Psychology Science, Volume 47, 2005 (1), p. 96 - 111

Some assumptions and facts about arithmetic facts

FRANK DOMAHS[1] & MARGARETE DELAZER[2]

Abstract

In the present review we will deal with the following questions: What are arithmetic facts? How are they related to other cognitive functions? What are characteristic features of fact retrieval performance in healthy adult subjects? Which typical patterns of breakdown can be observed in acalculia? How do current models account for the representation of and access to arithmetic facts?

Key words: simple calculation, mental arithmetic, fact retrieval

[1] Frank Domahs, Universitätsklinik für Neurologie, Innsbruck, Austria; Lehr- und Forschungsgebiet Neuropsychologie, Universitätsklinikum der RWTH, Aachen, Germany
[2] Universitätsklinik für Neurologie, Innsbruck, Austria

Simple arithmetic problems, such as 3+6 or 5x4, are frequently encountered and routinely solved in every-day life. Those problems whose solution does not require further computational processes or strategies but can directly be retrieved from long-term memory are commonly referred to as arithmetic facts. Usually, problems with one-digit operands (2+4; 3x5; 6-3) are subsumed under this label, though a precise definition is rarely given. In the following, we will describe dissociations and associations of arithmetic fact retrieval and related cognitive functions, reviewing neuropsychological evidence as well as findings from experimental and developmental psychology. In this way we will try to characterize in more detail what arithmetic facts actually are. Finally, some approaches will be presented to model the acquisition, representation, and retrieval of arithmetic facts.

Concepts, procedures, and facts

Three main types of arithmetic knowledge can be distinguished: concepts, procedures, and facts (Delazer, 2003). Conceptual knowledge provides understanding of arithmetic operations and principles. This type of knowledge is the prerequisite to make inferences and to relate different information involved in arithmetic. Conceptual knowledge is flexible, can be adapted and applied to new tasks and thus provides *adaptive expertise* (Hatano, 1988). Procedural knowledge guides the execution of algorithms. It can only be applied in familiar contexts. Accordingly, it can be characterised as *routine expertise* (Hatano, 1988). Arithmetic facts, finally, can be conceptualised as stored in and directly retrieved from (declarative) long-term memory (e.g., Ashcraft, 1987; Campbell, 1995; Dehaene & Cohen, 1995; Rickard & Bourne, 1996; Siegler, 1988; for a different view see Baroody, 1994). During development, arithmetic facts evolve from conceptual and procedural knowledge. For instance, it has been demonstrated that the pattern of response latencies and errors in the retrieval of simple multiplication facts of older children and adults reflects the number and type of errors made in the backup strategies used by children at an earlier developmental stage solving the same problems (Lemaire & Siegler, 1995; Siegler, 1988).

Neuropsychological studies support the assumption that arithmetic facts are stored separately from other numerical skills. In a seminal paper Warrington (1982) reported a patient who was no longer able to retrieve simple arithmetic problems from memory, but was able to give the approximate result of arithmetic problems, both in simple and more complex calculation, to estimate visually presented quantities of dots, to give adequate numerical cognitive estimates, to judge the relative size of a number and to give accurate definitions of arithmetic operations. Several later case reports confirmed the separate storage and selective vulnerability of arithmetic facts knowledge. However, other studies questioned the assumption that healthy, educated adults retrieve the solutions for *all* simple arithmetic problems from memory (i.e., as facts) (Lefevre, Bisanz, Daley, Buffone, Greenham, & Sadesky, 1996; Lefevre, Sadesky, & Bisanz, 1996; for methodological considerations see Kirk & Ashcraft, 2001). Lefevre and colleagues found that in single digit addition, non-retrieval strategies (e.g., counting) accounted for 29% of all trials and for about half the problems with sums above 10. In simple multiplication, up to 19% of all trials were solved by non-retrieval strategies (e.g., repeated addition). Furthermore, significant interindividual differences became apparent as the relative amount of retrieval on simple multiplication problems varied from 23% to 100% across participants.

Double dissociations between fact knowledge and conceptual knowledge point to the relative independence of the two components within the cognitive system. It has been shown that patients may lose all conceptual understanding of arithmetic, but preserve part of the memorised fact knowledge including nearly all simple multiplication problems (Dehaene & Cohen, 1997; Delazer & Benke, 1997). Other patients showed severe deficits in retrieving multiplication facts despite excellent conceptual knowledge of arithmetic (Hittmair-Delazer, Semenza, & Denes, 1994).

In the current literature there is general agreement that facts and procedures are processed separately and supported by different cognitive components. Dissociations between spared fact retrieval and impaired procedural skills, supporting this notion, have been reported by several authors (Girelli & Delazer, 1996; Lucchelli & De Renzi, 1993; McCloskey, Caramazza, & Basili, 1985). A double dissociation between arithmetic facts and procedures was reported for two developmental cases of dyscalculia (Temple, 1991, for similar evidence in cases of acquired acalculia see McCloskey et al., 1985). Finally, there is also evidence for dissociating procedural and conceptual knowledge (Girelli & Delazer, 1996, van Lehn, 1986).

