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Credibility models with error uniform dependence

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Abstract: Firstly, the errors were assumed to be uniformly dependent in normal-normal cases and the Bayes premiums were derived in this model which show that it is the exact credibility. Secondly, the Bühlmann's credibility models with uniform dependence were built, and the corresponding credibility estimators were derived. In addition, the models were extended to Bühlmann-Straub credibility in which the natural weights among contracts were introduced. However, the credibility estimators of individual premium under Bühlmann-Straub model have only the generalized form of "credibility".

Key words: credibility premium; error uniform dependent; orthogonal projection; credibility estimator.

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误差等相关的信度模型

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摘要: 首先，在假设误差呈现等相关的正态-正态分布下得到Bayes保费，表明正是精确信度的形式. 其次，建立了误差等相关的Bühlmann信度模型，得到了相应的信度估计. 在引进了合同之间的自然权重后将该模型推广到Bühlmann-Straub信度情形. 但是结果表明，在Bühlmann-Straub模型下信度估计仅拥有广义的“信度”形式.

关键词: 信度保费; 误差等相关; 正交投影; 信度估计.

0 Introduction

Credibility theory is a common approach to calculate insurance premium based on the policyholder's past experience and the experience of the entire group of policyholders. This

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method is widely used in commercial property of liability insurance, group health, and life insurance. The popular formulas in credibility theory take premium as weighted sum of the average experience of the policyholder and the average of the entire collection of policyholders. These formulas are easy to understand and simple to apply due to their linear properties.

In credibility theory, let X_i denote total claim amount of a policyholder in the ith policy period. The distribution of X_i depends on the risk parameter Θ, which is a random variable with the prior distribution $\pi(\theta)$. If $\Theta = \theta$ is given, $X_i, i = 1, 2, \cdots n$ are independent of each other with the same distribution function $f(x, \theta)$. The purpose of credibility theory is to calculate a premium for the $(n+1)^{th}$ period of a policyholder, given the policyholder's claim experience in the first n periods. If we constrain the estimator to be a linear function of the claim data, the estimator is a well known credibility formula

$$\widehat{\mu(\Theta)} = Z\overline{X} + (1 - Z)\mu,$$

where $Z = \frac{n}{n+\kappa}$ is called as credibility factor, and $\kappa = \frac{\sigma^2}{\tau^2}$ is the ratio of the expected value of conditional variance to the variance of conditional mean. In addition, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean, and μ is the collective mean.

However, the assumption of conditionally independent claim amounts is not practical in most cases. Certain conditional dependence over time has been recognized as more appropriate to fit the practice in some circumstances. For example, Bolance etc[1] estimated and tested autoregressive specifications for dynamic random effects in a frequency risk model and derived credibility predictors; Purcaru and Denuit[2,3] studied a type of dependence induced by the introduction of common latent variables in the annual numbers of claims reported by policyholders and revealed how the dependence structure affects the credibility estimate in the Poisson claim frequency models; Frees, Young and Luo[4,5] showed how to produce credibility predictors for linear longitudinal and panel data models, and more recently, Frees and Wang [6] considered a generalized linear model framework for modeling marginal claims distributions, which allowed dependence characterized by the Student-t copula to model the dependence over time for a class of risks.

In some cases, however, it has been recognized that there exist many important insurance applications where the dependence over risks are common, thus the risks are uniformly dependent. Examples include house insurance for which geographic proximity of the insureds may result in exposures to common catastrophes, and motor vehicle insurance where one accident may involve several insureds. See, also Yeo etc[7], L Wen. etc[8], among others. The uniform dependence is also introduced by Nikolai etc[9] in actuarial science, who considered the joint PGF of the uniform correlation claims and derived some desired properties.

Inspired by these papers, the assumption of conditionally independent claim amounts in Bühlmann's classical credibility model is replaced by a certain uniform conditional dependence for the claim amounts. A formula for the credibility estimator can be got under this dependence structure. Also a exact credibility is given under normal-normal case.

