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Derivations and 2-cocycles of the algebra of (r,s)-differential operators

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Abstract: This paper defined the (r,s)-differential operator of the algebra of Laurent polynomials over the complex numbers field. Let $\mathcal{D}_{r,s}$ be the associative algebra generated by $\{t^{\pm 1}\}$ and the (r,s)-differential operator, which is called (r,s)-differential operators algebra. In this paper, the derivation algebra of $\mathcal{D}_{r,s}$ and its Lie algebra $\mathcal{D}^-_{r,s}$ were described and all the non-trivial 2-cocycles were determined.
Key words: (r,s)-differential operator; Derivation; 2-cocycle.
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(r,s)-微分算子代数的导子及其二上圈

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摘要: 定义复数域$\mathbb{C}$上的 Laurent 多项式代数 $\mathbb{C}[t,t^{-1}]$ 的 (r,s)-微分算子 $\partial_{r,s}$. 给出该微分算子及 $\{t^{\pm 1}\}$ 生成的结合代数即 (r,s)-微分算子代数的一组基, 并在此基础上研究了 (r,s)-微分算子代数的导子代数及其非平凡二上圈.
关键词: (r,s)-微分算子; 导子; 二上圈

0 Introduction

Let $\mathbb{C}[t,t^{-1}]$ be the algebra of Laurent polynomials over the complex number field $\mathbb{C}$, and $\mathcal{D}=\mathrm{Diff}\ \mathbb{C}[t,t^{-1}]$ the associative algebra of all differential operators over $\mathbb{C}[t,t^{-1}]$, its $\mathbb{C}$-basis being $\{t^m\tilde{D}^n \mid m\in\mathbb{Z}, n\in\mathbb{Z}^+\}$ with multiplication:

$$(t^a\tilde{D}^b)\cdot(t^c\tilde{D}^d)=\sum_{i=0}^{b}\binom{b}{i}c^it^{a+c}\tilde{D}^{b+d-i},$$

where $\mathbb{Z}^+$ is the set of all non-negative integers and $\tilde{D}=t\partial, \partial=\dfrac{\mathrm{d}}{\mathrm{dt}}$.

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Let q be a complex number not 0 or 1, and define q-differential operator ∂_q by

$$\partial_q(P)=\frac{P(qt)-P(t)}{qt-t},\quad \forall P\in\mathbb{C}[t,t^{-1}].$$

Then $\partial_q(PQ)=\partial_q(P)Q+\tau_q(P)\partial_q(Q)$, $\forall\, P,Q\in\mathbb{C}[t,t^{-1}]$, where τ_q is the automorphism of $\mathbb{C}[t,t^{-1}]$ satisfying $\tau_q(t)=qt$, and ∂_q is also called a τ_q-derivation. Clearly, $\{t^i\partial_q\mid i\in\mathbb{Z}\}$ is a $\mathbb{C}$-basis of the vector space of all τ_q-derivations of $\mathbb{C}[t,t^{-1}]$.

Let $\mathcal{D}_q$ be the associative algebra generated by t,t^{-1} and ∂_q, so $\mathcal{D}_q$ is the q-differential operators algebra of $\mathbb{C}[t,t^{-1}]$.

In recent years, there have been many researches on the algebra of differential operators and q-differential operators. Concerning their derivation algebras, automorphisms, 2-cocycles and some representations, researches were undertaken by Frenkel et al. in [1](1995), Kac and Radul in [2](1993), Kassel in [3](1992), Li in [4](1989), Li and Wilson in [5] (1998), Su in [6] (1990), Zhao in [7] and [8](1993,1995), Hu in [9] (1999), Liu and Hu in [10](2004). For instance, in [7], the derivation algebras of $\mathcal{D}$ and of its Lie algebra $\mathcal{D}^-$ were determined. In [4], all non-trivial 2-cocycles on $\mathcal{D}^-$ were determined. In [10], the derivation algebras of $\mathcal{D}_q$ and of its Lie algebra $\mathcal{D}_q^-$ and all non-trivial 2-cocycles on $\mathcal{D}_q^-$ were determined.

Let $r,s\in\mathbb{C}$ with $r,s\neq 0,1$ and $r^2\neq s^2$. Define (r,s)-differential operator by

$$\partial_{r,s}(P)=\frac{P(rt)-P(st)}{rt-st},\quad \forall P\in\mathbb{C}[t,t^{-1}].$$

then

$$\partial_{r,s}(PQ)=\partial_{r,s}(P)\xi(Q)+\omega(P)\partial_{r,s}(Q),\quad \forall P,Q\in\mathbb{C}[t,t^{-1}],$$

where ω,ξ are two automorphisms of $\mathbb{C}[t,t^{-1}]$ satisfying $\omega(t)=rt,\xi(t)=st$, then $\partial_{r,s}$ is also called (ω,ξ)-derivation. It can be easily verified that

$$\partial_{r,s}\cdot t-rt\cdot\partial_{r,s}=\xi,\quad \partial_{r,s}\cdot t-st\cdot\partial_{r,s}=\omega.$$

By definition, the (r,s)-differential operators algebra $\mathcal{D}_{r,s}=\mathrm{Diff}_{r,s}\mathbb{C}[t,t^{-1}]$ is an associative algebra generated by t,t^{-1} and $\partial_{r,s}$. With easy calculation, we can obtain

$$t\partial_{r,s}\cdot\xi=\xi\cdot t\partial_{r,s}.$$

Let $D=\xi+(r-s)t\partial_{r,s}$. Then $D\in\mathcal{D}_{r,s}$ and we can get the following lemma.

Lemma 0.1 In $\mathcal{D}_{r,s}$, $D\xi=\xi D$, $\xi t=st\xi$, $Dt=rtD$.

Lemma 0.2 $\{\,t^mD^n\xi^q\mid m\in\mathbb{Z},n,q\in\mathbb{Z}^+\}$ is a $\mathbb{C}$-basis of $\mathcal{D}_{r,s}$, where r,s satisfy $r^xs^y\neq 1\,(x,y\in\mathbb{Z},\ (x,y)\neq(0,0))$.

