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Adjacent vertex-distinguishing edge partition of graphs

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Abstract: The minimum number of colors required to give a graph G an adjacent vertex-distinguishing edge partition was studied. Based on the classification of the degree of a graph, this paper proved that every graph without K_2 of minimum degree at least 188 permits an adjacent vertex-distinguishing 3-edge partition. The result is more superior than previous ones.

Key words: edge partition; adjacent vertex-distinguishing; non-proper

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图的邻点可区别边划分

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摘要: 研究了图的邻点可区别边划分所需要的最少边色数. 通过对图的度进行分类讨论, 证明了不包含K_2且最小度$\geqslant$ 188的图有邻点可区别点染色3边划分. 这个结论比已有结果更优越.

关键词: 边划分; 邻点可区别; 非正常

0 Introduction

Let G be a finite simple graph. A partition of the edges of a graph G into sets $\{S_1, \cdots, S_k\}$ defines a multiset X_v for each vertex v where the multiplicity of i in X_v (duplicate elements are allowed) is the number of edges incident to v in S_i. It is adjacent vertex-distinguishing if for every edge (u, v), the multiset X_u is distinct from X_v. Similarly, it is vertex-distinguishing if for all pairs of distinct vertices u and v, $X_u \neq X_v$. A k-edge-weighting of a graph G is an assignment of k colors to each edge e. An edge-weighting is vertex-coloring if for every edge (u, v), the colors c_u and c_v are distinct, where the colors of a vertex is defined as the sum of the weights on the edges incident to that vertex. Similarly, an edge-weighting is vertex-distinguishing if for all pairs of distinct vertices u and v, c_u is distinct from c_v.

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If a k-edge-weighting is vertex-coloring then it is adjacent vertex-distinguishing, though the converse may not hold. Clearly, a graph cannot have a vertex-coloring edge weighting or an adjacent vertex-distinguishing edge partition if it has a component which is isomorphic to K_2. The concept of vertex-coloring edge weighting was introduced in [1], where was conjectured that every graph without K_2 permits a vertex-coloring 3-edge-weighting. Then, [2] showed that every graph without K_2 permits a vertex-coloring 30-edge-weighting. The similar concept of vertex-distinguishing edge weighting has been considered in [3,4].

For a k-edge-coloring of a simple graph G, it is proper if no two adjacent edges are assigned the same color. An edge coloring of a graph is said to be adjacent vertex-distinguishing if for every vertex u adjacent to v, the set of colors assigned to the set of edges incident to u differs from the set of colors assigned to the set of edges incident to v. It is vertex-distinguishing if each pair of vertices is incident to a different set of colors. The concept of vertex-distinguishing proper edge coloring and non-proper vertex-distinguishing edge coloring have been studied in many papers (see for example [5-8]). Adjacent vertex-distinguishing edge colorings are studied in [9] (for non-proper colorings) and [10] (for proper colorings).

In paper [11], they proved that every graph without K_2 permits an adjacent vertex-distinguishing 4-edge partition and that graphs of minimum degree at least 1000 permits an adjacent vertex-distinguishing 3-edge partition. This paper will improve the condition of the theorem in [11], then we have the following theorem:

Theorem 1　Every graph without K_2 of minimum degree at least 188 permits an adjacent vertex-distinguishing 3-edge partition.

We shall prove Theorem 1 in Section 2. Therefore the following lemmas will be useful.

1　Preliminary results

If H is a spanning subgraph of G, or if W is a subset of $V = V(G)$, then $d_H(v)$ denotes the number of neighbors of v in H and $d_W(v)$ denotes $|N(v) \bigcap W|$ where $N(v)$ is the set of neighbors of v in G.

Lemma 1.1[11]　Every graph G can be partitioned into two subgraphs G_1 and G_2 such that for all v,

$$d_{G_1}(v) \in \left\{ \left\lfloor \frac{d_G(v)}{2} \right\rfloor, \left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right\}.$$

Lemma 1.2[11]　Suppose that for some graph G we have chosen, for every vertex v, two integers a_v^- and a_v^+ such that $\frac{d_G(v)}{3} \leqslant a_v^- \leqslant \frac{d_G(v)}{2}$ and $\frac{d_G(v)}{2} \leqslant a_v^+ \leqslant \frac{2d_G(v)}{3}$. Then there is a spanning subgraph H of G such that for every vertex v:

$$d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}.$$

In [11], we have the proofs of Lemma 1.1 and Lemma 1.2.