The rest of the paper is arranged as follows. In Section 1, models and assumptions are introduced and the Bayes estimator under normal-normal cases is derived. Section 2 derives the credibility formulae for the Bühlmann's models. The results are extended to the Bühlmann-Straub model in Section 3. Some conclusions are made in Section 4.

1 The exact credibility in normal-normal case

In credibility theory or Bayes analysis, the individual risk can be regarded as a black box that produces aggregate claim amounts X_i $(i = 1, 2, \cdots n)$, where X_i denotes the claim amount in year i.

On the basis of the observations in the previous periods, we want to determine the risk premium for the aggregate claims in a future period, for example, X_{n+1}.. In order to do this, we must make certain assumptions about the distribution function of the random variables X_i. We generalize the assumptions given by Bühlmann[10], and consider some uniform dependence which exists in error effects. Formally, the assumptions of the model are stated as below.

Assumption 1.1 $X_1, X_2, \cdots, X_n \cdots$ are characterized by a risk parameter Θ, and Θ itself is random variable with structure distribution $\pi(\theta)$;

Assumption 1.2 Given Θ, the X_i follows the linear model: $X_i = \mu(\Theta) + \varepsilon_i$, and the errors are conditionally uniformly dependent, i.e, $\operatorname{corr}(\epsilon_i, \epsilon_j) = \rho, i \neq j$ and $\rho < 1$, where "corr" indicate correlation coefficient. In addition, we assume that $\mathrm{E}(\varepsilon_i|\Theta) = 0$, $\operatorname{Var}(\varepsilon_i|\Theta) = \sigma^2(\Theta)$, and $\mathrm{E}[\mu(\Theta)] = \mu$, $\operatorname{Var}[\mu(\Theta)] = \tau^2$, $\mathrm{E}\big[\sigma^2(\Theta)\big] = \sigma^2$.

We further write $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ for the average of the claim experience. Now we address to find the credibility estimator of individual premium.

Definition 1.1 The individual premium of a risk with risk profile Θ is

$$P^{ind} = E\left(X_{n+1}|\Theta\right) := \mu(\Theta). \tag{1.1}$$

The individual premium is also referred to as the risk premium in credibility theory. However, in insurance practice, both Θ and $\mu(\Theta)$ are unknown. Therefore we have to find an estimator $\widehat{\mu(\Theta)}$ for $\mu(\Theta)$ as precisely as possible. The individual rating problem can then be described as the determination of the quantity $\mu(\Theta)$. One potential estimator is the collective premium μ, i.e. the premium for the considered particular risk is estimated by the "average" expected value over the whole collective. This estimator is appropriate when we are considering a new risk, about which there is no pre-existing claim experience.

If we have observed the risk over a period of n years and if $X = (X_1, \cdots, X_n)'$ denotes the vector of aggregate claim amounts associated with this period, then this information should contribute to the estimation process. This brings us to the concept of experience rating. The best experience premium depending on the individual claim experience vector X is called the Bayes premium, which we will now define.

Definition 1.2 The Bayes premium (best experience premium) is defined by $P^{Bayes} = \widetilde{\mu(\Theta)} := E\left(\mu(\Theta)|X\right)$.

In statistics, we know that Bayes premium is the best estimator of $\mu(\Theta)$ which minimize the Bayes risk. In general case. However, the Bayes premium can not be solved explicitly. In credibility theory, the case which Bayes premium is the linear forms of data sample is known as exact credibility. In Bühlmann's model, Jwell[11] proved that the exact credibility happen when the conditional distribution of X_i is exponential and Θ is also distributed as natural conjugate exponential family. In the following, we can derive the Bayes premium corresponding to Assumptions 1.1 and 1.2 under the normal-normal case, and found that this case is also exact credibility, which is shown in the following theorem.