Proof For $\mathcal{D}_{r,s}$ is generated by t,t^{-1} and $\partial_{r,s}$, and for $\partial_{r,s}\cdot t-rt\cdot\partial_{r,s}=\xi$, $\xi t=st\xi$, $t^{-1}\partial_{r,s}-r\partial_{r,s}t^{-1}=s^{-1}t^{-2}\xi$, $\partial_{r,s}\xi=s\xi\partial_{r,s}$, it is very clear that $\{\,t^m\partial_{r,s}^n\xi^q\mid m\in\mathbb{Z},n,q\in\mathbb{Z}^+\}$ is a $\mathbb{C}$-basis of $\mathcal{D}_{r,s}$.

Because $m\in\mathbb{Z}$, we can see that $\{\,t^m(t\partial_{r,s})^n\xi^q\mid m\in\mathbb{Z},n,q\in\mathbb{Z}^+\}$ is also a $\mathbb{C}$-basis of $\mathcal{D}_{r,s}$. For $D=\xi+(r-s)t\partial_{r,s}$ and $D\xi=\xi D$, $Dt=rtD$, then $t\partial_{r,s}=(r-s)^{-1}(D-\xi)$. As a result, $\{\,t^m(D)^n\xi^q\mid m\in\mathbb{Z},n,q\in\mathbb{Z}^+\}$ is a $\mathbb{C}$-basis of $\mathcal{D}_{r,s}$.

The main purpose of this paper is to determine all derivations and 2-cocycles of the algebras of (r,s)-differential operators.

The paper is organized as follows. In Section 1, we determine the derivation algebra of associative algebra $\mathcal{D}_{r,s}$. In Section 2, the derivation algebra of Lie algebra $\mathcal{D}^-_{r,s}$ is described, which is different from that of associative algebra of $\mathcal{D}_{r,s}$. In Section 3, we calculate $H^2(\mathcal{D}^-_{r,s},\mathbb{C})$ under the assumption that $r^xs^y\neq 1\,(x,y\in\mathbb{Z},\ (x,y)\neq(0,0)\,)$.

Throughout this paper we always suppose that r,s satisfy $r^xs^y\neq 1\,(x,y\in\mathbb{Z},\ (x,y)\neq (0,0))$.

1 Derivations of the algebra $\mathcal{D}_{r,s}$

At first, we prove some properties of $\mathcal{D}_{r,s}$ under the assumption that $r^xs^y\neq 1(\,x,y\in\mathbb{Z},\ \ (x,y)\neq(0,0)\,)$. For any nonempty set $B\subseteq\mathcal{D}_{r,s}$, we define the centralizer of B in $\mathcal{D}_{r,s}$ as $Z_{\mathcal{D}_{r,s}}(B)$.

Lemma 1.1 $Z_{\mathcal{D}_{r,s}}(t^p)=\mathbb{C}[t,t^{-1}](p\neq 0)$, $Z_{\mathcal{D}_{r,s}}(D)=\mathbb{C}[D,\xi]=Z_{\mathcal{D}_{r,s}}(\xi)$.

Proof It is clear that $\mathbb{C}[t,t^{-1}]\subseteq Z_{\mathcal{D}_{r,s}}(t^p)$.

For $p\neq 0$, let $d=\sum\limits_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}} d(m,n,q)t^mD^n\xi^q\in Z_{\mathcal{D}_{r,s}}(t^p)$, then

$$\left[\sum_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}} d(m,n,q)t^mD^n\xi^q,t^p\right]=\sum_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}}((r^ns^q)^p-1)d(m,n,q)t^{m+p}D^n\xi^q=0,$$

which implies that $d(m,n,q)=0$ for $(n,q)\neq(0,0)$, and then $d=\sum\limits_{m\in\mathbb{Z}} d(m,0,0)t^m\in\mathbb{C}[t,t^{-1}]$.

Similarly, we have $Z_{\mathcal{D}_{r,s}}(D)=\mathbb{C}[D,\xi]=Z_{\mathcal{D}_{r,s}}(\xi)$.

Lemma 1.2 $\mathcal{D}_{r,s}$ has outer derivations $\sigma_i(i=1,2,3)$, satisfying

$$\sigma_1(t)=t,\sigma_1(D)=\sigma_1(\xi)=0,\sigma_2(D)=D,\quad \sigma_2(t)=\sigma_2(\xi)=0,\quad \sigma_3(\xi)=\xi,\sigma_3(D)=\sigma_3(t)=0.$$

Proof Define linear maps $\sigma_1(t^mD^n\xi^q)=mt^mD^n\xi^q$, $\sigma_2(t^mD^n\xi^q)=nt^mD^n\xi^q$, $\sigma_3(t^mD^n\xi^q)=qt^mD^n\xi^q$. Obviously $\sigma_i\in\mathrm{Der}\,(\mathcal{D}_{r,s})(i=1,2,3)$.

If σ_1 is an inner derivation, there exists $y=\sum\limits_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}} c(m,n,q)t^mD^n\xi^q\in\mathcal{D}_{r,s}$, such that $\sigma_1=\mathrm{ad}\,y$. Then $\mathrm{ad}\,y(t)=(r^ns^q-1)\sum\limits_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}} c(m,n,q)t^{m+1}D^n\xi^q=t\in\mathcal{D}_{r,s}$, but it is impossible.

Similarly, σ_2,σ_3 are not inner derivations.

Lemma 1.3 For $\alpha\in\mathrm{Der}\,(\mathcal{D}_{r,s})$, there exists $x\in\mathcal{D}_{r,s}$ such that $\alpha(t)-\mathrm{ad}\,x(t)\in\mathbb{C}[t,t^{-1}]$.

