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An upper bound on the vertex-distinguishing IE-total chromatic number of graphs

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Abstract: The upper bound for the vertex-distinguishing IE-total chromatic number by the probability method is studied. If $\delta \geqslant 7$ and $16\Delta \leqslant n \leqslant \frac{\Delta^7}{32\times10^5(\Delta+1)}+1$, then $\chi_{vt}^{ie}(G) \leqslant 16\Delta$ is proved, where n is the order of G and δ is the minimum degree of G and Δ is the maximum degree of G.

Key words: probability method; positive probability; vertex-distinguishing IE-total chromatic number

图的点可区别IE-全色数的一个上界

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摘要:用概率方法研究图的点可区别IE-全色数的一个上界,得到:如果 $\delta \geqslant 7$ 且 $16\Delta \leqslant n \leqslant \frac{\Delta^7}{32\times10^5(\Delta+1)}+1$,则 $\chi_{vt}^{ie}(G) \leqslant 16\Delta$,这里 n 是 G 的阶,δ 是 G 中点的最小度数,Δ 是 G 中点的最大度数。

关键词:概率方法;正的概率;点可区别IE-全色数
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It is well known to compute the chromatic number of a graph is NP-hard. In [1], some results about it have been obtained by combinatorial. At ICM2002, Noga Alon advanced a theory that graph coloring could studied by probability methods. For instance, some conclusions have been gotten by probability methods[2-3].

A proper total coloring of graph G is an assignment of colors to all vertices and edges with three conditions: condition (v): each pair of incident vertices are colored with different colors; condition (e): each pair of incident edges are colored with different colors; condition (i): every vertex with its incident edges are colored with different colors.

All the graphs $G = G(V, E)$ discussed in this paper are finite, undirected, simple and connected. Let Δ be the maximum degree of G, δ be the minimum degree of G, and n be the order of G.

Definition 1 If the total coloring graph G only satisfies condition (v), then such coloring is called a IE-total coloring of graph G. If f is a IE-total coloring of graph G by using k colors, and for any vertices $u, v \in V(G)$, $u \neq v$, it has $C(u) \neq C(v)$, where $C(u) = \{f(u)\} \cup \{f(uu') \mid uu' \in E(G)\}$, then f is called a k-vertex-distinguishing IE-total coloring of G, or a k-VDIET-coloring of G for short. The minimum number of colors required for a VDIET-coloring of G is denoted by $\chi_{vt}^{ie}(G)$, and it is called the VDIET chromatic number of G.

For other terminologies and notations, you can refer to [4].

Lemma 2[5] Consider a set $\varepsilon = \{A_1, A_2, \cdots, A_n\}$ of (typically bad) events such that each A_i is mutually indepen-

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dent of $\varepsilon - (D_i \cup \{A_i\})$, for some $D_i \subseteq \varepsilon$. If we have reals $x_1, x_2, \cdots, x_n \in (0,1)$ such that for each $1 \leqslant i \leqslant n$, $\Pr(A_i) \leqslant x_i \prod_{A_j \in D_i}(1-x_j)$, then the probability that none of the events in ε occur is at least $\prod_{i=1}^{n}(1-x_i) > 0$.

Theorem 1　If $\delta \geqslant 7$ and $16\Delta \leqslant n \leqslant \frac{\Delta^7}{32 \times 10^5(\Delta+1)} + 1$, then $\chi_{vt}^{ie}(G) \leqslant 16\Delta$.

Proof　We easily know that $\chi_{vt}^{ie}(G) \leqslant n$. We assign to each edge and vertex of G a uniformly random coloring from $\{1,2,\cdots,16\Delta\}$ named this new coloring f. We will use lemma 2 to show that the probability that f is a vertex-distinguishing IE-total-coloring is positive. It is sufficient to show that the following two conditions should be satisfied:

(A) the vertex coloring is proper-no pair of incident vertices is colored with the same color.

(B) the coloring is vertex distinguishing-no pair of vertices meets the same color set.

Next we will show that the probability which the obtained coloring f is a vertex-distinguishing IE-total-coloring is positive. The following 'bad' events are defined.

Type 1　For each edge $e = uv$, let A_e be the event that both u and v are colored with the same color;

Type 2　For any two different vertices u and w, and $d(u) = d_1$, $d(w) = d_2$, $d = \min\{d_1, d_2\}$, where $d(u)$ is the degree of u. let $B_{u,w}$ be the event that u and w are colored with the different colors, and $C(u) = C(w)$, where $C(u) = \{f(u)\} \cup \{f(uu') \mid uu' \in E(G)\}$;

It remains to show that with positive probability none of these events occurs. We must estimates the probability of a given event firstly.