Exact and approximate simple calculation

The retrieval of arithmetic facts from memory always leads to an exact number. However, as already observed by Warrington (1982), exact and approximate calculation can dissociate. This was later replicated in a neuropsychological case study by Dehaene & Cohen (1991). They reported patient NAU who erred even with the simplest calculation problems (e.g., 2+2=3). In multiple choice tasks, however, he easily rejected distant false results (2+2=9), while he accepted false results close in magnitude to the correct one (2+2=5). In the light of these findings, Dehaene and Cohen suggested two distinct systems in number-processing - one processing numbers as exact symbols (involved in fact retrieval) and one processing numbers as approximate magnitudes (involved, for instance, in approximate calculation and estimation).

The distinction between exact and approximate calculation is somewhat related to the distinction between facts, procedures, and concepts: Whereas the exclusive use of fact retrieval or calculation procedures will always lead to an exact result, only conceptual knowledge related to magnitude information can be used for approximation. In this way, concept-based approximation can be used to monitor the output of exact fact retrieval or calculation procedures (Lochy, Domahs, & Delazer, 2004). However, approximate and conceptual knowledge are clearly distinguishable as the latter may also lead to exact results. For instance, knowing the principle of commutativity in multiplication one can directly deduce that a x b gives *exactly* (not approximately) the same as b x a.

Facts and rules

Not all simple calculation problems are considered as arithmetic facts. McCloskey and coworkers (e.g., Dagenbach & McCloskey, 1992; McCloskey et al., 1985) distinguished between three subsets of simple multiplication problems: 0's problems (all problems involving 0 as operand), 1's problems (involving 1 as operand) and 2-9 problems (2x2 through 9x9). While the first two subsets (involving 0 and 1) are thought to be answered by a stored rule, only the problems of the last subset are thought to be stored and retrieved individually from memory. First evidence for a special status of rule based problems came from their specific RT characteristics which differs from other problems (Parkman, 1972; Stazyk, Ashcraft, & Hamann, 1982). Furthermore, analyses of data from neurologically impaired and normal subjects revealed that problems involving 0 or 1 typically show consistent error patterns and may be disrupted selectively in multiplication (e.g., nx0=n; nx1=1), addition (n+0; 0+n), or division (n:1) (Delazer, Domahs, Lochy, Karner, Benke & Poewe, 2004; McCloskey, Aliminosa, & Sokol, 1991; Pesenti, Depoorter, & Seron, 2000).

Different operations – separate networks?

Are the four basic operations of arithmetic represented in complete separation, are their representations interrelated or do they form a single common network of representations? Compelling evidence points to at least interrelated representations for the two operations most commonly thought to be solved by memory retrieval, i.e., multiplication and addition. Two main findings support this assumption. First, there is a substantial number of *cross-operation errors*, i.e. 'correct' multiplication results stated for addition problems or vice versa (e.g., 2+4=8; 3x4=7). Miller, Perlmutter, and Keating (1984), for example, found that such cross-operation errors constituted about one quarter of the mistakes that adults made while performing separate blocks of simple additions and multiplications. Furthermore, when offered in verification tasks, 'correct' multiplication results lead to significant interference for addition problems and vice versa (Winkelman & Schmidt, 1974). Second, the acquisition of multiplication in school leads to a temporary increase of response latencies for (already acquired) addition (Miller & Parades, 1990).

Concerning the relationship between multiplication and division, learning studies with healthy subjects yielded somewhat conflicting results. Campbell (1997) reported highly correlated RTs and error characteristics for multiplication and division as well as priming of multiplication errors by previous division trials, compatible with the notion of multiplication at least used to check division. Similar results have been described by Lefevre and Morris (1999), who found closely related error and latency patterns and cross-operational facilitation by complementary problems (more so from division to multiplication). Moreover, on large division problems, participants reported that they 'recast' problems as multiplication. These findings are taken to support the hypothesis that multiplication and division are stored in separate mental representations but that solution of difficult division problems sometimes involves access to multiplication. On the other hand, little if any transfer from multiplication training to division was observed by Rickard, Healy, and Bourne (1994, but see Campbell, 1999 a). Individual differences may play some role in a unifying interpretation of these data, as discussed, for example, by Rickard et al. (1994).

A number of neuropsychological case studies highlighted the relative autonomy of representations for different arithmetic operations. A detailed description of a profound deficit in fact retrieval differently affecting basic operations was provided by Singer and Low (1933, see also Girelli, 2003). Their patient suffered a carbon monoxide poisoning which lead to several neuropsychological deficits including apraxia, agraphia and acalculia. In 'mental figuring' (fact retrieval) the patient was unable, even after six months of daily instruction, to make the simplest subtractions and divisions. In additions, he preserved the ability to add two digits the sum of which was less than 10 (4+3=7) and to add a digit to 10 or 20 (10+8=18). Only in multiplication he showed an improvement after four months of intensive daily training. In fact, the patient was able to answer problems such as 7x2 or 6x12 with very few mistakes. Thus, performance clearly varied across operations in terms of error rates, but also in terms of reaction times, multiplication being fast and automatic, addition being slow.