Proof For a given $\alpha\in\mathrm{Der}\,(\mathcal{D}_{r,s})$, let $\alpha(t)=\sum\limits_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}} c(m,n,q)t^mD^n\xi^q$ with $c(m,n,q)\in\mathbb{C}$. Take $x=\sum\limits_{\substack{n,q\in\mathbb{Z}^+\\ n+q\neq 0,m\in\mathbb{Z}}}(r^ns^q-1)^{-1}c(m,n,q)t^{m-1}D^n\xi^q$, then we have

$$\alpha(t)-\mathrm{ad}\,x(t)=\sum_{m\in\mathbb{Z}} c(m,0,0)t^m\in\mathbb{C}[t,t^{-1}].$$

Theorem 1 The derivation algebra of $\mathcal{D}_{r,s}$ is $\mathrm{ad}\,(\mathcal{D}_{r,s})\bigoplus\sum\limits_{i=1}^{3}\mathbb{C}\sigma_i$.

Proof (i) For a given $\alpha_0\in\mathrm{Der}\,(\mathcal{D}_{r,s})$, by Lemma 1.3, there exists an $x\in\mathcal{D}_{r,s}$, such that $\alpha_0(t)-\mathrm{ad}\,x(t)\in\mathbb{C}[t,t^{-1}]$. Denote $\alpha=\alpha_0-\mathrm{ad}\,x$, and we can assume that $\alpha(t)=\sum\limits_{m\in\mathbb{Z}} a_mt^m$.

Write $\alpha(D)=\sum\limits_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}} c(m,n,q)t^mD^n\xi^q$, $\alpha(\xi)=\sum\limits_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}} e(m,n,q)t^mD^n\xi^q$, where $c(m,n,q),e(m,n,q)\in\mathbb{C}$. Acting α on both sides of $Dt=rtD$, we have

$$\alpha(D)t+D\alpha(t)=r\alpha(t)D+rt\alpha(D).\tag{1.1}$$

Thus we get

$$\sum_{\substack{m\in\mathbb{Z}\\ n,q\in\mathbb{Z}^+}}(r^ns^q-r)c(m,n,q)t^{m+1}D^n\xi^q=\sum_{m\in\mathbb{Z}}(r-r^m)a_mt^mD,\tag{1.2}$$

which implies that $c(m,n,q)=0$ for $n\neq 1$ or $q\neq 0$, and $r^ns^q-r=0$ for $n=1$ and $q=0$. That is to say, $\sum\limits_{m\in\mathbb{Z}}(r-r^m)a_mt^mD=0$. Hence $a_m=0$ for $m\neq 1$.

Similarly, acting α on both sides of $\xi t=st\xi$, we have $e(m,n,q)=0$ for $n\neq 0$ or $q\neq 1$. Thus, we obtain that

$$\alpha(t)=a_1t,\quad \alpha(D)=\sum_{m\in\mathbb{Z}}c(m)t^mD,\quad \alpha(\xi)=\sum_{m\in\mathbb{Z}}e(m)t^m\xi.$$

(ii) Furthermore, using $D\xi=\xi D$, we can get $\alpha(D)\xi+D\alpha(\xi)=\alpha(\xi)D+\xi\alpha(D)$, that is

$$\sum_{m\in\mathbb{Z}}(c(m)+r^me(m))t^mD\xi=\sum_{m\in\mathbb{Z}}(s^mc(m)+e(m))t^mD\xi,$$

then $e(m)=\dfrac{1-s^m}{1-r^m}c(m)$ for $m\neq 0$. Hence we have

$$\alpha(D)=\sum_{m\in\mathbb{Z},m\neq 0}c(m)t^mD+cD,\quad \alpha(\xi)=\sum_{m\in\mathbb{Z},m\neq 0}\frac{1-s^m}{1-r^m}c(m)t^m\xi+e\xi.$$

(iii) We can easily get $\mathrm{ad}\,t^m(D)=(1-r^m)t^mD$ and $\mathrm{ad}\,t^m(\xi)=(1-s^m)t^m\xi$ for $m\neq 0$. Take $y=\sum\limits_{m\in\mathbb{Z},m\neq 0}\dfrac{c(m)}{1-r^m}t^m$, and denote $\gamma=\alpha-\mathrm{ad}\,y$, then we have $\gamma(t)=a_1t$, $\gamma(D)=cD$ and $\gamma(\xi)=e\xi$. Since γ is a derivation, $\gamma=a_1\sigma_1+c\sigma_2+e\sigma_3$.

This means $\alpha_0=\mathrm{ad}\,(x+y)+a_1\sigma_1+c\sigma_2+e\sigma_3$. The proof is complete.

2 Derivations of the Lie algebra $\mathcal{D}^-_{r,s}$

Lie algebra $\mathcal{D}^-_{r,s}$ of the associative algebra $\mathcal{D}_{r,s}$ is called the (r,s)-differential operators Lie algebra. Clearly

$$\mathrm{Der}\,(\mathcal{D}_{r,s})\subseteq \mathrm{Der}\,(\mathcal{D}^-_{r,s}),\qquad \mathrm{ad}\,(\mathcal{D}_{r,s})=\mathrm{ad}\,(\mathcal{D}^-_{r,s}).$$

In this section, we will determine the derivation algebra of $\mathcal{D}^-_{r,s}$.

Lemma 2.1 The Lie algebra $\mathcal{D}^-_{r,s}$ is generated by $\{t^m,D,\xi\mid m\in\mathbb{Z}\}$.

Proof Let $\mathcal{A}$ denote the Lie subalgebra of $\mathcal{D}^-_{r,s}$ generated by $\{t^m,D,\xi\mid m\in\mathbb{Z}\}$. By Lemma 1.2, we only need to prove that $t^mD^n\xi^q\in\mathcal{A}(m\in\mathbb{Z},n,q\in\mathbb{Z}^+)$.

