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Numerical analysis of beams on unilateral elastic foundation

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ABSTRACT

Purpose: The main issue of this paper is to present results of finite element analysis of beams elements on unilateral elastic foundation received with a use of special finite elements of zero thickness designated for foundation modelling.

Design/methodology/approach: Computer strength analysis with a use of Finite Element Method (FEM) was carried out.

Findings: The paper presents possibilities of special finite elements of zero thickness which enable taking into consideration unilateral contact in construction-foundation interaction as well as an impact of surrounding construction environment to its behaviour.

Research limitations/implications: Further researches should concentrate on taking into consideration a multi-layer aspects as well as elasto-plasticity of foundation.

Practical implications: Modern engineering construction on elastic foundation analyze require not only standard analysis on Winkler (one parameter) foundation but also calculation of construction on two-parameter foundation which will take into consideration a possibility of loosing contact between construction and foundation (unilateral contact).

Originality/value: The paper can be useful for person who performs strength analysis of beams on elastic foundation with a use of finite element method.

Keywords: Analysis and modelling; Computational mechanics; Finite element method; Elastic foundation; Unilateral contact.

METHODOLOGY OF RESEARCH, ANALYSIS AND MODELLING

1. Introduction

Modern engineering analyze methods carried out with a use of FEM tools need taking into consideration a complex numerical models reflecting real construction behaviour [1-5].

Two approaches are generally applied for description of beams on elastic foundation.

Theories for analysis of beam behaviour include: (i) Euler-Bernoulli beam theory (C^1 and C^2 class finite beam elements without transverse shear deformation) [6]; (ii) Timoshenko beam theory (C^0 class finite beam element with transverse shear deformation effects) [7].

For the elastic foundation, according to the first approach, the foundation reaction *p(x)* is directly proportional to the vertical beam deflection *w(x)*. This foundation is well known as the Winkler foundation [6]. Physically, this foundation consists of independent spring elements:

$$p(x) = k_0 w(x)\ . \qquad (1)$$

The second approach introduces shear interactions between the beam and foundation (different vertical deformations), see Pasternak [8], Filonenko-Borodich [9] and Vlasov [10]. It should be noted that these models are mathematically equivalent. The only difference is the definition of the parameters. Therefore, these models can be written in the general form

$$p(x) = k_0 w(x) - k_1 \left(\frac{d^2 w(x)}{dx^2} \right) \qquad (2)$$

where k_0 and k_1 are the first (Winkler modulus) and the second foundation parameters, respectively. The modern FEM analysis of beams on an elastic foundation has been widely reported in the literature [11-14].

2. Stiffness matrices formulation

For the beam on foundation formulation the total potential Π is expressed by

$$\Pi = (U_e + U_f) - W \qquad (3)$$

where U_e is the beam strain energy, U_f is the foundation strain energy, W is the potential of the loads. The potential of the loads W is given by

$$W = \int_l qw dx \,. \qquad (4)$$

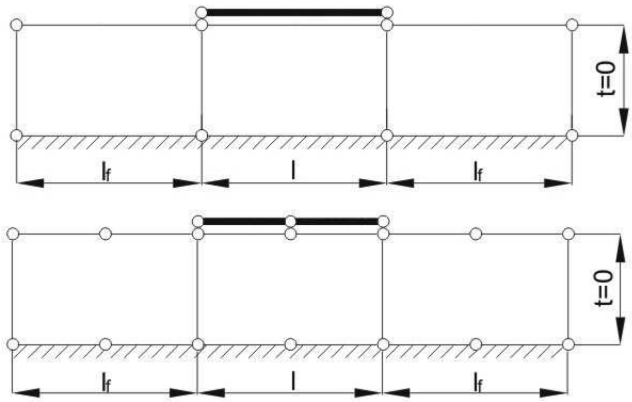

Fig. 1. The four and six-node zero thickness foundation elements connected to the beam two- and three-node elements