Some decades later, McCloskey et al. (1991) tested a group of acalculic patients and found that for most patients performance was consistently worse for multiplication than for addition and subtraction. One might assume that multiplication is simply more difficult than subtraction and addition and that the pattern of preserved and impaired performance reflects differences among operations in premorbid 'strength' of the stored facts. However, various patterns of impaired and preserved operations have been observed and a simple operation-difficulty effect cannot account for the results. While, indeed, impaired multiplication tables with better or entirely preserved addition and / or subtraction have been repeatedly reported (see also Hittmair-Delazer et al., 1994; van Harskamp & Cipolotti, 2001), the reverse dissociation has been observed as well (Dehaene & Cohen, 1997; Delazer & Benke, 1997; Singer & Low, 1933; van Harskamp & Cipolotti, 2001). Furthermore, subtraction has occasionally been found to be better preserved than multiplication and addition (Dagenbach & McCloskey, 1992; McNeil & Warrington, 1994; Pesenti, Seron, & Van der Linden, 1994). The opposite dissociation, i.e., selectively impaired subtraction was reported by van Harskamp and Cipolotti (2001). Van Harskamp & Cipolotti also observed a case of selectively impaired addition, but, in contrast to other cases, this patient (FS) made predominantly operation errors, producing in most cases the 'correct' multiplication result to the addition operands. In their reanalysis of this case Dehaene, Piazza, Pinel, and Cohen (2003) argue that the problem of FS may have been the correct choice of operation or the insufficient inhibition of multiplication rather than poor addition itself. This interpretation is supported by the fact that despite his frequent operation errors, FS was able to add correctly in as much as 100/108 trials in which no operation error occurred. Finally, a selective deficit for division was described by Cipolotti and de Lacy Costello (1995). However, a complementary dissociation between impaired simple multiplication and spared simple division has not yet been observed. This leaves open the possibility that division problems are not represented as facts in memory but may be solved by translating them into multiplication as has also been suggested by some of the above described experimental findings with healthy subjects.

Divergent explanations have been offered to account for the operation specific deficits observed. While Dagenbach and McCloskey (1992) explained selective deficits of their patient in terms of damage to segregated representations for the arithmetic operations, McNeil and Warrington (1994) interpreted operation and modality specific deficits by a visual and a verbal calculator dedicated to different operations. Addition and multiplication would be preferentially elaborated in the verbal calculation system, subtractions in the visual/Arabic system. Thus, damage to one of the systems (or impaired access to it) would result in operation and modality specific deficits. A somewhat related interpretation was proposed by Dehaene and Cohen (1995). However, these authors emphasise the different processing components employed in answering the four basic operations. Multiplications are taught systematically and depend heavily on rote memory, whereas subtractions and divisions, which are not taught systematically, rely on back-up strategies. Thus, patterns of selectively preserved and selectively disrupted operations are thought to reflect problems in specific processing components rather than damage to selectively stored representations. Accordingly, disruption of memory representations should result in a severe deficit in multiplication, but not in subtraction. A problem in executing back-up strategies, on the other hand, should lead to problems in subtraction and division, but not in the overlearned multiplication tables. Since the operations are thought to differ in their representation format (multiplications processed in a verbal-auditory code, subtractions in an analog magnitude code) as well as in the processing component used, patterns of modality- and operation-specific deficits can be accounted for (Dehaene & Cohen, 1997). According to this approach simple additions can be solved both via fact retrieval from memory and via quantity manipulations (e.g., Dehaene et al., 2003). Therefore, individual performance is hard to predict. However, one prediction is derived from their model by the Dehaene et al. themselves: Addition cannot dissociate from both subtraction and multiplication together as it relies on at least one of the systems used for those operations. Until now, we are not aware of any evidence challenging this prediction.

It is clear from this brief review of neuropsychological data, that operation difficulty cannot account for all reported operation specific deficits. So far, two main hypotheses have been proposed to explain the observed dissociations, the first assuming segregated memory networks (e.g., Dagenbach & McCloskey, 1992), the second assuming that the four operations are subserved by two different main processing components (e.g., Dehaene & Cohen, 1995; Dehaene & Cohen, 1997). This latter hypothesis seems consistent with findings from developmental and experimental psychology, briefly reviewed in the initial part of this section, which have shown that networks at least of addition and multiplication are interrelated. While not completely excluding the existence of such relations, the former hypothesis had to be modified to account for these data.

Arithmetic facts and language functions

In the last decades of the 19^{th} century it was generally assumed that calculation problems were only one aspect of the complex constellation of deficits present in aphasia (for a historical review see Boller & Grafman, 1983). The first observation of a specific calculation disorder independent of aphasia was reported by Lewandowsky and Stadelmann (1908). They suggested that calculation may be impaired without reduced general intelligence or aphasic problems and described numerical difficulties as a distinct and isolated neuropsychological deficit. Later on, also Henschen (1919; 1920) stated that acalculia constitutes an independent symptom although he observed that acalculia and aphasia are often related. Moreover, he recognised that agraphia and acalculia are frequently associated symptoms which arise after lesions to the left angular gyrus.