Theorem 1.1 Conditionally, given $\Theta=\theta$, the X_i's $(i=1,2,\cdots n)$ are uniform dependent and normally distributed, that is $X_i \sim N\left(\Theta, \sigma^2\right)$, $\operatorname{cov}(X_i, X_j|\Theta)=\rho$, where $i \neq j$, and $\rho<1$. In addition, the parameter Θ itself is a random variable, with the normal distribution $N\left(\mu, \tau^2\right)$, then the Bayes estimator of Θ are given by

$$\widetilde{\Theta}=\frac{n\tau^2}{n\tau^2+(1-\rho+n\rho)\sigma^2}\overline{X}+\frac{(1-\rho+n\rho)\sigma^2}{n\tau^2+(1-\rho+n\rho)\sigma^2}\mu. \tag{1.2}$$

Proof Conditionally, given $\Theta=\theta$, the distribution of random vector X is normal. The mean vector and covariance matrix of X are

$$\mu(\theta)=\mathrm{E}\left(X|\theta\right)=\theta\mathbf{1}_n,\ \Sigma(\theta)=\operatorname{Cov}(X,X|\theta)=\sigma^2\begin{pmatrix}1 & \rho & \cdots & \rho\\ \rho & 1 & \cdots & \rho\\ \cdots & \cdots & \cdots & \cdots\\ \rho & \rho & \cdots & 1\end{pmatrix}:=\Sigma. \tag{1.3}$$

Notes that the matrix $\Sigma(\theta)$ above is independent of θ. Then the conditional density of X is

$$f\left(x_1,x_2,\cdots x_n|\theta\right)=\frac{1}{\left(\sqrt{2\pi}\right)^n|\Sigma|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(x_1-\theta,\cdots,x_n-\theta\right)\Sigma^{-1}\left(x_1-\theta,\cdots,x_n-\theta\right)'\right\}$$

By the well known formula in linear models,

$$(A+BCD)^{-1}=A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}, \tag{1.4}$$

where A,B,C and D are matrix with adaptable order (see in detail, Radhakrishna Rao[12]), the matrix inverse of Σ is given by

$$\Sigma^{-1}=\frac{1}{\sigma^2}\left[\rho\mathbf{1}_n\mathbf{1}_n'+(1-\rho)I_n)\right]^{-1}=\frac{1}{\sigma^2(1-\rho)}\left(I_n-\frac{\rho\mathbf{1}_n\mathbf{1}_n^T}{1-\rho+n\rho}\right).$$

Then

$$\begin{aligned}&\left(x_1-\theta,\cdots,x_n-\theta\right)\Sigma^{-1}\left(x_1-\theta,\cdots,x_n-\theta\right)'\\&=\frac{1}{\sigma^2(1-\rho)}\left[\sum_{i=1}^n(x_i-\theta)^2-\left(\sum_{i=1}^n(x_i-\theta)\right)^2\frac{\rho}{1-\rho+n\rho}\right]\\&=\frac{1}{\sigma^2(1-\rho)}\left(\sum_{i=1}^n x_i^2-\frac{\rho n^2}{1-\rho+n\rho}\overline{X}^2+\frac{n\left(1-\rho\right)}{1-\rho+n\rho}\theta^2-\frac{2n\overline{X}\left(1-\rho\right)}{1-\rho+n\rho}\theta\right).\end{aligned}$$

Therefore, the joint density of random vector $(X_1, X_2, \cdots X_n, \Theta)$ can be derived as

$$\begin{aligned}
f &= f(x_1, \cdots, x_n, \theta) = \pi(\theta) f(x_1, \cdots, x_n|\theta) \\
&\propto \frac{1}{\sqrt{2\pi}\tau} \exp\left\{-\frac{(\theta-\mu)^2}{2\tau^2}\right\} \frac{1}{\left(\sqrt{2\pi}\right)^n |\Sigma|^{\frac{1}{2}}} \\
&\exp\left\{-\frac{1}{2}(x_1-\theta, \cdots, x_n-\theta)\Sigma^{-1}(x_1-\theta, \cdots, x_n-\theta)'\right\} \\
&\propto \exp\left\{\frac{\mu\theta - \frac{1}{2}\theta^2}{2\tau^2} - \frac{1}{2\sigma^2(1-\rho)}\left(\frac{n(1-\rho)}{1-\rho+n\rho}\theta^2 - \frac{2n\overline{X}(1-\rho)}{1-\rho+n\rho}\theta\right)\right\} \\
&\propto \exp\left\{\frac{\sigma^2(1-\rho)(1-\rho+n\rho)\mu + n\tau^2(1-\rho)\overline{X}}{\tau^2\sigma^2(1-\rho)(1-\rho+n\rho)}\theta - \frac{\left[\sigma^2(1-\rho)(1-\rho+n\rho)+n\tau^2(1-\rho)\right]}{\tau^2\sigma^2(1-\rho)(1-\rho+n\rho)}\frac{\theta^2}{2}\right\}
\end{aligned}$$

