. Hence $\mathfrak{K}(W)$ is a simple $u_{\overline{\chi}}(\mathcal{L}_{[0],I})$-module.

Thus, the adjunction morphisms $\mathcal{J}\circ\mathfrak{K}\longmapsto\mathrm{Id}_{u_{\overline{\chi}}(\mathcal{L})\textbf{-Mod}}$ and $\mathfrak{K}\circ\mathcal{J}\longmapsto\mathrm{Id}_{u_{\overline{\chi}}(\mathcal{L}_{[o],I})\textbf{-Mod}}$ are both isomorphisms for simple objects. The proposition is proved.

Remark In the case when $I=\{1,2,\cdots,m\}$, $\mathcal{L}_{[0],I}=L_{[0],I}=L_{[0]}$ where $L_{[0]}$ is the canonical grade-0 component of L, arising from the gradation of the divided power algebra[10]. The corresponding results have been known[9,11].

3 The representations corresponding to loop algebras

Set $L = W(m;\mathbf{n})$. For the divided power algebra $\mathfrak{A} = \mathfrak{A}(s;\mathbf{n})$ and a restricted Lie algebra $(\mathfrak{g}, [p])$, we can define a loop algebra $O(\mathfrak{g}) = \mathfrak{A} \otimes \mathfrak{g}$ with Lie product:

$$[x^\alpha \otimes X, x^\beta \otimes Y] = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta} \otimes [X, Y].$$

Then $O(\mathfrak{g})$ is still a restricted Lie algebra with

$$(x^\alpha \otimes X)^{[p]} = \begin{cases} 0, & \text{if } \alpha \neq 0, \\ 1 \otimes X^{[p]}, & \text{if } \alpha = 0. \end{cases} \tag{3.1}$$

When $\mathfrak{g}$ is centerless, there is a unique $[p]$-mapping for $O(\mathfrak{g})$. In fact, in this case $O(\mathfrak{g})$ is also centerless[4].

In the rest of this paper, we only deal with $\mathfrak{gl}(I)$ for I as in the preceding sections. Here $\mathfrak{gl}(I)$ is the classical Lie subalgebra of $\mathfrak{gl}(m)$ corresponding to the index set $I = \{i_1, i_2, \cdots, i_l\} \subset \{1, 2, \cdots, m\}$, which is isomorphic to $\mathfrak{gl}(\#I)$. Denote by $\mathfrak{A}(\hat{I})$ the divided power subalgebra of $\mathfrak{A}(m)$ with generators $x^{\alpha_{\hat{I}}}$ for all $\alpha_{\hat{I}} \in A(\hat{I})$ (when $\hat{I} = \varnothing$, let naturally define $\mathfrak{A}(\hat{I})$ to be $\mathcal{K}$). In the sequel, we always suppose that all loop algebras $O(\mathfrak{g})$ are associated with the divided power algebra $\mathfrak{A}(\hat{I})$. Set $\widehat{\mathfrak{gl}}(I) = O(\mathfrak{gl}(I))$, $\mathcal{L}_{\bar{0}} = \mathcal{L}_{[0],I}$. Then $\mathcal{L}_{\bar{0}} = S_{\hat{I}} \oplus L_I^0 \oplus \mathfrak{g}_{\hat{I}}$, where

$$\mathfrak{g}_{\hat{I}} = \sum_{\alpha_{\hat{I}} \in A(\hat{I})} \sum_{i,j \in I} \mathcal{K} x^{\alpha_{\hat{I}} + \varepsilon_i} D_j.$$

Lemma 3.1 $\mathfrak{g}_{\hat{I}}$ is Lie-isomorphic to the loop algebra $\widehat{\mathfrak{gl}}(I)$.

Proof Set $\phi : x^{\alpha_{\hat{I}} + \varepsilon_i} D_j \mapsto x^{\alpha_{\hat{I}}} \otimes E_{ij}$, where E_{ij} is the $m \times m$ matrix with (i,j)-entry being 1 and with the others being 0. This is an isomorphism.