2 The proof of Theorem 1

Let G be a graph of minimum degree $\geqslant 188$. We can give a greedy vertex-coloring of G such that for each vertex v, the color $c(v)$ of G is between 0 and $d_G(v)$. We think of $c(v)$ as a pair $(p(v), r(v))$ where

$$p(v) = \left\lfloor \frac{c(v)}{\lceil \sqrt{d_G(v)} \rceil} \right\rfloor \quad \text{and} \quad r(v) = c(v) \text{mod} \left\lceil \sqrt{d_G(v)} \right\rceil.$$

By Lemma 1.1, G can be partitioned into two subgraphs G_1 and G_2 such that

$$d_{G_i}(v) \in \left\{ \left\lfloor \frac{d_G(v)}{2} \right\rfloor - 1, \left\lfloor \frac{d_G(v)}{2} \right\rfloor, \left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right\}.$$

We set

$$a_v^-(1) = \left\lfloor \frac{d_G(v)}{4} \right\rfloor - p(v) - 1, \ a_v^+(1) = \left\lfloor \frac{d_G(v)}{4} \right\rfloor + p(v) + 1.$$

Since $0 \leqslant p(v) \leqslant \left\lfloor \frac{d_G(v)}{\lceil \sqrt{d_G(v)} \rceil} \right\rfloor \leqslant \left\lfloor \sqrt{d_G(v)} \right\rfloor \leqslant \sqrt{d_G(v)}$ and $d_G(v) \geqslant 188$, we have

$$a_v^-(1) \leqslant \frac{1}{2} \left\lfloor \frac{d_G(v)}{2} \right\rfloor - p(v) - 1 = \frac{1}{2} \left(\left\lfloor \frac{d_G(v)}{2} \right\rfloor - 1 \right) - p(v) - \frac{1}{2} \leqslant \frac{1}{2} \left(\left\lfloor \frac{d_G(v)}{2} \right\rfloor - 1 \right) \leqslant \frac{d_{G_1}(v)}{2},$$

and

$$a_v^+(1) \geqslant \frac{1}{2} \left\lfloor \frac{d_G(v)}{2} \right\rfloor - \frac{1}{2} + p(v) + 1 = \frac{1}{2} \left(\left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right) + p(v) \geqslant \frac{1}{2} \left(\left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right) \geqslant \frac{d_{G_1}(v)}{2}.$$

We can also prove that

$$a_v^-(1) \geqslant \frac{d_{G_1}(v)}{3}, \quad a_v^+(1) \leqslant \frac{2d_{G_1}(v)}{3}.$$

In fact, we consider the values of $d_G(v)$ according to the following four cases:

Case 1 $d_G(v) = 4k$ $(k \in Z^+)$, then we have $k \geqslant 47$, and

$$a_v^-(1) \geqslant k - \sqrt{4k} - 1, \ \ a_v^+(1) \leqslant k + \sqrt{4k} + 1.$$

As well,

$$\frac{d_{G_1}(v)}{3} \leqslant \frac{1}{3} \left(\left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right) = \frac{1}{3}(2k+1),$$

$$\frac{2d_{G_1}(v)}{3} \geqslant \frac{2}{3} \left(\left\lfloor \frac{d_G(v)}{2} \right\rfloor - 1 \right) = \frac{2}{3}(2k-1).$$

To ensure that $a_v^-(1) \geqslant \frac{d_{G_1}(v)}{3}, a_v^+(1) \leqslant \frac{2d_{G_1}(v)}{3}$, it is enough that

$$k - \sqrt{4k} - 1 \geqslant \frac{1}{3}(2k+1), \quad k + \sqrt{4k} + 1 \leqslant \frac{2}{3}(2k-1).$$

That is $k - 6\sqrt{k} - 4 \geqslant 0$, $\quad k - 6\sqrt{k} - 5 \geqslant 0$. By calculating, when $k \geqslant 47$, both of the above two inequalities hold.

Case 2 $d_G(v)=4k+1$ $(k\in Z^+)$, then we have $k\geqslant 47$, and

$$a_v^-(1)\geqslant k-\sqrt{4k+1}-1,\quad a_v^+(1)\leqslant k+\sqrt{4k+1}+1.$$

As well,

$$\frac{d_{G_1}(v)}{3}\leqslant\frac{1}{3}\left(\left\lfloor\frac{d_G(v)}{2}\right\rfloor+1\right)=\frac{1}{3}(2k+1),$$
$$\frac{2d_{G_1}(v)}{3}\geqslant\frac{2}{3}\left(\left\lfloor\frac{d_G(v)}{2}\right\rfloor-1\right)=\frac{2}{3}(2k-1).$$

To ensure that $a_v^-(1)\geqslant\frac{d_{G_1}(v)}{3}, a_v^+(1)\leqslant\frac{2d_{G_1}(v)}{3}$, it is enough that

$$k-\sqrt{4k+1}-1\geqslant\frac{1}{3}(2k+1),\qquad k+\sqrt{4k+1}+1\leqslant\frac{2}{3}(2k-1).$$

That is $k-3\sqrt{4k+1}-4\geqslant 0,\quad k-3\sqrt{4k+1}-5\geqslant 0$. By calculating, when $k\geqslant$47, both of the above two inequalities hold.