From this, we know that the posterior distribution of Θ is also normal, with the expectation

$$\begin{aligned}
\widetilde{\Theta} &= E(\Theta|X_1, X_2, \cdots, X_n) \\
&= \frac{n\tau^2}{n\tau^2 + (1-\rho+n\rho)\sigma^2}\overline{X} + \frac{(1-\rho+n\rho)\sigma^2}{n\tau^2+(1-\rho+n\rho)\sigma^2}\mu
\end{aligned}$$

which gives the result.

From this theorem, we see that the Bayes premium is the linear form of the samples. We called this case exact credibility. See, for example, Bühlmann and Gisler[13]. In other distribution assumption, the conclusions can not be derived. However, we can find the best linear unbiased estimator for premium. That is, the estimators are limited to the linear function of the sample, called the credibility estimators or credibility premium in credibility theory.

2 Bühlmann's credibility models with error uniform dependence

In this section, we address to consider the following credibility estimator of individual premium under Assumptions of 1.1 and 1.2.

Firstly, note that assumption 1.2 implies $\mathrm{Cov}(X_i, X_j|\Theta) = \rho\sigma^2(\Theta)$ and $\mathrm{Var}(X_i|\Theta) = \sigma^2(\Theta)$. Our goals are to calculate a premium for the $(n+1)^{th}$ period, denoting by $\widehat{\mu(\Theta)}^*$, based on the linear estimator class $L(X,1) := \left\{\widehat{\mu(\Theta)} = c_0 + \sum_{i=1}^n c_i X_i,\ \text{with } c_0,\ c_i \in R\right\}$. That is, we must solve the following optimalization problem.

$$\underset{c_0, c_i}{\mathrm{Min}}\ \mathrm{E}\left[\mu(\Theta) - c_0 - \sum_{i=1}^n c_i X_i\right]^2 \tag{2.1}$$

In [13], the optimal estimator of $\mu(\Theta)$ is defined as the orthogonal projection of $\mu(\Theta)$ on linear space $L(X,1)$, and was denoted by $\widehat{\mu(\Theta)}^* = pro(\mu(\Theta)|L(X,1))$. The following lemma gives the calculation formula of the credibility estimator, the proof can be referred to [8].

Lemma 2.1 The credibility estimator (predictor) of a random vector Y on $L(X,1)$ is exactly the projection of Y on the linear space $L(X,1)$, and the following formula holds true.

$$\widehat{Y}^* = \mathrm{E}(Y) + \Sigma_{YX}\Sigma_{XX}^{-1}(X - \mathrm{E}(X)). \tag{2.2}$$

where $\Sigma_{YX} = \mathrm{Cov}\,(Y, X)$ is the covariance of Y and X, and $\Sigma_{XX} = \mathrm{Cov}(X, X)$ indicates the covariance of $X = (X_1, \cdots, X_n)'$.

Consequently, the credibility estimator $\widehat{\mu(\Theta)}^*$ can be derived by calculation of following formula:

$$\widehat{\mu(\Theta)}^* = \mathrm{E}\left[\mu\left(\Theta\right)\right] + \Sigma_{\mu(\Theta)X}\Sigma_{XX}^{-1}(X - \mathrm{E}X) \tag{2.3}$$

Theorem 2.1 Under Assumptions 1.1, 1.2 and the notation above, the optimal credibility estimator of X_{n+1} is given by

$$\widehat{\mu(\Theta)}^* = Z\overline{X} + (1 - Z)\mu, \tag{2.4}$$

where $Z = \frac{n\tau^2}{n\tau^2 + (1-\rho+n\rho)\sigma^2}$ is so-called credibility factor.