Clearly $t^m\in\mathcal{A}(m\in\mathbb{Z})$. Suppose we have proved $t^mD^n\xi^q\in\mathcal{A}(m\neq 0)$, then $[t^mD^n\xi^q,D]=(1-r^m)t^mD^{n+1}\xi^q\in\mathcal{A}$ and $[t^mD^n\xi^q,\xi]=(1-s^m)t^mD^n\xi^{q+1}\in\mathcal{A}$, then we have $t^mD^n\xi^q\in\mathcal{A}(m\neq 0)$.

Furthermore, for $[tD^n\xi^q,t^{-1}]=(1-r^ns^q)D^n\xi^q$ $(n+q\neq 0)$, $D^n\xi^q\in\mathcal{A}$.

Lemma 2.2 Lie algebra $\mathcal{D}^-_{r,s}$ has outer derivations $\zeta_i(i\in\mathbb{Z}\setminus\{0\})$ satisfying $\zeta_i(t^j)=\delta_{i,j},\zeta_i(D)=\zeta_i(\xi)=0$.

Proof Define linear maps $\zeta_i(i\in\mathbb{Z}\setminus\{0\})$ by $\mathcal{D}^-_{r,s}$ as: $\zeta_i(t^mD^n\xi^q)=\delta_{i,m}\delta_{n+q,0}$.

$$\begin{aligned}\zeta_i([t^mD^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1}])&=(r^{nm_1}s^{qm_1}-r^{n_1m}s^{q_1m})\delta_{i,m+m_1}\delta_{n+n_1+q+q_1,0}\\&=[\zeta_i(t^mD^n\xi^q),t^{m_1}D^{n_1}\xi^{q_1}]+[t^mD^n\xi^q,\zeta_i(t^{m_1}D^{n_1}\xi^{q_1})]\\&=0.\end{aligned}$$

Thus $\zeta_i\in\mathrm{Der}\,(\mathcal{D}^-_{r,s})$.

If ζ_i is not an outer derivation, then there exists $y_i \in \mathcal{D}^-_{r,s}$ such that $\mathrm{ad}\, y_i = \zeta_i$. Thus $\mathrm{ad}\, y_i(D) = 0$ and $\mathrm{ad}\, y_i(t^m) = 0 (i \neq m)$. By lemma 1.1, we can conclude that $y_i \in \mathbb{C}[t, t^{-1}] \cap \mathbb{C}[D, \xi] = \mathbb{C}$ and $\mathrm{ad}\, y_i(t^i) = 0$, contradicting with $\zeta_i(t^i) = 1$. As a result, $\zeta_i (i \in \mathbb{Z} \setminus \{0\})$ are outer derivations of $\mathcal{D}_{r,s}$.

Lemma 2.3 For $\alpha \in \mathrm{Der}\,(\mathcal{D}^-_{r,s})$, there exists $x \in \mathcal{D}^-_{r,s}$ such that $\alpha(t) - \mathrm{ad}\, x(t) \in \mathbb{C}[t, t^{-1}]$.

Proof The proof is similar to that of Lemma 1.3.

Theorem 2 Derivation algebra of $\mathcal{D}^-_{r,s}$ is $\mathrm{ad}\,(\mathcal{D}_{r,s}) \oplus \sum\limits_{i=1}^{3} \mathbb{C}\sigma_i \oplus \sum\limits_{j \in \mathbb{Z}\setminus\{0\}} \zeta_j$.

Proof (i) Given any $\alpha_0 \in \mathrm{Der}\,(\mathcal{D}^-_{r,s})$, owing to lemma 2.3, there exists $x \in \mathcal{D}^-_{r,s}$, such that $\alpha_0(t) - \mathrm{ad}\, x(t) \in \mathbb{C}[t, t^{-1}]$. Set $\alpha = \alpha_0 - \mathrm{ad}\, x$. Since $[t, t^m] = 0$, we can obtain $\alpha(t^m) \in \mathbb{C}[t, t^{-1}]$, thus $\alpha(t) = \sum\limits_{m \in \mathbb{Z}} a_m t^m, \alpha(t^{-1}) = \sum\limits_{m \in \mathbb{Z}} b_m t^m$. Write

$$\alpha(D) = \sum_{\substack{m \in \mathbb{Z} \\ n,q \in \mathbb{Z}^+}} c(m,n,q) t^m D^n \xi^q, \alpha(\xi) = \sum_{\substack{m \in \mathbb{Z} \\ n,q \in \mathbb{Z}^+}} e(m,n,q) t^m D^n \xi^q.$$

Acting α on $[t, D] = (1-r)tD$ and $[t^{-1}, tD] = (1-r^{-1})D$, respectively we have

$$\alpha(\,[t, D]\,) = [\,\alpha(t), D\,] + [\,t, \alpha(0)\,] = (1-r)\alpha(tD), \tag{2.1}$$

$$\alpha(\,[t^{-1}, tD]\,) = [\,\alpha(t^{-1}), tD\,] + [\,t^{-1}, \alpha(tD)\,] = (1-r^{-1})\alpha(D). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$[\,\alpha(t^{-1}), tD\,] + \left[t^{-1}, \frac{1}{1-r}\alpha(\,[t, D]\,)\right] = \left(1 - \frac{1}{r}\right)\alpha(D). \tag{2.3}$$

In (2.3),

$$[\,\alpha(t^{-1}), tD\,] = \sum_{m \in \mathbb{Z}} b_m [\,t^m, tD\,] = \sum_{m \in \mathbb{Z}} b_m (1 - r^m) t^{(m+1)} D,$$

$$\begin{aligned} [\,t^{-1}, \frac{1}{1-r}\alpha(\,[t, D]\,)\,] &= [\,t^{-1}, \frac{1}{1-r}\,(\,[\,\alpha(t), D\,] + [\,t, \alpha(D)\,]\,)\,] \\ &= \frac{1}{1-r}(\,[\,t^{-1}, [\,\alpha(t), D\,]\,] + [\,t^{-1}, [\,t, \alpha(D)\,]\,]\,), \end{aligned}$$

Using equations above and (2.3), we obtain

$$\sum_{\substack{m \in \mathbb{Z} \\ n,q \in \mathbb{Z}^+}} \frac{1}{r(1-r)} [r(r^{\frac{n}{2}} s^{\frac{q}{2}} - r^{\frac{-n}{2}} s^{\frac{-q}{2}})^2 - (1-r)^2] c(m,n,q) t^m D^n \xi^q$$

$$= \sum_{m \in \mathbb{Z}} [\frac{r^{m+1} - 1}{r} a_{m+1} + (1 - r^{m-1} b_{m-1})] t^m D. \tag{2.4}$$

In (2.4), we have $[r(r^{\frac{n}{2}} s^{\frac{q}{2}} - r^{\frac{-n}{2}} s^{\frac{-q}{2}})^2 - (1-r)^2] c(m,n,q) = 0$, for $n \neq 1$ or $q \neq 0$.