Lemma 3　The following two statements hold:

(1) For each event A_e of type 1, we have $\Pr(A_e) = \frac{1}{16\Delta}$;

(2) For each event $B_{u,w}$ of type 2, we have $\Pr(B_{u,w}) < \frac{2 \times 10^5}{\Delta^7}$.

Proof　(1) is trivial. For (2), let B_i be the event of type 2 and satisfies that $|C(u)| = |C(w)| = i(1 \leqslant i \leqslant d+1)$. Then the probability $\Pr(B_i)$ that the event B_i happens is following:

1°　If u and w are not adjacent, then $$\Pr(B_i) = \frac{\binom{16\Delta}{i}\left[\binom{d_1+1}{i} \cdot i! \cdot i^{d_1+1-i}\right]\left[\binom{d_2+1}{i} \cdot i! \cdot i^{d_2+1-i}\right]}{(16\Delta)^{d_1+1} \cdot (16\Delta)^{d_2+1}};$$

2°　If u and w are adjacent, then $$\Pr(B_i) < \frac{\binom{16\Delta}{i}\left[\binom{d_1+1}{i} \cdot i! \cdot i^{d_1+1-i}\right]\left[\binom{d_2+1}{i} \cdot i! \cdot i^{d_2+1-i}\right]}{(16\Delta)^{d_1+d_2+1}};$$

From 1° and 2°, we can see that $$\Pr(B_i) < \frac{\binom{16\Delta}{i}\left[\binom{d_1+1}{i} \cdot i! \cdot i^{d_1+1-i}\right]\left[\binom{d_2+1}{i} \cdot i! \cdot i^{d_2+1-i}\right]}{(16\Delta)^{d_1+d_2+1}}.$$

Using $\binom{n}{k} \leqslant \left(\frac{en}{k}\right)^k$ and Stirling formular, $n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot e^{\frac{\theta}{12\pi}} (0 \leqslant \theta \leqslant 1)$,

$$\Pr(B_i) < \frac{\left(\frac{16e\Delta}{i}\right)^i \cdot \left[\left(\frac{e(d_1+1)}{i}\right)^i\right] \cdot \left[\left(\frac{e(d_2+1)}{i}\right)^i\right] \cdot \left[\sqrt{2\pi i} \cdot \left(\frac{i}{e}\right)^i \cdot e^{\frac{\theta}{12\pi}}\right]^2 \cdot i^{d_1+d_2+2-2i}}{(16\Delta)^{d_1+d_2+1}} < \frac{4\pi e^i \cdot (d_1+1)^i \cdot (d_2+1)^i}{i^{2i-2} \cdot \left(\frac{16\Delta}{i}\right)^{d_1+d_2+1-i}},$$

Let $d_1 \geqslant d_2 = d$, and since $\left\{f(k) = \frac{4\pi e^i (k+1)^i (d+1)^i}{i^{2i-2} \cdot \left(\frac{16\Delta}{i}\right)^{k+d+1-i}}\right\} (d \leqslant k \leqslant \Delta)$ and $\left\{h(k) = \frac{4\pi e^{k+1}(k+1)^3}{\left(\frac{16\Delta}{k+1}\right)^k}\right\} (7 \leqslant \delta \leqslant k \leqslant \Delta)$ are decreasing function $\left\{g(k) = \frac{4\pi e^k (d+1)^{2k}}{k^{2k-2} \cdot \left(\frac{16\Delta}{k}\right)^{2d+1-k}}\right\} (1 \leqslant k \leqslant d+1)$, is increasing function, then

$$\Pr(B_{u,w}) = \sum_{i=1}^{d+1} \Pr(B_i) < \sum_{i=1}^{d+1} \frac{4\pi e^i \cdot (d_1+1)^i \cdot (d_2+1)^i}{i^{2i-2} \cdot \left(\frac{16\Delta}{i}\right)^{d_1+d_2+1-i}} < \sum_{i=1}^{d+1} \frac{4\pi e^i \cdot (d+1)^{2i}}{i^{2i-2} \cdot \left(\frac{16\Delta}{i}\right)^{2d+1-i}} <$$

$$\sum_{i=1}^{d+1}\frac{4\pi e^{d+1}\cdot(d+1)^2}{\left(\frac{16\Delta}{d+1}\right)^d}=\frac{4\pi e^{d+1}\cdot(d+1)^3}{\left(\frac{16\Delta}{d+1}\right)^d}<\frac{4\pi e^8\cdot 8^3}{\left(\frac{16\Delta}{8}\right)^7}<\frac{2\times 10^5}{\Delta^7}.$$

Therefore $\Pr(B_{u,w})<\frac{2\times 10^5}{\Delta^7}$.

In this paper, event X is incident to event Y, if X and Y contain at least one common edge or vertex. Note that for lemma 2, if we estimate the number of events of each type which are incident to one given event A_i, then we easily obtain the value of D_i. So we need to estimate the number of events of each type which are incident to any given event.