2.1. C¹ class beam element

For beam elements modelling two and three-node beam elements of C^1 type were used. The strain energy in the beam is expressed as [6]

$$U_e = \frac{EI}{2} \int_l [w''(x)]^2 dx + \frac{EA}{2} \int_l [u'(x)]^2 dx. \qquad (5)$$

The 'strains' are defined in terms of the nodal displacements and shape functions derivatives by the expressions

$$\varepsilon_b = w''(x), \quad \varepsilon_a = u'(x) \qquad (6)$$

or in other form as

$$\varepsilon_b = \boldsymbol{B}_b \boldsymbol{a}, \; \varepsilon_a = \boldsymbol{B}_a \boldsymbol{a}. \qquad (7)$$

The nodal displacement vectors for the two-node beam element are given as

$$\boldsymbol{a} = [u_1 \quad w_1 \quad \theta_1 \quad u_2 \quad w_2 \quad \theta_2]. \qquad (8)$$

For the two-node beam element

$$\boldsymbol{B}_b = \frac{4}{l^2} \left[0 \quad N_1''(x) \quad \overline{N}_1''(x)\frac{l}{2} \quad 0 \quad N_2''(x) \quad \overline{N}_2''(x)\frac{l}{2} \right],$$

$$\boldsymbol{B}_a = \frac{2}{l} \left[N_3'(x) \quad 0 \quad 0 \quad N_4'(x) \quad 0 \quad 0 \right]. \qquad (9)$$

Shape functions $N_1(x) - N_4(x)$ for the two-node beam element are given by Torbacki and Buczkowski [15]. Using Eq. (5-9) the stiffness matrix of the two-node element can be evaluated from

$$\mathbf{K}_e^{ij} = \frac{l}{2} \int_{-1}^{1} \mathbf{B}_i^{\mathrm{T}} \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix} \mathbf{B}_i \, d\xi \,. \qquad (10)$$

Thanks to similar procedure stiffness matrix for the three-node beam element can be received.

2.2. Foundation under C¹ class beam element

For foundation modelling special four and six-node finite elements of zero thickness were used (see Fig. 1).

The strain energy in foundation is expressed as [15]

$$U_f = \frac{k_0}{2} \int_l [w(x)]^2 dx + \frac{k_1}{2} \int_l [w'(x)]^2 dx + \frac{k_1}{2} \int_{l_f} [w'(x)]^2 dx. \qquad (11)$$

where k_0 is the first parameter in vertical direct (Winkler foundation modulus), k_1 is the second parameter, which represents the shear interaction of the foundation layer, l_f is the length of surroundings outside the beam element,

$$w(x) = \mathbf{B}_w \mathbf{a}, \quad w'(x) = \mathbf{B}_w' \mathbf{a}. \qquad (12)$$

For the four-node foundation element we have

$$\mathbf{B}_w = [\overline{\mathbf{B}} \quad -\overline{\mathbf{B}}]_{1x12}, \quad \mathbf{B}_w' = [\overline{\mathbf{B}'} \quad -\overline{\mathbf{B}'}]_{1x12} \qquad (13)$$

and for the six-node foundation element

$$\mathbf{B}_w = [\overline{\mathbf{B}} \quad -\overline{\mathbf{B}}]_{1x18}, \quad \mathbf{B}_w' = [\overline{\mathbf{B}'} \quad -\overline{\mathbf{B}'}]_{1x18} \,, \qquad (14)$$

whereas the submatrices $\overline{\mathbf{B}}$ for the four-node elements are given by

$$\overline{\mathbf{B}} = [0 \quad N_1 \quad \overline{N}_1 det\mathbf{J} \quad 0 \quad N_2 \quad \overline{N}_2 det\mathbf{J}],$$

$$\overline{\mathbf{B}'} = \frac{1}{det\mathbf{J}} [0 \quad N_1' \quad \overline{N_1'} det\mathbf{J} \quad 0 \quad N_2' \quad \overline{N_2'} det\mathbf{J}], \qquad (15)$$