Assume $n, q \in \mathbb{Z}^+$ satisfying $[r(r^{\frac{n}{2}} s^{\frac{q}{2}} - r^{\frac{-n}{2}} s^{\frac{-q}{2}})^2 - (1-r)^2] = 0$, then we have $r^n s^q + r^{-n} s^{-q} = r + r^{-1}$. It is easy to obtain that $n = 1$ and $q = 0$. As a result, $c(m,n,q) = 0$ for $n \neq 1$ or $q \neq 0$, that is, $\sum\limits_{m \in \mathbb{Z}} [\frac{r^{m+1} - 1}{r} a_{m+1} + (1 - r^{m-1} b_{m-1})] t^m D = 0$. Denote $(m)_r = \dfrac{1 - r^m}{1 - r}$, and we finally get

$$\alpha(D) = \sum_{m \in \mathbb{Z}} c(m) t^m D, \quad (m+1)_r a_{m+1} = r(m-1)_r b_{m-1}.$$

Similar treatment to $\alpha(\xi)$, we obtain

$$\alpha(\xi)=\sum_{m\in\mathbb{Z}}e(m)t^m\xi,\quad (m+1)_s a_{m+1}=s(m-1)_s b_{m-1},\quad (m)_s=\frac{1-s^m}{1-s}.$$

(ii) Similar to the proof of part (ii) of Theorem 1, let α act on $[D,\xi]=0$, and get $e(m)=\frac{1-s^m}{1-r^m}c(m)$ for $m\neq 0$. Then we have

$$\alpha(D)=\sum_{m\in\mathbb{Z},m\neq 0}c(m)t^mD+cD,\quad \alpha(\xi)=\sum_{m\in\mathbb{Z},m\neq 0}\frac{1-s^m}{1-r^m}c(m)t^m\xi+e\xi.$$

(iii) Choose $y=\frac{c(m)}{1-r^m}t^m$, then $(\alpha-\operatorname{ad}y)(t^m)=\alpha(t^m),(\alpha-\operatorname{ad}y)(D)=cD,(\alpha-\operatorname{ad}y)(\xi)=e\xi$. Denote $\gamma=\alpha-\operatorname{ad}y-a_1\sigma_1-c\sigma_2-e\sigma_3$, we have $\gamma(D)=\gamma(\xi)=0$, $\gamma(t)=\alpha(t)$ and $\gamma(t^{-1})=\alpha(t^{-1})$.

Acting γ on $[t,D]=(1-r)tD,[t^{-1},D]=(1-r^{-1})t^{-1}D,[t,D^2]=(1-r^2)tD^2$, and $[tD,t^{-1}D]=(r^{-1}-r)D^2$, respectively we have

$$\gamma(tD)=\sum_{m\in\mathbb{Z}}(m)_r a_m t^mD,\quad \gamma(t^{-1}D)=\sum_{m\in\mathbb{Z}}r(m)_r b_m t^mD, \tag{2.5}$$

$$(r^{-1}-r)\gamma(D^2)=[\gamma(tD),t^{-1}D]+[tD,\gamma(t^{-1}D)], \tag{2.6}$$

$$(1+r)[\gamma(tD),D]=[\gamma(t),D^2]+[t,\gamma(D)^2]. \tag{2.7}$$

By (2.5) and (2.6), we have

$$\gamma(D^2)=\frac{r}{1-r^2}\left(\sum_{m\in\mathbb{Z}}(m)_r\left(\frac{1}{r}-r^m\right)a_mt^{m-1}D^2+\sum_{m\in\mathbb{Z}}r(m)_r(r-r^m)b_mt^{m+1}D^2\right),$$

In (2.6), we can calculate that

$$[\gamma(t),D^2]=\sum_{m\in\mathbb{Z}}(1-r^{2m})a_mt^mD^2,$$

$$\begin{aligned}[t,\gamma(D^2)] &= \sum_{m\in\mathbb{Z}}(m)_r(1-r^{m+1})a_mt^mD^2+\sum_{m\in\mathbb{Z}}r(m-2)_r(r^2-r^{m-1})b_{m-2}t^mD^2\\ &= (m)_ra_m\sum_{m\in\mathbb{Z}}(1-r^{m+1}+r^2-r^{m-1})(m)_ra_mt^mD^2,\end{aligned}$$

$$(1+r)[\gamma(tD),D]=(1+r)\sum_{m\in\mathbb{Z}}(1-r^m)(m)_ra_mt^mD^2.$$

Using all the equations above and (2.7), we obtain that

$$(m)_ra_m(1+r)(1-r^m)=(m)_ra_m[(1+r^m)(1-r)+1+r^2-r^{m+1}-r^{m-1}].$$

Simplifying the equation above, we have

$$(m)_r(1-r^{m-1})(r-1)^2a_m=0,$$

which implies that $a_m=0(m\neq 0,1)$, $b_m=0(m\neq 0,-1)$ and $a_1=b_{-1}$. Thus

$$\gamma(t)=a_1t+a_0,\quad \gamma(t^{-1})=-a_1t+b_0.$$

(iv) Denote $\beta=\gamma-a_1\sigma_1-a_0\zeta_1-b_0\zeta_{-1}$, then $\beta(t)=\beta(t^{-1})=\beta(D)=\beta(\xi)=0$, and $\beta(t^m)\in\mathbb{C}[t,t^{-1}]$. For $[t,D]=(1-r)tD,[t^{-1},D]=(1-r^{-1})t^{-1}D$, $\beta(tD)=\beta(t^{-1}D)=0$. Then $\beta([t^2,D])=(1-r^2)\beta(t^2D)=(1-r^2)(1-r)\beta([t,tD])=0$. By lemma 2.1, we have $\beta(t^2)\in\mathbb{C}[t,t^{-1}]\cap\mathbb{C}[D,\xi]=\mathbb{C}$. Similarly, $\beta(t^{-2})\in\mathbb{C}$.