Proof One can obtain

$$\mathrm{E}(\mu(\Theta)) = \mu, \quad \mathrm{Cov}(\mu(\Theta), X_i) = \mathrm{E}[\mathrm{Cov}(\mu(\Theta), X_i|\Theta)] + \mathrm{Cov}(\mu(\Theta), \mathrm{E}(X_i|\Theta)) = \tau^2,$$

and

$$\mathrm{Cov}(X_i, X_j) = \mathrm{E}\left[\mathrm{Cov}(X_i, X_j|\Theta)\right] + \mathrm{Cov}(\mathrm{E}\left(X_i|\Theta\right), \mathrm{E}\left(X_j|\Theta\right)) = \begin{cases} \rho\sigma^2 + \tau^2, & i \neq j, \\ \sigma^2 + \tau^2, & i = j, \end{cases} \tag{2.5}$$

which gives $\Sigma_{\mu(\Theta)X} = \mathrm{Cov}(\mu(\Theta), X) = \tau^2\mathbf{1}_n'$ and

$$\Sigma_{XX} = \begin{pmatrix} \rho\sigma^2 + \tau^2 & \cdots & \rho\sigma^2 + \tau^2 \\ \cdots & \cdots & \cdots \\ \rho\sigma^2 + \tau^2 & \cdots & \rho\sigma^2 + \tau^2 \end{pmatrix} = \left(\rho\sigma^2 + \tau^2\right)\mathbf{1}_n\mathbf{1}_n' + (1 - \rho)\sigma^2 I_n \tag{2.6}$$

By the formula (1.4) we have

$$\begin{aligned} \Sigma_{XX}^{-1} &= \left(\left(\rho\sigma^2 + \tau^2\right)\mathbf{1}_n\mathbf{1}_n^T + (1 - \rho)\sigma^2 I_n)\right)^{-1} \\ &= \frac{1}{(1-\rho)\sigma^2}I_n - \frac{1}{(1-\rho)\sigma^2}I_n\mathbf{1}_n\left(\frac{1}{\rho\sigma^2 + \tau^2} + \mathbf{1}_n'\frac{1}{(1-\rho)\sigma^2}I_n\mathbf{1}_n\right)^{-1}\mathbf{1}_n^T\frac{1}{(1-\rho)\sigma^2}I_n \\ &= \frac{1}{(1-\rho)\sigma^2}\left(I_n - \frac{\rho\sigma^2 + \tau^2}{(1-\rho)\sigma^2 + n(\rho\sigma^2 + \tau^2)}\mathbf{1}_n\mathbf{1}_n'\right). \end{aligned}$$

Therefore, the credibility estimator of of X_{n+1} is

$$\begin{aligned} \widehat{\mu(\Theta)}^* &= E\left(\mu(\Theta)\right) + \Sigma_{\mu(\Theta)X}\Sigma_{XX}^{-1}(X - EX) \\ &= \mu + \tau^2\mathbf{1}_n^{\mathrm{T}}\frac{1}{(1-\rho)\sigma^2}\left(I_n - \frac{\rho\sigma^2 + \tau^2}{(1-\rho)\sigma^2 + n(\rho\sigma^2 + \tau^2)}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right)(X - \mu\mathbf{1}_n,) \\ &= \mu + \frac{\tau^2}{(1-\rho)\sigma^2 + n(\rho\sigma^2 + \tau^2)}\sum_{i=1}^{n}(X_i - \mu) \\ &= \mu + \frac{n\tau^2}{(1-\rho)\sigma^2 + n(\rho\sigma^2 + \tau^2)}\left(\overline{X} - \mu\right) \\ &= Z\overline{X} + (1 - Z)\mu. \end{aligned}$$

Comparing the credibility estimator in Theorem 2.1 with the classical Bühlmann credibility, our results can be regarded as a generalization of Bühlmann's model. From the expression of credibility in Theorem 2.1, if we assume that $\rho = 0$, then (2.4) is reduced to Bühlmann's model. Furthermore, we also find that the (2.4) is the same as exact credibility given by (1.2) in Section 1.