Dissociations between language and fact retrieval performance in both directions were described afterwards in a number of case studies. While, for instance, Warrington (1982) and Delazer et al. (2004) presented cases of impaired simple calculation in the light of spared linguistic abilities, the opposite pattern was reported by Rossor, Warrington, and Cipolotti (1995). Rossor and colleagues examined a severely aphasic patient whose ability to answer simple as well as complex problems (addition, subtraction, multiplication) was not compromised by his language impairment. This patient possibly compensated verbally supported calculation skills by non-verbal ones. Also Whalen, McCloskey, Lindemann, and Bouton (2002) argue against a purely language based account of fact retrieval. Their patients successfully retrieved answers to simple arithmetic problems from memory even when they were unable to generate the phonological representation of either the problem itself or its answer.

While these cases provide compelling evidence for the view that language and calculation skills are, in principle, functionally separable, it should not be neglected that language disorders and impairments of arithmetic fact retrieval frequently co-occur and can be systematically related (for a review see Delazer & Bartha, 2001). In the following, some evidence supporting this notion will be shortly reviewed.

Berger (1926) reported calculation problems due to language deficits and classified them as secondary acalculia. Similarly, Hécaen, Angelergues, and Houllier (1961) reported calculation problems due to alexia or agraphia for numerals. Benson and Denckla (1969) identified paraphasias as a source of calculation errors in two aphasic patients. Their calculation deficits appeared rather severe when verbal or written answers had to be given. When confronted with multiple choice tasks, however, the patients were able to select correct solutions. Calculation difficulties in different aphasic groups were investigated in the study of Dahmen, Hartje, Bussing, and Sturm (1982), as well as in the study of Rosselli & Ardila (1989). A more recent study aimed to evaluate the relation between language impairment on the one hand and number processing, mental and written calculation on the other hand (Delazer, Girelli, Semenza, & Denes, 1999). In contrast to previous studies tasks were designed taking into account current calculation models and assessed the single components separately. Importantly, all aphasic patients performed worse than controls in answering arithmetic facts. However, these difficulties could not be explained by verbal production problems, since different answer modalities were allowed. Overall, the error rate in various tasks correlated with the severity of the language deficit, global aphasics being most impaired. More interestingly, qualitative differences were found between aphasic groups, partially reflecting the nature of the specific language problems. In all patient groups (Amnesic aphasia, Broca's aphasia, Wernicke's aphasia, Global aphasia) addition was better preserved than subtraction and multiplication. Multiplication tables were particularly difficult for Broca's aphasics who scored significantly lower in multiplication than in subtraction. This result is in line with previous reports that indicate a high incidence of multiplication deficits in patients with language impairment. It is also consistent with the assumption of verbally supported multiplication tables as suggested by McNeil and Warrington (1994) or Dehaene and Cohen (1995).

A recent training study with healthy, bilingual subjects conducted by Spelke & Tsivkin (2001) also demonstrated that there is some language influence on numerical facts. While their Russian-English bilinguals retrieved information about approximate numbers and non-numerical facts with equal efficiency in both languages, they retrieved information about exact numbers more effectively in the language of training.

In sum, although there are, in general, systematic correlations between performance in linguistic tasks and simple arithmetic problems, single neuropsychological cases point to a relative autonomy of both domains.

Some characteristics of arithmetic fact retrieval

Some stable effects have been observed in simple arithmetic, neither reaction times nor error rates are randomly distributed over problems. In general, problems involving large operands (8x7) yield longer reaction times and higher error rates than problems involving small numbers (2x3). This *problem size effect* is well known in the developmental and experimental literature (Ashcraft, 1992; Ashcraft, 1995; Campbell & Graham, 1985; Groen & Resnick, 1972; Siegler, 1988) and frequently observed in acalculic patients (e.g., Hittmair-Delazer et al., 1994). However, the term *problem-size* effect has been heavily criticised for several reasons. Ashcraft (1992), for example, states that larger numbers have nothing inherent that makes them more difficult to process, but problems with large operands are simply less frequently presented and thus more difficult to answer. Siegler (1988), on the other hand, argues that the problem-size effect arises from the more complex and thus more error-prone back-up strategies used for large problems during development. Interestingly, the problem size effect does not hold for all simple arithmetic problems in the same manner. Rather, tie problems (e.g., 7x7) are solved faster and more accurately than would be predicted by their problem size (Blankenberger, 2001; Campbell & Gunter, 2002). Moreover, there is a size x tie interaction, i.e., the size effect is weaker for ties than for non-ties.

Furthermore, errors in fact retrieval are highly systematic, too. In multiplication, a large proportion of them are multiples of one of the operands, e.g. 5x4=24 (Campbell & Graham, 1985; Siegler, 1988; Sokol, McCloskey, Cohen, & Aliminosa, 1991). These *operand errors* are mostly close to the correct result - typically they do not differ by more than two operands (Campbell & Graham, 1985; McCloskey, 1992). Operand errors are also the most frequent error type in acalculic patients. However, error patterns may differ across patients. Some patients also presented with a high incidence of close miss errors (5x6=31; Girelli, Delazer, Semenza, & Denes, 1996) or of non-table errors (3x4=37; Domahs, Bartha, & Delazer, 2003). Interestingly, such highly implausible error types may disappear in favour of more plausible operand errors during remediation (Domahs et al., 2003; Girelli et al., 1996).