Suppose that we have proved $\beta(t^m)\in\mathbb{C}$ and $\beta(t^mD)=0$. Owing to $[t,t^mD]=(1-r)t^{m+1}D$, $\beta(t^{m+1}D)=0$,and then we have $\beta(t^{m+1})\in\mathbb{C}$ since $[\beta(t^{m+1}),D]=(1-r^{m+1})\beta(t^{m+1}D)=0$. Similarly, $\beta(t^{m-1})\in\mathbb{C}$.

Let $h_m=\beta(t^m)$ and $\eta=\beta-\sum\limits_{\substack{i\neq 0,\pm1\\ i\in\mathbb{Z}}}h_i\zeta_i$, then $\eta(t^m)=\eta(D)=\eta(\xi)=0$. By lemma 2.1, we have $\eta=0$. Finally,

$$\alpha_0=\mathrm{ad}\,(x+y)+a_1\sigma_1+c\sigma_2+e\sigma_3+a_0\zeta_1+b_0\zeta_{-1}+\sum_{i\neq 0,1,-1}h_i\zeta_i.$$

The proof is complete.

3 $H^2(\mathcal{D}^-_{r,s},\mathbb{C})$

Let L be a Lie algebra over $\mathbb{C}$. Recall that a 2-cocycle on L is a bilinear $\mathbb{C}$-valued form ψ satisfying the following conditions:

(1) $\psi(a,b)=-\psi(b,a)$,

(2) $\psi([a,b],c)+\psi([b,c],a)+\psi([c,a],b)=0\quad$ for all $a,b,c\in L$.

If f is a linear function on L, we define

$$\alpha_f(x,y)=f([x,y]),$$

for $x,y\in L$, then α_f is a 2-cocycle. This 2-cocycle is called a trivial 2-cocycle. A 2-cocycle φ is equivalent to a 2-cocycle ψ, if $\varphi-\psi$ is trivial.

Given a 2-cocycle α on L, we can construct a central extension of L. If the Lie bracket on L is $[\,,]$, we define a new Lie bracket $[\ ,\ ]_0$ on $L\oplus\mathbb{C}c$ as

$$[x+\lambda c,y+\mu c]_0=[x,y]+\alpha(x,y)c,\qquad \forall x,y\in L,\ \lambda,\mu\in\mathbb{C}.$$

It is well-known that $L\oplus\mathbb{C}c$ is a Lie algebra with this Lie bracket and every 1-dimentional central extension of L can be obtained in this way. Denote this central extension of L by $L(\alpha)$.

Let γ be a trivial 2-cocycle induced by $f\in L^*$ and α a cocycle, then the mapping

$$x\mapsto x+f(x)c,\quad c\mapsto c,\ \forall x\in L,$$

gives an isomorphism from $L(\alpha)$ to $L(\alpha+\gamma)$.

In [4] and [6], all non-trivial 2-cocycle on the algebras of differential operators were determined. In [10], all non-trivial 2-cocycle of $\mathcal{D}_q$ also be determined. $H^2(\mathcal{D}^-_{r,s},\mathbb{C})$ is different from $H^2(\mathcal{D}^-,\mathbb{C})$, for the number of generators of Lie algebra $\mathcal{D}^-_{r,s}$ is infinite. It is similar to $H^2(\mathcal{D}^-_q,\mathbb{C})$, but the calculation is much more complicated. In this section, we will try to determine all the 2-cocycles on $\mathcal{D}^-_{r,s}$ under the assumption that $r^xs^y\neq 1(x,y\in\mathbb{Z},(x,y)\neq(0,0))$.

Let ψ be a 2-cocycle on $\mathcal{D}^-_{r,s}$. We define a linear function f_ψ on $\mathcal{D}^-_{r,s}$ as

$$f_\psi(D^n\xi^q)=\frac{1}{r^ns^q-1}\psi(t^{-1}D^n\xi^q,t),\quad n+q>0,$$

$$f_\psi(t^mD^n\xi^q)=\left\{\begin{array}{ll}(1-r^m)^{-1}\psi(t^mD^{n-1}\xi^q,D), & n>0\\(1-s^m)^{-1}\psi(t^mD^n\xi^{q-1},\xi), & q>0\end{array}\right.\ (m\neq 0).$$

It is easy to check that, for $n>0$ and $q>0$, we have

$$(1-r^m)^{-1}\psi(t^mD^{n-1}\xi^q,D)=(1-s^m)^{-1}\psi(t^mD^n\xi^{q-1},\xi),$$

that is, f_ψ is well-defined.

Denote $\theta=\psi-\alpha_{f_\psi}$, then we have

$$\begin{array}{rcl}\theta(t^mD^{n-1}\xi^q,D) &=& 0\quad (n>0),\\ \theta(t^mD^n\xi^{q-1},\xi) &=& 0\quad (q>0),\\ \theta(t^{-1}D^n\xi^q,t) &=& 0\quad (n+q>0),\\ \theta(tD^n\xi^q,t^{-1}) &=& (1-r^{-n}s^{-q})/(1-r^ns^q)\ \theta(t^{-1}D^n\xi^q,t)\\ &=& 0\quad (n+q>0).\end{array}\tag{3.1}$$

Lemma 3.1 $\theta(t^mD^n\xi^q,D^{n_1}\xi^{q_1})=0$, for $m\neq 0$ and $n+q>0$.