and for the six-node elements

$$\overline{\mathbf{B}} = [0 \quad N_1 \quad \overline{N}_1 det\mathbf{J} \quad 0 \quad N_2 \quad \overline{N}_2 det\mathbf{J} \quad 0 \quad N_3 \quad \overline{N}_3 det\mathbf{J}], \quad (16)$$

$$\overline{\mathbf{B}'} = \frac{1}{det\mathbf{J}}[0 \quad N_1' \quad \overline{N_1'} det\mathbf{J} \quad 0 \quad N_2' \quad \overline{N_2'} det\mathbf{J} \quad 0 \quad N_3' \quad \overline{N_3'} det\mathbf{J}],$$

where for the foundation element under the beam element $det\mathbf{J} = l/2$ and for the foundation element beyond the beam element $det\mathbf{J} = l_f/2$. The foundation stiffness matrix $\mathbf{K}_f$ is

$$\mathbf{K}_f = \mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_b . \quad (17)$$

By substituting Eqns. (12-16) into the first integral of Eq. (11) the Winkler foundation stiffness matrix $\mathbf{K}_0$ is derived

$$\mathbf{K}_0 = k_0 \int_l \mathbf{B}_w^T(x)\mathbf{B}_w(x)dx \quad \text{or} \quad \mathbf{K}_0 = \frac{lk_o}{2}\int_{-1}^{1} \mathbf{B}_w^T(\xi)\mathbf{B}_w(\xi)d\xi. \quad (18)$$

The shear foundation stiffness matrix $\mathbf{K}_1$ is derived

$$\mathbf{K}_1 = k_1 \int_l [\mathbf{B}_w'(x)]^T \mathbf{B}_w'(x)dx \text{ or } \mathbf{K}_1 = \frac{2k_1}{l}\int_{-1}^{1} [\mathbf{B}_w'(\xi)]^T \mathbf{B}_w'(\xi)d\xi \quad (19)$$

The shear foundation stiffness matrix of the foundation element beyond the beam region $\mathbf{K}_b$ can be expressed as

$$\mathbf{K}_b = k_1 \int_{l_f} [\mathbf{B}_w'(x)]^T \mathbf{B}_w'(x)dx \text{ or } \mathbf{K}_b = \frac{2k_1}{l_f}\int_{-1}^{1} [\mathbf{B}_w'(\xi)]^T \mathbf{B}_w'(\xi)d\xi \quad (20)$$

Using the three-point Gauss integration formula for the solution of $\mathbf{K}_0$, $\mathbf{K}_1$ and $\mathbf{K}_b$ gives the following forms for the four-node and the six-node elements, respectively

$$\mathbf{K}_0 = \begin{bmatrix} \overline{\mathbf{K}}_0 & -\overline{\mathbf{K}}_0 \\ -\overline{\mathbf{K}}_0 & \overline{\mathbf{K}}_0 \end{bmatrix}_{12x12}, \quad \mathbf{K}_0 = \begin{bmatrix} \overline{\mathbf{K}}_0 & -\overline{\mathbf{K}}_0 \\ -\overline{\mathbf{K}}_0 & \overline{\mathbf{K}}_0 \end{bmatrix}_{18x18}$$
$$\mathbf{K}_1 = \begin{bmatrix} \overline{\mathbf{K}}_1 & -\overline{\mathbf{K}}_1 \\ -\overline{\mathbf{K}}_1 & \overline{\mathbf{K}}_1 \end{bmatrix}_{12x12}, \quad \mathbf{K}_1 = \begin{bmatrix} \overline{\mathbf{K}}_1 & -\overline{\mathbf{K}}_1 \\ -\overline{\mathbf{K}}_1 & \overline{\mathbf{K}}_1 \end{bmatrix}_{18x18} \quad (21)$$
$$\mathbf{K}_b = \begin{bmatrix} \overline{\mathbf{K}}_b & -\overline{\mathbf{K}}_b \\ -\overline{\mathbf{K}}_b & \overline{\mathbf{K}}_b \end{bmatrix}_{12x12}, \quad \mathbf{K}_b = \begin{bmatrix} \overline{\mathbf{K}}_b & -\overline{\mathbf{K}}_b \\ -\overline{\mathbf{K}}_b & \overline{\mathbf{K}}_b \end{bmatrix}_{18x18}$$