Representation and retrieval of arithmetic facts according to cognitive models

Various models have been advanced to explain the storage and retrieval of arithmetic facts. Though each single model can account for relevant empirical findings, they differ with respect to the perspective taken and to phenomena neglected.

Siegler's (Lemaire & Siegler, 1995; Siegler, 1988) *distribution of associations model*, for example, is able to describe the acquisition of arithmetic facts and to account for the size effect and operand related errors. Both are described as the result of the backup strategies used during development. According to this account, more complex strategies are used for large than for small problems during an early developmental stage, causing more errors for large than for small problems. As all results ever produced for a specific problem create or strengthen associations with this problem, more errors produced during early developmental stages result in more interference in the retrieval of that specific arithmetic fact later on – leading to an effect of problem size. Furthermore, errors made during the execution of backup strategies are highly systematic, leading to a majority of operand related errors which are preserved through the distribution of problem-answer associations. The tie effect, according to Siegler, is related to the frequency of exposure, ties being more frequently encountered in elementary textbooks of arithmetic. In addition to explaining the development of memory retrieval, the strategy selection process which decides between memory retrieval and the use of backup strategies is addressed by this model.

In other network retrieval models, as proposed by Ashcraft (1987) and by McCloskey and Lindemann (1992) retrieval is conceptualized as spreading activation between operand nodes and answer nodes. Spreading activation to neighboring answer nodes naturally accounts for operand related errors. The size effect is attributed to the lower presentation frequency of problems with larger operands during education. Evidence for higher frequency of exposure for small as compared to large problems is, indeed, provided by Ashcraft and Christy (1995). The same explanation should also hold for the tie effect. Yet, there are some problems for frequency-based explanations of the size and tie effects. As noted by McCloskey, Harley, and Sokol (1991), it is not clear how frequency of text book presentation in childhood can account for problem difficulty observed in adults. More importantly, however, a closer look reveals that ties actually do not appear to be more frequent than non-ties in elementary textbooks (for further discussion, see Verguts & Fias, in press).

In Campbell's *network interference model* (Campbell, 1995), so called problem nodes are activated depending on the activation of their constituents. If a subject is, for example, confronted with the problem 3x8, problem nodes like {3, 7, 21}, {4, 8, 32}, or {3, 8, 24} become activated, the latter one (in case of a correct response) most strongly. The activation of one of the former problem nodes, however, would result in an operand error. In addition to the activation of problem nodes, a magnitude code approximating the size of the result is calculated. Thus, the presentation of a small problem activates the problem nodes of several other small problems, while the presentation of a large problem activates the problem nodes of several large problems. According to Campbell, the existence of a magnitude code can explain the problem size effect, provided that it is represented in a compressed way (e.g., Dehaene, 2003). Given a compressed magnitude representation, a large magnitude code would activate more problems than a small magnitude code, leading to more interference and inferior performance. However, the explanation of the tie effect needs an additional assumption. Campbell (1995) argues for a separate representation of ties. As pointed out by Verguts and Fias (in press), this assumption is problematic, because it would predict that most errors to tie problems should result in answers to other tie problems, a prediction which is not borne out by empirical data.

Only recently, Verguts and Fias (in press) proposed a model which does not rely on frequency based explanations of the size and tie effects and which does not assume independent representations of ties. In the model's core component, the *semantic field*, the representation of multiplication problems is internally organized according to the size of their operands. Only one of the complement problems is stored (*max x min* or *min x max* order). For the model itself it does not matter which operand order is stored as long as this is consistent within a person. The semantic field activates answer nodes which are separate for decades and units. Following this assumption, the size effect can result from more consistent cooperative activation of decades from smaller operands than from larger operands, consistent meaning that the same decade is activated. For example, the problems 5x2, 6x3, and 4x3, which are neighbors in the semantic field to the target problem 5x3, consistently activate the same decade (1) whereas only the neighboring problem 5x4 activates another decade (2). For large problems, less consistent cooperative activation of a decade in the answer nodes will occur, leading to more competition of different decades and accordingly to slower and more error-prone responses, i.e., the size effect. In this model, the tie effect is due to representational characteristics of the semantic field, tie problems having fewer (competing) neighbors than non-tie problems. The Verguts and Fias model is silent about how the assumed representations are acquired. On the one hand, its frequency-independent explanation of the size and tie effect can be seen as the model's biggest advantage. On the other hand, it is not clear how the recent version of it can account for the findings of Graham (1987) who showed that the size effect can be influenced by the order in which arithmetic facts are acquired.

Most of the models reviewed so far neither address the representation of different operations nor the specific format of storage and retrieval. Concerning the latter point, it can be assumed that storage relies on some kind of abstract representations (for an exception to this view see Campbell, 1994). The neuropsychological models, which will be presented in the following, deal with these questions, largely ignoring the acquisition and RT phenomena of arithmetic fact retrieval like the problem size effect or attributing them to peripheral factors.