Hence we can identify $\mathfrak{g}_{\hat{I}}$ with $\widehat{\mathfrak{gl}}(I)$ in the following discussion. In particular, $\mathfrak{g}_{\hat{I}}$ is a p-subalgebra of $(L, [p])$ with $[p]$-mapping satisfying

$$(x^{\alpha_{\hat{I}} + \varepsilon_i} D_j)^{[p]} = \begin{cases} x^{\varepsilon_i} D_i, & \text{if } \alpha_{\hat{I}} = 0 \text{ and } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

This is admissible to the $[p]$-mapping as defined in (3.1) for $\widehat{\mathfrak{gl}}(I)$. Notice that for $k \in \hat{I}$, $i, j \in I$, $1 \leqslant d_k \leqslant n_k - 1$,

$$[x^{\beta_{\hat{I}}} D_k, x^{\alpha_{\hat{I}}} \otimes E_{ij}] = \tbinom{\beta_{\hat{I}} + \alpha_{\hat{I}} - \varepsilon_k}{\beta_{\hat{I}}} x^{\beta_{\hat{I}} + \alpha_{\hat{I}} - \varepsilon_k} \otimes E_{ij},$$

$$[D_k^{[p]^{d_k}}, x^{\alpha_{\hat{I}}} \otimes E_{ij}] = x^{\alpha_{\hat{I}} - p^{d_k} \varepsilon_k} \otimes E_{ij}.$$

Then we obtain the following lemma.

Lemma 3.2 If $\mathfrak{g}'$ is a subalgebra of $\widehat{\mathfrak{gl}}(I)$, then the loop algebra $\mathfrak{A}(\hat{I}) \otimes \mathfrak{g}'$ is normalized by L_I^0 and $S_{\hat{I}}$.

Let Φ be the root system of $\mathfrak{g} = \mathfrak{gl}(I)$. For $\alpha \in \Phi$, let $e_\alpha \in \mathfrak{g}$ be a nonzero root vector. We can assume that those root vectors are normalized so that if $h_\alpha = [e_\alpha, e_{-\alpha}]$ then the system $\{e_{\pm\alpha},\ h_\alpha\}$ satisfies $[h_\alpha, e_{\pm\alpha}] = \pm 2 e_{\pm\alpha}$. We have the decomposition of root spaces, $\mathfrak{g} = \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, Canonically, denote $\mathfrak{n}^{\pm} = \oplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}_\alpha$. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ associated with Φ^+.

Denote $\mathfrak{b}_{\bar{0}} = O(\mathfrak{b}) \oplus L_I^0 \oplus S_{\hat{I}}, \mathfrak{n}_{\bar{0}}^{\pm} = O(\mathfrak{n}^{\pm})$ Then $\mathfrak{b}_{\bar{0}} = S_{\hat{I}} \oplus L_I \oplus \mathfrak{n}_{\bar{0}}^{+} = \mathcal{L}_I \oplus \mathfrak{n}_{\bar{0}}^{+}$. By the arguments in section 3.1, $(x^{\mathbf{a}} \otimes e_\alpha)^{[p]} = 0$ for all $\mathbf{a}, \alpha$ and $(x^{\mathbf{a}} \otimes h_\alpha)^{[p]} = 0$ for $\mathbf{a} \neq 0$ and all α. Thus all elements of $\mathfrak{n}_{\bar{0}}^{\pm}$ act nilpotently on any $\overline{\chi}$-reduced modules of $\mathfrak{n}_{\bar{0}}^{\pm}$ because of $\overline{\chi}(\mathfrak{n}_{\bar{0}}^{\pm}) = 0$. In particular, for any simple module V of $\mathcal{L}_I$, V can act as a $\mathfrak{b}_{\bar{0}}$- module with the trivial $\mathfrak{n}_{\bar{0}}^{+}$-action. Here we have an induced module:

$$\operatorname{Ind} V = u_{\overline{\chi}}(\mathcal{L}_{\bar{0}}) \otimes_{u_{\overline{\chi}}(\mathfrak{b}_{\bar{0}})} V.$$

Definition 3.1 Let $\overline{\chi} \in \mathcal{L}^*$. We call $\overline{\chi}$ regular semisimple if $\overline{\chi}(G_\alpha) \neq 0$ for all $\alpha \in \Phi^+$ and $\overline{\chi}(\mathcal{L}^{\pm}(I)) = 0$, where $G_\alpha = x^{\pi_{\hat{I}}} \otimes h_\alpha$.

Lemma 3.3 Suppose $\overline{\chi} \in \mathcal{L}^*$ is regular semisimple. Then for any simple $u_{\overline{\chi}}(\mathcal{L}_{\bar{0}})$-module M, M^N is a simple B- submodule and the natural mapping $\operatorname{Ind} M^N \longrightarrow M$ is a $u_{\overline{\chi}}(\mathcal{L}_{\bar{0}})$-module isomorphism. Any simple $u_{\overline{\chi}(\mathcal{L}_{\bar{0}})}$-module is N-projective. Here the functor Ind is defined as above. $B = u_{\overline{\chi}}(\mathfrak{b}_{\bar{0}})$ and $N = u(O(\mathfrak{n}^+))$. (**Note**: $\overline{\chi}(O(\mathfrak{n}^+)) = 0$.)