with the submatrices $\overline{\mathbf{K}}_0$ and $\overline{\mathbf{K}}_1$ assembled using the three-point Gauss integration. Submatrices $\overline{\mathbf{K}}_b$ for the four and the six-node foundations element has the similar form of $\overline{\mathbf{K}}_1$. The only difference is l_f instead of l in submatrices $\overline{\mathbf{K}}_b$.

3. Numerical example

Beam element freely sitting on an elastic foundation including unilateral contact was analyzed. Total length of beam element amounts $l = 6$ [m] and the flexural rigidity amounts $EI = 17 \cdot 10^4$ [kNm^2]. Element is loaded with a centrally, vertically placed concentrated force $P = 30$ [kN]. Three examples of foundation parameters were assumed. First of them is Winkler foundation with module $k_0 = 25 \cdot 10^4$ [kN/m^2] and $k_1 = 0$ [kN]. For such model there exists an analytical solution of vertical deflection of beam in a following form [16]:

$$w(x) = \frac{P}{8EI\lambda^3}(\sin\lambda x \cosh\lambda x - 2\frac{\sin^2\lambda l_k + \sinh^2\lambda l_k}{\sin 2\lambda l_k + \sinh 2\lambda l_k}\sin\lambda x \sinh\lambda x +$$
$$+ 2\frac{\cos^2\lambda l_k + \cosh^2\lambda l_k}{\sin 2\lambda l_k + \sinh 2\lambda l_k}\cos\lambda x \cosh\lambda x \quad (22)$$

where it is assumed that beginning of x axis is situated in a middle of beam element and l_k represents half of length zone of beam and foundation contact, $\lambda l_k = \pi/2$, $\lambda = \sqrt[4]{4EI/k_0}$. Length of analytically distinguished contact zone amounts $2l_k = 4.034$ [m]. Analytical result was confirmed numerically (see Table 1 and Fig. 2a). Contact zone in Tab. 1 and Fig. 2a includes points with negative deflection values. Due to symmetry only half of the system was presented on a graph. On a Fig. 2a a line of beam deflection and on a Fig. 2b distribution of reactions to three analysed examples were presented.