Proof For $n+q>0$, we can assume that $n>0$. Then we have:

$$\begin{array}{rcl}\theta(t^mD^n\xi^q,D^{n_1}\xi^{q_1}) &=& \dfrac{1}{1-r^m}\theta([t^mD^{n-1}\xi^q,D],D^{n_1}\xi^{q_1})\\ &=& \dfrac{-1}{1-r^m}\theta([D^{n_1}\xi^{q_1},t^mD^{n-1}\xi^q],D])\\ &=& \dfrac{1-r^{n_1m}s^{q_1m}}{1-r^m}\ \theta(t^mD^{n+n_1-1}\xi^{q+q_1},D)=0.\end{array}$$

Lemma 3.2 $\theta(1,t^mD^n\xi^q)=0$, for $n+q>0$.

Proof $\theta(1,t^mD^n\xi^q)=\frac{1}{1-r^{-n}s^{-q}}\ \theta(1,[t^{-1},t^{m+1}D^n\xi^q])=0$.

Lemma 3.3 $\theta(t^mD^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1})=0$, for $m+m_1\neq 0$, $n+q>0$ or $n_1+q_1>0$.

Proof (i) For $n+q>1(n>0,m\neq 0)$,

$$\begin{array}{rcl}\theta(t^mD^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1}) &=& \dfrac{1}{1-r^m}\theta([t^mD^{n-1}\xi^q,D],t^{m_1}D^{n_1}\xi^{q_1})\\ &=& \dfrac{-1}{1-r^m}(\theta([D,t^{m_1}D^{n_1}\xi^{q_1}],t^mD^{n-1}\xi^q)\\ && +\theta([t^{m_1}D^{n_1}\xi^{q_1},t^mD^{n-1}\xi^q],D))\\ &=& -\dfrac{1-r^{m_1}}{1-r^m}\theta(t^mD^{n-1}\xi^q,t^{m_1}D^{n_1+1}\xi^{q_1})\\ &=& (-1)^{n+q}\dfrac{(1-r^{m_1})^n}{(1-r^m)^n}\dfrac{(1-s^{m_1})^q}{(1-s^m)^q}\theta(t^m,t^{m_1}D^{n_1+n}\xi^{q_1}),\end{array}$$

$$\begin{array}{rcl}\theta(t^mD^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1}) &=& \dfrac{1}{1-r^{nm}s^{qm}}\theta([t^m,D^n\xi^q],t^{m_1}D^{n_1}\xi^{q_1})\\ &=& \dfrac{1}{1-r^{nm}s^{qm}}(\theta([D^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1}],t^m)\\ && +\theta([t^{m_1}D^{n_1}\xi^{q_1},t^m],D^n\xi^q))\\ &=& -\dfrac{1-r^{nm_1}s^{qm_1}}{1-r^{nm}s^{qm}}\theta(t^m,t^{m_1}D^{n_1+n}\xi^{q_1+q}).\end{array}$$

Then we have

$$\left((-1)^{n+q}\frac{(1-r^{m_1})^n}{(1-r^m)^n}\frac{(1-s^{m_1})^q}{(1-s^m)^q}+\frac{1-r^{nm_1}s^{qm_1}}{1-r^{nm}s^{qm}}\right)\theta(t^m,t^{m_1}D^{n_1+n}\xi^{q_1+q})=0.$$

It is easy to obtain that $(-1)^{n+q}\frac{(1-r^{m_1})^n}{(1-r^m)^n}\frac{(1-s^{m_1})^q}{(1-s^m)^q}+\frac{1-r^{nm_1}s^{qm_1}}{1-r^{nm}s^{qm}}=0$ if and only if $n+q$ is odd, $m=m_1$ or $n+q=1$ under the assumption that $r^xs^y\neq 1(x,y\in\mathbb{Z},\ (x,y)\neq(0,0))$. Consequently, $\theta(t^mD^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1})=0$ for $n+q>1, m\neq 0, m\neq m_1$.

For $m=m_1, n+q>1$, choose some non-zero integer i with $i\neq m, 2m$, such that

$$\theta(t^mD^n\xi^q,t^mD^{n_1}\xi^{q_1})=\frac{1}{1-r^{ni}s^{qi}}\theta([t^i,t^{m-i}D^n\xi^q],t^mD^{n_1}\xi^{q_1})$$

$$\begin{aligned}&=\frac{-1}{1-r^{ni}s^{qi}}(\theta([t^{m-i}D^n\xi^q,t^mD^{n_1}\xi^{q_1}],t^i)+\theta([t^mD^{n_1}\xi^{q_1},t^i],t^{m-i}D^n\xi^q))\\&=\frac{-1}{1-r^{ni}s^{qi}}((r^{nm_1}s^{qm_1}-r^{n(m-i)}s^{q(m-i)})\theta(t^{2m-i}D^{n+n_1}\xi^{q+q_1},t^i)\\&\quad+(r^{n_1i}s^{q_1i}-1)\theta(t^{m+i}D^{n_1}\xi^{q_1},t^{m-i}D^n\xi^q)).\end{aligned}$$

By lemma 3.1, we have $\theta(D^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1})=0$ for $n_1+q_1\neq 0$. If $n_1+q_1=0$, choose no-zero integer $i\neq m_1$, then

$$\begin{aligned}\theta(D^n\xi^q,t^{m_1})&=\frac{1}{1-r^{ni}s^{qi}}\theta([t^i,t^{-i}D^n\xi^q],t^{m_1})\\&=\frac{1}{1-r^{ni}s^{qi}}\theta([t^{m_1},t^{-i}D^n\xi^q],t^i)=0.\end{aligned}$$

As a result, we obtain $\theta(t^mD^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1})=0$ for $n+q>1, m+m_1\neq 0$.

(ii) For $n_1+q_1>1$ and $m+m_1\neq 0$, the situation is similar to (i).