Case 3 $d_G(v)=4k+2$ $(k\in Z^+)$, then we have $k\geqslant 47$, and

$$a_v^-(1)\geqslant k-\sqrt{4k+2}-1,\quad a_v^+(1)\leqslant k+\sqrt{4k+2}+1.$$

As well,

$$\frac{d_{G_1}(v)}{3}\leqslant\frac{1}{3}\left(\left\lfloor\frac{d_G(v)}{2}\right\rfloor+1\right)=\frac{1}{3}(2k+1+1)=\frac{2}{3}(k+1),$$
$$\frac{2d_{G_1}(v)}{3}\geqslant\frac{2}{3}\left(\left\lfloor\frac{d_G(v)}{2}\right\rfloor-1\right)=\frac{2}{3}(2k+1-1)=\frac{4k}{3}.$$

To ensure that $a_v^-(1)\geqslant\frac{d_{G_1}(v)}{3}, a_v^+(1)\leqslant\frac{2d_{G_1}(v)}{3}$, it is enough that

$$k-\sqrt{4k+2}-1\geqslant\frac{2}{3}(k+1),\qquad k+\sqrt{4k+2}+1\leqslant\frac{4k}{3}.$$

That is, $k-3\sqrt{4k+2}-5\geqslant 0,\quad k-3\sqrt{4k+2}-3\geqslant 0$. By calculating, when $k\geqslant$47, both of the above two inequalities hold.

Case 4 $d_G(v)=4k+3$ $(k\in Z^+)$, then we have $k\geqslant 47$, and

$$a_v^-(1)\geqslant k-\sqrt{4k+3}-1,\quad a_v^+(1)\leqslant k+\sqrt{4k+3}+1.$$

As well,

$$\frac{d_{G_1}(v)}{3}\leqslant\frac{1}{3}\left(\left\lfloor\frac{d_G(v)}{2}\right\rfloor+1\right)=\frac{1}{3}(2k+1+1)=\frac{2}{3}(k+1),$$
$$\frac{2d_{G_1}(v)}{3}\geqslant\frac{2}{3}\left(\left\lfloor\frac{d_G(v)}{2}\right\rfloor-1\right)=\frac{2}{3}(2k+1-1)=\frac{4k}{3}.$$

To ensure that $a_v^-(1)\geqslant\frac{d_{G_1}(v)}{3}, a_v^+(1)\leqslant\frac{2d_{G_1}(v)}{3}$, it is enough that

$$k-\sqrt{4k+3}-1\geqslant\frac{2}{3}(k+1),\qquad k+\sqrt{4k+3}+1\leqslant\frac{4k}{3}.$$

That is $k - 3\sqrt{4k+3} - 5 \geqslant 0$, $k - 3\sqrt{4k+3} - 3 \geqslant 0$. By calculating, when $k \geqslant 47$, both of the above two inequalities hold.

So, by Lemma 1.2, we find a spanning subgraph H_1 of G_1 such that for every vertex v:

$$d_{H_1}(v) \in \{a_v^-(1), a_v^-(1)+1, a_v^+(1), a_v^+(1)+1\}.$$

We set

$$a_v^-(2) = \lfloor \frac{d_G(v)}{4} \rfloor - r(v) - 1, \quad a_v^+(2) = \lfloor \frac{d_G(v)}{4} \rfloor + r(v) + 1.$$

Since $0 \leqslant r(v) \leqslant \sqrt{d_G(v)}$, by Lemma 1.2, as the proof of the above, we can find a spanning subgraph H_2 of G_2 such that for every vertex v:

$$d_{H_2}(v) \in \{a_v^-(2), a_v^-(2)+1, a_v^+(2), a_v^+(2)+1\}.$$

We label the edges of H_i with color i and the remaining edges with label 0. If u and v are not distinguished by our labeling, then $d_G(u) = d_G(v)$, $d_{H_1}(u) = d_{H_1}(v)$, $d_{H_2}(u) = d_{H_2}(v)$ from which it follows that $p(u) = p(v)$ and $r(u) = r(v)$, that is, $c(u) = c(v)$. But, by greedily coloring, this implies $uv \overline{\in} E(G)$.

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