Proposition 3.1 Suppose $\overline{\chi} \in \mathcal{L}^*$ is regular semisimple. Then $u_{\overline{\chi}}(\mathcal{L}_I)$ and $u_{\overline{\chi}}(\mathcal{L}_{\bar{0}})$ are Morita equivalent. In particular, the modules Ind$V$ constitute the corresponding set for $u_{\overline{\chi}}(\mathcal{L}_{\bar{0}})$ when V runs over a set of representatives of the isomorphism class of simple $u_{\overline{\chi}}(\mathcal{L}_I)$-modules.

The proof of this proposition is mainly from lemma 3.3 and its proof[4].

4 The main result

Let $L = W(m; \mathbf{n})$, $\chi \in L^*$. $\mathcal{L} = L + \sum_{i=1}^{m} \sum_{d_i=1}^{n_i-1} \mathcal{K} D_i^{[p]^{d_i}}$. For any $\overline{\chi} \in \mathcal{L}^*$, there exists the smallest index set $I(\chi)$ such that $\overline{\chi}(\mathcal{L}^{\pm}(I(\chi))) = 0$ (Proposition 1.1). Associated with I, we have the subalgebra $\mathcal{L}_I$ of $\mathcal{L}$: $\mathcal{L}_I = S_{\hat{I}} \oplus W(\hat{I}) \oplus L_I^c$, where $L_I^c \cong \mathfrak{A}(\hat{I}) \otimes \mathfrak{h}(I)$, $\mathfrak{h}(I)$ is the canonical torus of $W(I)$. Let $B(I) = \mathcal{L}_I \oplus \mathcal{L}^+(I)$. For any given simple $\mathcal{L}_I$-module V, V can be regarded as a simple $B(I)$-module with the trivial $\mathcal{L}^+(I)$-action. Then we have the induced module $\operatorname{Ind}_{B(I)}^{\mathcal{L}}(V) = u_{\overline{\chi}}(\mathcal{L}) \otimes_{u_{\overline{\chi}}(B(I))} V$. Hence，we similarly have a functor $\operatorname{Ind}_{B(I)}^{\mathcal{L}}$ from $u_{\bar{\chi}}(\mathcal{L}_I)$-module category to $u_{\bar{\chi}}(\mathcal{L})$-module category.

Theorem 4.1 Suppose $\overline{\chi} \in \mathcal{L}^*$ is regular semisimple. Then the functor $\operatorname{Ind}_{B(I)}^{\mathcal{L}}$ is one-to-one between the isomorphism classes of simple $u_{\overline{\chi}}(\mathcal{L}_I)$-module and simple $u_{\overline{\chi}}(\mathcal{L})$-module.

Proof Note that $B(I) = \mathcal{L}_I \oplus \mathcal{L}^+(I) = \mathfrak{b}_{\bar{0}} \oplus \mathcal{L}_{1,I} \subset \mathcal{L}_{0,I} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{1,I}$.

$$\begin{aligned} \operatorname{Ind}_{B(I)}^{\mathcal{L}}(V) &= u_{\overline{\chi}}(\mathcal{L}) \otimes_{u_{\overline{\chi}}(\mathcal{L}_{0,I})} (u_{\overline{\chi}}(\mathcal{L}_{0,I}) \otimes_{u_{\overline{\chi}}(B(I))} V) \\ &= u_{\overline{\chi}}(\mathcal{L}) \otimes_{u_{\overline{\chi}}(\mathcal{L}_{0,I})} (u_{\overline{\chi}}(\mathcal{L}_{\bar{0}}) \otimes_{u_{\overline{\chi}}(\mathfrak{b}_{\bar{0}})} V) \\ &= \operatorname{Ind}_{\mathcal{L}_{0,I}}^{\mathcal{L}}(\operatorname{Ind} V). \end{aligned}$$

The second equality holds because $u_{\overline{\chi}}(\mathcal{L}_{0,I}) \otimes_{u_{\overline{\chi}}(B(I))} V$ is isomorphic to $\operatorname{Ind} V$ as $u_{\overline{\chi}}(\mathcal{L}_{\bar{0}})$-modules, with the trivial $\mathcal{L}_{1,I}$-action on both. The theorem is proved due to Proposition 2.1 and Proposition 3.1.

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