McCloskey and colleagues (1985; see also McCloskey, 1992) assume a central semantic system which is accessed in all calculation processes independently from the format of the input. All number formats (spoken, written, Arabic) are converted into abstract representations which specify the magnitude of the number and serve as input for the calculation system (consisting of an arithmetic fact component, procedural knowledge and the processing of arithmetic signs). Accordingly, the McCloskey model predicts that the processing of arithmetic facts is independent from the input format, e.g. *3x4* (Arabic script) and *three times four* (spoken or written number words) are processed in the same central system. Only the input and output are processed in modality specific components. Within the calculation module of the central system, operations are assumed to be represented separately. However, there is evidence questioning both the assumption of a central, amodal calculation system (e.g., Campbell, 1999 b; Campbell & Fugelsang, 2001) and the separate representation of operations (e.g., Dehaene et al., 2003).

An alternative to McCloskey et al.'s model has been proposed by Dehaene and colleagues (1992; Dehaene & Cohen, 1995). They suggest a *triple-code model* of number processing, comprising a visual-Arabic number code, an auditory-verbal code and an analog-magnitude representation. Each code is dedicated to specific tasks in number processing and calculation. The *visual-Arabic code* is thought to mediate digital input and output, multi-digit operations and parity judgements. The *analog magnitude code* represents the quantity associated with a number as local distributions of activation on an oriented and compressed number line. The analog magnitude representation underlies number comparison, approximate calculation, estimations and contributes to subitizing. The *auditory-verbal code*, finally, represents numbers as syntactically organised sequences of words (Dehaene & Cohen, 1995 following McCloskey, Sokol, & Goodman, 1986). The *auditory-verbal code* mediates verbal input and output, counting and memorised arithmetic facts. The model postulates that multiplication tables and some additions are stored as verbal associations which cannot be retrieved unless the problem is converted into a verbal code. Thus, problems presented in Arabic numerals (3x4) are (at least subvocally) converted into a verbal format (*three times four*) before the answer is retrieved. The direct verbal route is used for overlearned calculations, in particular for multiplication problems. However, Dehaene and Cohen (1997) also propose a second, indirect route for answering simple calculation problems. Operands are not only represented as verbal forms, but also as quantities on the oriented number line, on which operations can be performed. This indirect semantic route is employed when no verbal association is available, typically for subtraction problems. The core assumption of the triple-code-model concerning storage and retrieval of arithmetic facts is that it exclusively relies on the verbal modality. However, there are some problems with this assumption, which were already discussed in the 'Arithmetic facts and language functions' section.

Noël and Seron (1993) introduced a *preferred-entry code hypothesis*. As they observed in neuropsychological case studies, subjects may prefer a particular code in order to access number meaning and to perform numerical tasks. Some subjects prefer a verbal entry code (as did the patients described by Noël & Seron, 1993; 1995), others a visual entry code. If transcoding from a specific notation (e.g., Arabic) to the preferred entry code (e.g., verbal) is impaired, all numerical tasks (e.g., number comparison or calculation) presented in this notation (e.g., Arabic) will be impaired. However, as this hypothesis lacks a satisfying degree of elaboration, specific predictions about acquired disturbances of arithmetic cannot be derived from it. For instance, it remains unclear whether damage to the preferred entry code should result in the complete inability to perform calculations or whether a non-preferred entry code could – at least partly – take over this function.

In sum, models developed from developmental or experimental perspectives focus on typical findings observed in fact retrieval. They differ, for instance, in their explanation of the size effect. Neuropsychological calculation models, on the other hand, are in particular concerned with the relationship of different operations and the representational format of arithmetic facts. They disagree as to whether facts are stored as verbal sequences or as abstract memory representations.

Concluding remarks

Due to space restrictions, this brief review was doomed to neglect some domains of evidence concerning simple arithmetic. For example, neuro-imaging studies have not been discussed, but will certainly be of increasing importance in the coming years (for an overview see Dehaene et al., 2003, and references cited therein). Furthermore, the rehabilitation of facts knowledge in acquired disorders of arithmetic could not be addressed. Interested readers may find the overviews of Girelli and Seron (2001) and Lochy et al. (2004) helpful.

It has to be questioned whether all the studies investigating fact retrieval truly assess simple calculation as used in every day life or whether they just assess performance under laboratory conditions. In fact, several differences may be found between studies specifically designed to study fact retrieval and simple calculation in every day life. First of all, in experimental studies simple calculation is presented without context and without meaning. Thus, arithmetic facts may be recited without reference to the meaning (for example quantity, money, distance) they represent. It is questionable whether 'normal' fact retrieval is done in that way. One the one hand, some errors frequently appearing in experimental studies seem less likely in real life situations. For example, it may be speculated that cross-operation errors occur less frequently when a real context is given. Evidence showing that real life context can, indeed, help doing calculation is, for instance, provided in an investigation of Brazilian street children by Nunes, Schliemann, and Carraher (1993). On the other hand, the complexity of every day situations can also have detrimental effects on number processing as shown in a study with Alzheimer's patients by Martini, Domahs, Benke, and Delazer (2003).

Second, experimental studies commonly present hundreds of problems within one session. Such a high number of trials may critically influence the characteristics of fact retrieval. Cognitive models depict memory for arithmetic facts as related networks with fluctuating levels of activation. Importantly, it is assumed that repeated retrieval changes the activation level of the network. This is, for example, instructively illustrated in studies of priming performed by Campbell and colleagues (Campbell, 1991; Campbell & Arbuthnott, 1996). Thus, constant activation of the network or parts of it may alter the characteristics of retrieval.