(iii) For $n+q=n_1+q_1=1, m+m_1\neq 0$, we can assume $n=1, q_1=1$, then

$$\begin{aligned}\theta(t^mD,t^{m_1}\xi)&=\frac{1}{1-r^m}\theta([t^m,D],t^{m_1}\xi)\\&=\frac{-1}{1-r^m}(\theta([D,t^{m_1}\xi],t^m)+\theta([t^{m_1}\xi,t^m],D))\\&=\frac{-1}{1-r^m}((r^{m_1}-1)\theta(t^{m_1}D\xi,t^m)+(r^m-1)\theta(t^{m+m_1}\xi,D))\\&=0.\end{aligned}$$

(iv) For $n+q=1, n_1+q_1=0$,

$$\begin{aligned}\theta(t^mD^n\xi^q,t^{m_1})&=\frac{1}{1-(r^ns^q)^{m+m_1}}\theta([t^{-m_1}D^n\xi^q,t^{m+m_1}],t^{m_1})\\&=\frac{-1}{1-(r^ns^q)^{m+m_1}}\theta([t^{-m_1}D^n\xi^q,t^{m_1}],t^{m+m_1}))\\&=\frac{1-(r^ns^q)^{m+m_1}}{1-(r^ns^q)^{m+m_1}}\theta(D^n\xi^q,t^{m+m_1})=0,\end{aligned}$$

Finally, if $n+q>0$ or $n_1+q_1>0$, then $\theta(t^mD^n\xi^q,t^{m_1}D^{n_1}\xi^{q_1})=0$, for $m+m_1\neq 0$.

Lemma 3.4 $\theta(D^n\xi^q,D^{n_1}\xi^{q_1})=0$.

Proof For $n+q=0$ or $n_1+q_1=0$, we obtain $\theta(D^n\xi^q,D^{n_1}\xi^{q_1})=0$ by lemma 3.2.

For $n+q>0$ and n_1+q_1, we have

$$\theta(t^{-1}D^n\xi^q,tD^{n_1}\xi^{q_1})=\frac{1}{1-r^{n_1}s^{q_1}}\theta(t^{-1}D^n\xi^q,[t,D^{n_1}\xi^{q_1}])=-\frac{1-r^ns^q}{1-r^{n_1}s^{q_1}}\theta(D^n\xi^q,D^{n_1}\xi^{q_1}),$$

$$\theta(t^{-1}D^n\xi^q,tD^{n_1}\xi^{q_1})=\frac{1}{1-r^{-n}s^{-q}}\theta([t^{-1},D^n\xi^q],tD^{n_1}\xi^{q_1})=-\frac{1-r^{-n_1}s^{-q_1}}{1-r^{-n}s^{-q}}\theta(D^n\xi^q,D^{n_1}\xi^{q_1}),$$

then $\left(\frac{1-r^n s^q}{1-r^{n_1}s^{q_1}}-\frac{1-r^{-n_1}s^{-q_1}}{1-r^{-n}s^{-q}}\right)\theta(D^n\xi^q, D^{n_1}\xi^{q_1})=0$.

It is easy to conclude that $\frac{1-r^n s^q}{1-r^{n_1}s^{q_1}}=\frac{1-r^{-n_1}s^{-q_1}}{1-r^{-n}s^{-q}}$ if and only if $n=n_1$ and $q=q_1$. As a result, for $n\neq n_1$ or $q\neq q_1$, we have $\theta(D^n\xi^q, D^{n_1}\xi^{q_1})=\theta(t^{-1}D^n\xi^q, tD^{n_1}\xi^{q_1})=0$.

Lemma 3.5 For $m\neq 0$, $n+q\neq 0$ or $n_1+q_1\neq 0$, $\theta(t^{-m}D^n\xi^q, t^m D^{n_1}\xi^{q_1})=0$.

Proof For $n_1+q_1=0$,

$$\theta(t^{-m}D^n\xi^q, t^m)=\frac{1}{(r^n s^q)^{-1}-1}\theta([t^{-m+1}D^n\xi^q, t^{-1}], t^m)=\frac{1-(r^n s^q)^m}{1-(r^n s^q)^{-1}}\theta(tD^n\xi^q, t^{-1})=0.$$

For $n+q\neq 0$ or $n_1+q_1\neq 0$,

$$\begin{aligned}\theta(t^{-m}D^n\xi^q, t^m D^{n_1}\xi^{q_1}) &= \frac{1}{(r^n s^q)^{-1}-1}\theta([t^{-m+1}D^n\xi^q, t^{-1}], t^m D^{n_1}\xi^{q_1})\\ &= (-1)^m\left(\frac{1-r^{-n_1}s^{-q_1}}{1-r^{-n}s^{-q}})^m\theta(D^n\xi^q, D^{n_1}\xi^{q_1})\right)\\ &= 0.\end{aligned}$$

Theorem 3 $\dim H^2(\mathcal{D}^-_{r,s},\mathbb{C})=\infty$. Every 2-cocycle on $\mathcal{D}^-_{r,s}$ is equivalent to one of the following 2-cocycle,

$$\theta(t^m D^n\xi^q, t^{m_1}D^{n_1}\xi^{q_1}) = \begin{cases} a_{m,m_1}, & \text{if } n=n_1=q=q_1=0 \textit{ and } m\neq m_1,\\ 0, & \text{otherwise.}\end{cases}$$

where a_{m,m_1} are arbitrary constants with $a_{m,m_1}=-a_{m_1,m}$.

Proof We have $\theta(t^m D^n\xi^q, t^{m_1}D^{n_1}\xi^{q_1})=0$, for $\forall\, m, m_1\in\mathbb{Z},\ n, n_1, q, q_1\in\mathbb{Z}^+$ $(n+q, n_1+q_1)\neq(0,0)$ from the lemmas above. By the definition of 2-cocycle, we conclude that the values of $\theta(t^m, t^{m_1})$ are independent.

So the proof of theorem is complete.

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