3 Credibility estimator with natural weights

The credibility model with weights was developed by Bühlmann and Straub[14] and is known as the Bühlmann-Straub model. There have been broad applications of this model in insurance practice and thus it has been one of the building blocks of credibility theory. In this section, we extend the Bühlmann model with uniform dependence in error effects to the case of Bühlmann-Straub model. The assumptions are stated as follows.

Assumption 3.1 $X_1, X_2, \cdots, X_n \cdots$ are characterized by a risk parameter Θ, and Θ itself is random variable with structure distribution $\pi(\theta)$;

Assumption 3.2 Given Θ, the X_i follows the linear model: $X_i = \mu(\Theta) + \varepsilon_i$, and the errors are conditionally uniformly dependent, i.e, $\mathrm{corr}(\epsilon_i, \epsilon_j) = \rho, i \neq j$ and $\rho < 1$.

Assumption 3.3 The conditional moments are given by $\mathrm{E}(\varepsilon_i|\Theta) = 0$, $\mathrm{Var}(\varepsilon_i|\Theta) = \frac{\sigma^2(\Theta)}{w_i}, i = 1, 2, \cdots, n+1$, where w_i are known weights.

We also denote $\mathrm{E}\left[\mu(\Theta)\right] = \mu$, $\mathrm{E}\left[\sigma^2(\Theta)\right] = \sigma^2$, $\mathrm{Var}(\mu(\Theta)) = \tau^2$. Note that Assumption 3.2 imply $\mathrm{Cov}(X_i, X_j|\Theta) = \frac{\rho\sigma^2(\Theta)}{\sqrt{w_i w_j}}$. Then we derive the following conclusion.

Theorem 3.1 Under Assumptions 3.1-3.3, the credibility estimator of $\mu(\Theta)$, denoting $\widehat{\mu(\Theta)}^*$, is

$$\widehat{\mu(\Theta)}^* = Z_1 \overline{\overline{X}}^W - Z_2 \overline{\overline{X}}^{W_a} + (1 - Z_1 + Z_2)\,\mu, \tag{3.1}$$

where

$$Z_1 = \frac{W\tau^2}{(1+\Lambda\tau^2)(1-\rho)\sigma^2},\ Z_2 = \frac{\tau^2\rho W_a^2}{(1-\rho+n\rho)(1+\Lambda\tau^2)(1-\rho)\sigma^2} \tag{3.2}$$

are called generalized credibility factors, and

$$W = \sum_{i=1}^n w_i,\ W_a = \sum_{i=1}^n \sqrt{w_i},\ \Lambda = \frac{(1-\rho+n\rho)\,W - \rho W_a^2}{(1-\rho+n\rho)(1-\rho)\sigma^2}.$$

In addition,

$$\overline{\overline{X}}^W = \frac{1}{W}\sum_{i=1}^n w_i X_i,\ \ \overline{\overline{X}}^{W_a} = \frac{1}{W_a}\sum_{i=1}^n \sqrt{w_i} X_i$$

are the weighted means of the samples by weights w_i and $\sqrt{w_i}$ respectively.

Proof From the assumptions of 3.1-3.3, one can write

$$\mathrm{E}\,(\mu(\Theta)) = \mu,\ \ \mathrm{Cov}(\mu(\Theta), X_i) = \tau^2 \tag{3.3}$$

and

$$\mathrm{Cov}(X_i, X_j) = \mathrm{E}\left[\mathrm{Cov}(X_i, X_j|\Theta)\right] + \mathrm{Cov}(\mathrm{E}\,(X_i|\Theta), \mathrm{E}\,(X_j|\Theta)) = \begin{cases} \frac{\rho\sigma^2}{\sqrt{w_i w_j}} + \tau^2, & i \neq j \\ \frac{\sigma^2}{w_i} + \tau^2, & i = j \end{cases}. \tag{3.4}$$