Finally, operations are frequently presented in blocked order. Only few studies presented calculation problems in mixed order (Miller & Parades, 1990; Rubinstein, Meyer, & Evans, 2001). In these studies, it has been shown that the presentation in mixed order influences accuracy and error types. It may thus be questioned whether the presentation of long blocks of just one operation is representative of 'true' fact retrieval.

After all, it may be asked why, over several decades, dozens of researchers investigate the representation of a few memory associations. In fact, most studies focus on the small set of 64 multiplication facts only. This question is all the more so justified, as some of the basic problems are still not unequivocally answered, as for example the role of language in fact retrieval, the origin of the problem size effect or the role of intentional control. One of the reasons why arithmetic facts are continuously investigated may be that arithmetic consists of semantic knowledge shared by adults in virtually all cultures, though the level of expertise clearly varies interindividually. Moreover, arithmetic fact knowledge is the only domain where not only the single pieces of information are exactly defined, but also the relations between these single pieces. Thus, cognitive models may be developed at a level of precision which is not possible in other domains. Accordingly, the assessment of fact retrieval allows to study in great detail a small portion of semantic knowledge, the characteristics of retrieval, as well as the acquisition in childhood and the dissolution in acquired disorders of arithmetic.

Reference

Ashcraft, M. H. (1987). Children's knowledge of simple arithmetic: A developmental model and simulation. In R. Brainerd & J. Bisanz, Formal methods in developmental research (pp.302-338). New York: Springer-Verlag.

Ashcraft, M. H. (1992). Cognitive arithmetic: a review of data and theory. Cognition, 44, 75-106.

Ashcraft, M. H. (1995). Cognitive psychology and simple arithmetic: A review and summery of new directions. Mathematical Cognition, 1, 3-34.

Ashcraft, M. H. & Christy, K. S. (1995). The frequency of arithmetic facts in elementary texts: addition and multiplication in grades 1-6. Journal for Research in Mathematics Education, 5, 396-421.

Baroody, A. J. (1994). An evaluation of evidence supporting fact-retrieval models. Learning and Individual Differences, 6, 1-36.

Benson, D. F. & Denckla, M. B. (1969). Verbal paraphasia as a source of calculation disturbance. Archives of Neurology, 21, 96-102.

Berger, H. (1926). Über Rechenstörungen bei Herderkrankungen des Großhirns. Archiv für Psychiatrie und Nervenkrankheiten, 78, 238-263.

Blankenberger, S. (2001). The arithmetic tie effect is mainly encoding-based. Cognition, 82, B15-B24.
Boller, F. & Grafman, J. (1983). Acalculia: historical development and current significance. Brain and Cognition, 2, 205-223.
Campbell, J. I. (1991). Conditions of error priming in number-fact retrieval. Memory and Cognition, 19, 197-209.
Campbell, J. I. (1994). Architectures for numerical cognition. Cognition, 53, 1-44.
Campbell, J. I. (1995). Mechanisms of simple addition and multiplication: A modified network-interference theory and simulation. Mathematical Cognition, 1, 121-165.
Campbell, J. I. (1997). On the relation between skilled performance of simple division and multiplication. Journal of Experimental Psychology: Learning, Memory and Cognition, 23, 1140-1159.
Campbell, J. I. (1999 a). Division by multiplication. Memory and Cognition, 27, 791-802.
Campbell, J. I. (1999 b). The surface form x problem size interaction in cognitive arithmetic: evidence against an encoding locus. Cognition, 70, B25-B33.
Campbell, J. I. & Arbuthnott, K. D. (1996). Inhibitory processes in sequential retrieval: evidence from variable-lag repetition priming. Brain and Cognition, 30, 59-80.
Campbell, J. I. & Gunter, R. (2002). Calculation, culture, and the repeated operand effect. Cognition, 86, 71-96.
Campbell, J. I. D. & Graham, D. J. (1985). Mental multiplication skills: Structures, Process and Acquisition. Canadian Journal of Psychology, 39, 338-366.
Campbell, J. I. D. & Fugelsang, J. (2001). Strategy choice for arithmetic verification: effects of numerical surface form. Cognition, 80, B21-B30.
Cipolotti, L. & de Lacy Costello A. (1995). Selective impairment for simple division. Cortex, 31, 433-449.
Dagenbach, D. & McCloskey, M. (1992). The organization of arithmetic facts in memory: evidence from a brain-damaged patient. Brain and Cognition, 20, 345-366.
Dahmen, W., Hartje, W., Bussing, A., & Sturm, W. (1982). Disorders of calculation in aphasic patients: Spatial and verbal components. Neuropsychologia, 20, 145-153.
Dehaene, S. (1992). Varieties of numerical abilities. Cognition, 44, 1-42.
Dehaene, S. (2003). The neural basis of the Weber-Fechner law: a logarithmic mental number line. Trends in Cognitive Science, 7, 145-147.
Dehaene, S. & Cohen, L. (1991). Two mental calculation systems: a case study of severe acalculia with preserved approximation. Neuropsychologia, 29, 1045-1054.
Dehaene, S. & Cohen, L. (1995). Towards an anatomical and functional model of number processing. Mathematical Cognition, 1, 83-120.
Dehaene, S. & Cohen, L. (1997). Cerebral pathways for calculation: double dissociation between rote verbal and quantitative knowledge of arithmetic. Cortex, 33, 219-250.
Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. Cognitive Neuropsychology, 20, 487-506.
Delazer, M. (2003). Neuropsychological findings on conceptual knowledge of arithmetic. In A. J. Baroody & A. Dowker, The development of arithmetic concepts and skills (pp.385-408). Mahwah, NJ: Lawrence Earlbaum Associates.
Delazer, M. & Bartha, L. (2001). Transcoding and calculation in aphasia. Aphasiology, 15, 649-679.
Delazer, M. & Benke, T. (1997). Arithmetic facts without meaning. Cortex, 33, 697-710.
Delazer, M., Domahs, F., Lochy, A., Karner, E., Benke, T., & Poewe, W. (2004). Number Processing and Calculation in a Case of Fahr's Disease. Neuropsychologia, 42, 1050-1062.