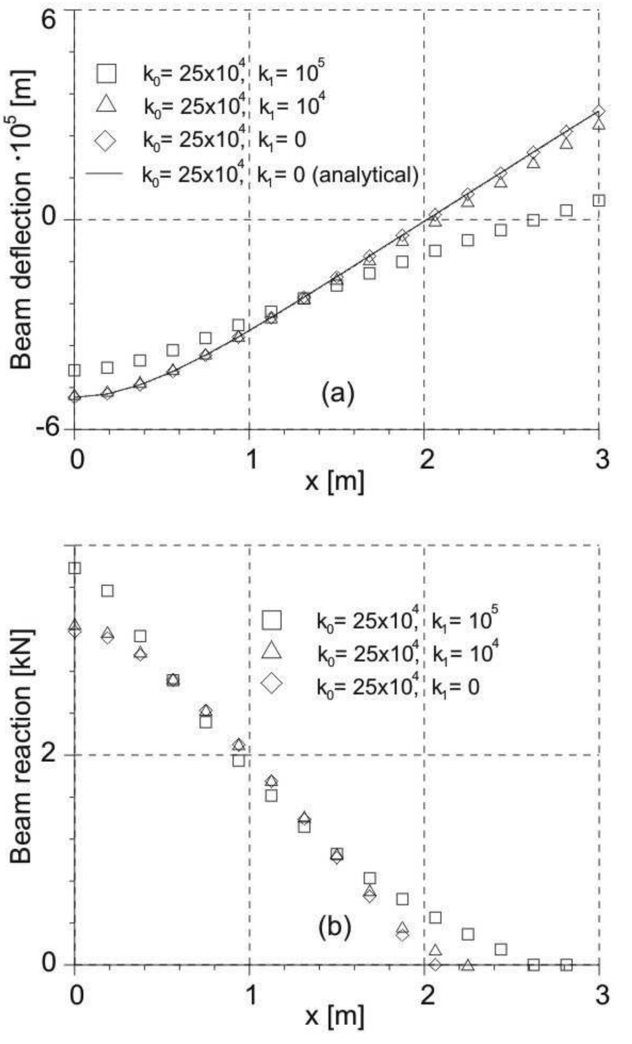

Fig. 2. Free beam on unilateral foundation: line of beam deflection (a) and beam reaction (b), k_0 [kN/m^2] and k_1 [kN]

Table 1.
Composition of own and analytical results of foundation deflection: all presented values *w(x)*[m] are multiplied by 10^5 (k_0 [kN/m^2] and k_1 [kN])

	Analytical method [16]		Present method	
k_0	$25 \cdot 10^4$	$25 \cdot 10^4$	$25 \cdot 10^4$	$25 \cdot 10^4$
k_1	0	0	10^4	10^5
x [m]	*w(x)* [m]			
0.000	-5.0941	-5.0829	-4,9902	-4,3103
0.188	-4.9943	-4.9880	-4.8980	-4.2336
0.375	-4.7312	-4.7278	-4.6458	-4.0275
0.563	-4.3465	-4.3468	-4.2780	-3.7344
0.750	-3.8803	-3.8811	-3.8302	-3.3878
0.938	-3.3556	-3.3590	-3.3303	-3.0011
1.125	-2.7992	-2.8016	-2.7990	-2.0130
1.313	-2.2200	-2.2240	-2.2510	-2.2503
1.500	-1.6340	-1.6361	-1.6957	-1.8832
1.688	-1.0409	-1.0439	-1.1386	-1.5331
1.875	-0.4500	-0.4505	-0.5820	-1.2014
2.063	0.1446	0.1432	-0.0263	-0.8871
2.250	0.7359	0.7369	0.5293	-0.5874
2.438	1.3300	1.3305	1.0848	-0.2982
2.625	1.9183	1.9242	1.6403	-0.0144
2.813	2.5036	2.5518	2.1958	0.2680
3.000	3.0717	3.1115	2.7514	0.5504

Second example is a two-parameter foundation with coefficients $k_0 = 25 \cdot 10^4$ [kN/m^2] and $k_1 = 10^4$ [kN]. It can be noticed that by adding second parameter and incensement of foundation stiffness contact zone of beam element with foundation was enlarged in comparison to Winkler foundation. Third example also represents two-parameter foundation in which first parameter remain the same but the value of the second was increased ($k_0 = 25 \cdot 10^4$ [kN/m^2] and $k_1 = 10^5$ [kN]). It resulted in progressive enlargement of contact zone.

4. Summary

The performance of the solution for the beam element according to Euler-Bernoulli C^1 class beam element resting on elastic foundation represented by zero-thickness foundation elements has been presented and tested.

The results obtained using these kinds of beam elements compare quite well with the theoretical values. The model adopted can be used to analyse the beams on the elastic two-parameter foundation with any type common boundary and contact unilateral conditions or loading combinations. Taking into consideration the influence of the surroundings on settlement of the beams is also possible.

These beam elements on foundation can be further extended to layered inelastic beam elements resting on non-linear foundation. A solution of this problem should be a subject of future investigation.

Explicit expressions for the stiffness matrices both the beam elements and the zero-thickness foundation elements, obtained by using Gauss integration schemes, can be evaluated. Validity and accuracy of the suggested method is verified by a numerical example.

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