So we have

$$\Sigma_{\mu(\Theta)X} = \mathrm{Cov}(\mu(\Theta), X) = \tau^2 \mathbf{1}_n' \tag{3.5}$$

and

$$\Sigma_{XX} = \begin{pmatrix} \frac{\sigma^2}{w_1}+\tau^2 & \frac{\rho\sigma^2}{\sqrt{w_1w_2}}+\tau^2 & \cdots & \frac{\rho\sigma^2}{\sqrt{w_1w_n}}+\tau^2 \\ \frac{\rho\sigma^2}{\sqrt{w_2w_1}}+\tau^2 & \frac{\sigma^2}{w_2}+\tau^2 & \cdots & \frac{\rho\sigma^2}{\sqrt{w_2w_n}}+\tau^2 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\rho\sigma^2}{\sqrt{w_nw_1}}+\tau^2 & \frac{\rho\sigma^2}{\sqrt{w_nw_2}}+\tau^2 & \cdots & \frac{\sigma^2}{w_1}+\tau^2 \end{pmatrix}$$
$$= \tau^2 \mathbf{1}_n \mathbf{1}_n^{\mathrm{T}} + \rho\sigma^2 \begin{pmatrix} \frac{1}{\sqrt{w_1}} & \cdots & \frac{1}{\sqrt{w_n}} \end{pmatrix}' \begin{pmatrix} \frac{1}{\sqrt{w_1}} & \cdots & \frac{1}{\sqrt{w_n}} \end{pmatrix}$$
$$+ (1-\rho)\sigma^2 \mathrm{diag}\left(\frac{1}{w_1}, \cdots, \frac{1}{w_n}\right).$$

We denote

$$E = \rho\sigma^2 \begin{pmatrix} \frac{1}{\sqrt{w_1}} & \cdots & \frac{1}{\sqrt{w_n}} \end{pmatrix}' \begin{pmatrix} \frac{1}{\sqrt{w_1}} & \cdots & \frac{1}{\sqrt{w_n}} \end{pmatrix} + (1-\rho)\sigma^2 \mathrm{diag}\left(\frac{1}{w_1}, \cdots, \frac{1}{w_n}\right).$$

By the formula (1.4) and some matrix calculation, we get

$$E^{-1} = \left(\left(\rho\sigma^2+\tau^2\right)\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}} + (1-\rho)\sigma^2 I_n\right)\right)^{-1}$$
$$= \frac{1}{(1-\rho)\sigma^2} I_n - \frac{1}{(1-\rho)\sigma^2} I_n \mathbf{1}_n \left(\frac{1}{\rho\sigma^2+\tau^2} + \mathbf{1}_n' \frac{1}{(1-\rho)\sigma^2} I_n \mathbf{1}_n\right)^{-1} \mathbf{1}_n^{\mathrm{T}} \frac{1}{(1-\rho)\sigma^2} I_n$$
$$= \frac{1}{(1-\rho)\sigma^2}\left[\mathrm{diag}\left(w_1, \cdots, w_n\right) - \frac{\rho}{1-\rho+n\rho} \begin{pmatrix} \sqrt{w_1} & \cdots & \sqrt{w_n} \end{pmatrix}' \begin{pmatrix} \sqrt{w_1} & \cdots & \sqrt{w_n} \end{pmatrix}\right].$$

So we can write

$$\Sigma_{XX}^{-1} = \left(\tau^2 \mathbf{1}_n \mathbf{1}_n' + E\right)^{-1} = E^{-1} - E^{-1}\mathbf{1}_n \left(\frac{1}{\tau^2} + \mathbf{1}_n^T E^{-1} \mathbf{1}_n\right)^{-1} \mathbf{1}_n' E^{-1}. \tag{3.6}$$