Lemma 4 The following two statements hold:

(1) Each event of type 1 is incident to at most 2Δ events of type 1 which consist one set D_{11}, $2n-2$ events of type 2 which consist one set D_{12};

(2) Each event of type 2 is incident to at most 2Δ events of type 1 which consist one set D_{21}, $(2\Delta+2)n-(\Delta+1)(2\Delta+3)$ events of type 2 which consist one set D_{22}.

Proof (1) For each event A_e of type 1, for any given edge $e=uv$, at most 2Δ vertices are incident to u or v. Because events of type 1 which contain one vertex u or v are at most $n-1$, so each event of type 1 is incident to at most $2n-2$ events of type 2.

The proof of (2) is similar to (1).

Next we must determine the real constant $x_i\,(0<x_i<1)$ for applying lemma 2. Let $\{A_1,A_2,\cdots,A_{m_1}\}$ be the set of events of type 1 and $\{A_{m_1+1},A_{m_1+2},\cdots,A_{m_2}\}$ be the set of events of type 2. Note that type 1 and type 2 have no common event. Note that type 1 and type 2 have no common event. For lemma 2, $\varepsilon=\{A_1,A_2,\cdots,A_{m_1},A_{m_1+1},A_{m_1+2},\cdots,A_{m_2}\}$, then each $A_i\,(1\leqslant i\leqslant m_1)$ is mutually independent of $\varepsilon-[(D_{11}\cup D_{12})\cup A_i]$, and $A_j\,(m_1+1\leqslant j\leqslant m_2)$ is mutually independent of $\varepsilon-[(D_{21}\cup D_{22})\cup A_j]$, where D_{ij} is defined in lemma 4. Let $x_i=\frac{1}{8\Delta}\,(1\leqslant i\leqslant m_1)$ and $x_j=\frac{4\times 10^5}{\Delta^7}\,(m_1+1\leqslant j\leqslant m_2)$. Using lemma 2, we conclude that with positive probability no events of type 1 and type 2 occur, provided that:

$$\frac{1}{16\Delta}\leqslant\frac{1}{8\Delta}\left(1-\frac{1}{8\Delta}\right)^{2\Delta}\cdot\left(1-\frac{4\times 10^5}{\Delta^7}\right)^{2n-2}\leqslant\frac{1}{8\Delta}\cdot\left(\prod_{A_i\in D_{11}}\left(1-\frac{1}{8\Delta}\right)\right)\cdot\left(\prod_{A_j\in D_{12}}\left(1-\frac{4\times 10^5}{\Delta^7}\right)\right);\tag{1}$$

$$\frac{2\times 10^5}{\Delta^7}\leqslant\frac{4\times 10^5}{\Delta^7}\left(1-\frac{1}{8\Delta}\right)^{2\Delta}\cdot\left(1-\frac{4\times 10^5}{\Delta^7}\right)^{(2\Delta+2)n-(\Delta+1)(2\Delta+3)}\leqslant\frac{4\times 10^5}{\Delta^7}\cdot\left(\prod_{A_i\in D_{21}}\left(1-\frac{1}{8\Delta}\right)\right)\cdot\left(\prod_{A_j\in D_{22}}\left(1-\frac{4\times 10^5}{\Delta^7}\right)\right).\tag{2}$$

Since $\left(1-\frac{1}{z}\right)^z\geqslant\frac{1}{4}$ for all real $z\geqslant 2$, and $n\leqslant\frac{\Delta^7}{32\times 10^5(\Delta+1)}+1$, we can prove that the inequalities (1) and (2) are true. So G has (16Δ)-VDIET-coloring, when $\delta\geqslant 7$ and $16\Delta\leqslant n\leqslant\frac{\Delta^7}{32\times 10^5(\Delta+1)}+1$. This completes the proof.

References:

[1] ZHANGg Z F, QIU P X, XU B G, et al. Vertex-distinguishing total coloring of graphs[J]. Ars Combinayoria, 2008, 87: 33-45.

[2] LIU X S, DENG K, An upper bound on the star chromatic index of graphs with $\Delta\geqslant 7$[J]. Journal of Lanzhou University: Natural Science, 2008, 44(2): 88-89.

[3] LIU X S, AN M Q, GAO Y, An upper bound of the adjacent vertex distinguishing acyclic edge chromatic number of a graph [J]. Acta Mathematicae Sinica, 2009, 25(1): 137-140.

[4] BONDY J A, MARTY U S R. Graph theory with applications[M]. New York: The Macmillan Press Ltd, 1976.

[5] MICHAEL M, BRUCE R. Graph coloring and the probabilistic method [M]. New York: Springer-Verlag, 2002.

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