Note that

$$\mathbf{1}_n^{\mathrm{T}} E^{-1} = \frac{1}{(1-\rho)\sigma^2}\left[\begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix} - \frac{\rho W_a}{1-\rho+n\rho} \begin{pmatrix} \sqrt{w_1} & \cdots & \sqrt{w_n} \end{pmatrix}\right] \tag{3.7}$$

and

$$\mathbf{1}_n^{\mathrm{T}} E^{-1} \mathbf{1}_n = \frac{(1-\rho+n\rho)\, W - \rho W_a^2}{(1-\rho+n\rho)\,(1-\rho)\sigma^2} := \Lambda. \tag{3.8}$$

Inserting (3.3), (3.5)−(3.7) into (2.3), then the credibility estimator of $\mu(\Theta)$ is given by

$$\begin{aligned}
\widehat{\mu(\Theta)}^* &= \mathrm{E}\left(\mu(\Theta)\right) + \Sigma_{\mu(\Theta)X}\Sigma_{XX}^{-1}(X - \mathrm{E}X) \\
&= \mu + \tau^2 \mathbf{1}_n^{\mathrm{T}} \left[E^{-1} - E^{-1}\mathbf{1}_n \left(\frac{1}{\tau^2} + \mathbf{1}_n^T E^{-1} \mathbf{1}_n \right)^{-1} \mathbf{1}_n^T E^{-1} \right] (X - \mu \mathbf{1}_n) \\
&= \mu + \tau^2 \left(\mathbf{1}_n^{\mathrm{T}} E^{-1} - \frac{\Lambda\tau^2}{1+\Lambda\tau^2} \mathbf{1}_n^T E^{-1} \right) (X - \mu\mathbf{1}_n) \\
&= \mu + \frac{\tau^2}{1+\Lambda\tau^2} \mathbf{1}_n^{\mathrm{T}} E^{-1} (X - \mu\mathbf{1}_n) \\
&= \mu + \frac{\tau^2}{(1+\Lambda\tau^2)(1-\rho)\sigma^2} \left[\sum_{i=1}^n w_i (X_i - \mu) - \frac{\rho W_a}{1-\rho+n\rho} \sum_{i=1}^n \sqrt{w_i}(X_i - \mu) \right] \\
&= \mu + \frac{W\tau^2}{(1+\Lambda\tau^2)(1-\rho)\sigma^2} \left(\overline{\overline{X}}^W - \mu \right) - \frac{\tau^2 \rho W_a^2}{(1-\rho+n\rho)(1+\Lambda\tau^2)(1-\rho)\sigma^2} \left(\overline{\overline{X}}^{W_a} - \mu \right) \\
&= Z_1 \overline{\overline{X}}^W - Z_2 \overline{\overline{X}}^{W_a} + (1 - Z_1 + Z_2)\mu
\end{aligned}$$

The proof is completed.

Seeing from the (3.1), the credibility estimator of $\mu(\Theta)$ is not the strict weight form any longer. However, we can think this form the general credibility since the credibility weight factors still satisfy $Z_1 - Z_2 +$ $(1 - Z_1 + Z_2) = 1$. The general credibility with natural weights is the generalization of Bühlmann credibility in Section 2, i.e., if we take all $w_i = 1$, then (3.1) is degenerated to (2.4).

4 Conclusion

In this paper, the exact credibility is derived when the error effects are uniformly dependent under normal-normal case. In the second, the Bühlmann credibility model is investigated, and find that the credibility estimator is the same as exact credibility formula under normal-normal case. The model is also extended to Bühlmann-Straub case. However, as is shown in section 3, the credibility estimator of individual premium under Bühlmann-Straub model have only the generalized form of "credibility".

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献率的真实值并不知道, 但是三个不同模拟点数的Monte-Carlo计算值比较稳定, 因此可以将贡献率的M-C计算值作为全局分析中对称函数的敏感性度量指标. 另外, 从以上两个表我们不难发现：系统函数方差确实等于分解得到的各个对称函数方差之和, 这正是本文定理 7所要求的. 因此, 选择文献[7, 8]中对称函数的定义来研究对称性也就合情合理.

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