Delazer, M., Girelli, L., Semenza, C., & Denes, G. (1999). Numerical skills and aphasia. Journal of the International Neuropsychological Society, 5, 213-221.
Domahs, F., Bartha, L., & Delazer, M. (2003). Rehabilitation of arithmetic abilities: Different intervention strategies for multiplication. Brain and Language, 87, 165-166.
Girelli, L. (2003). Singer and Low's case of acalculia: Foresight of modern theories on number processing. In C. Code, C. W. Wallesch, Y. Joanette, & A. Roch Lecours, Classic Cases in Neuropsychology (pp.37-55). Hove: Psychology Press.
Girelli, L. & Delazer, M. (1996). Subtraction bugs in an acalculic patient. Cortex, 32, 547-555.
Girelli, L., Delazer, M., Semenza, C., & Denes, G. (1996). The representation of arithmetical facts: evidence from two rehabilitation studies. Cortex, 32, 49-66.
Girelli, L. & Seron, X. (2001). Rehabilitation of number processing and calculation skills. Aphasiology, 15, 695-712.
Graham, D. J. (1987). An associative retrieval model of arithmetic memory: How children learn to multiply. In J. Sloboda & D. Rogers, Cognitive processes in mathematics (pp.123-141). Oxford, UK: Oxford University Press.
Groen, G. J. & Resnick, J. M. (1972). A chronometric analysis of simple addition. Psychological Review, 79, 329-343.
Hatano, G. (1988). Social and motivational bases for mathematical understanding. In G. B. Saxe & M. Gearhart, Children's mathematics (pp.55-70). San Francisco: Jossey-Bass.
Hécaen, H., Angelergues, R., & Houllier, S. (1961). Les variétes cliniques de acalculies au cours de lesions retrorolandiques: Approche statistique du problème. Revue Neurologique, 105, 85-103.
Henschen, S. E. (1919). Über Sprach-, Musik- und Rechenmechanismen und ihre Lokalisation im Großhirn. Zeitschrift für die gesamte Neurologie und Psychiatrie, 52, 273-298.
Henschen, S. E. (1920). Klinische und anatomische Beiträge zur Pathologie des Gehirns. Stockholm: Nordiska Bokhandeln.
Hittmair-Delazer, M., Semenza, C., & Denes, G. (1994). Concepts and facts in calculation. Brain, 117 (Pt 4), 715-728.
Kirk, E. P., & Ashcraft, M. H. (2001). Telling stories: The perils and promise of using verbal reports to study math strategies. Journal of Experimental Psychology: Learning, Memory, and Cognition, 27, 157-175.
Lefevre, J. A., Bisanz, J., Daley, K., Buffone, L., Greenham, St., & Sadesky, G. (1996). Multiple routes to solution of single-digit multiplication problems. Journal of Experimental Psychology: General, 125, 284-306.
Lefevre, J. A. & Morris, J. (1999). More on the relation between division and multiplication in simple arithmetic: evidence for mediation of division solutions via multiplication. Memory and Cognition, 27, 803-812.
Lefevre, J. A., Sadesky, G., & Bisanz, J. (1996). Selection of procedures in mental addition: Reassessing the problem-size effect in adults. Journal of Experimental Psychology: Learning, Memory and Cognition, 22, 216-230.
Lemaire, P. & Siegler, R. S. (1995). Four aspects of strategic change: contributions to children's learning of multiplication. Journal of Experimental Psychology: General, 124, 83-97.
Lewandowsky, M. & Stadelmann, E. (1908). Über einen bemerkenswerten Fall von Hirnblutung und über Rechenstörungen bei Herderkrankung des Gehirns. Journal für Psychologie und Neurologie, 11, 249-265.
Lochy, A., Domahs, F., & Delazer, M. (2004). Assessment and Rehabilitation of Acquired Calculation and Number Processing Disorders. to appear in J. I. D. Campbell (ed.) Handbook of Mathematical Cognition (pp.469-485). New York: Psychology Press.

Lucchelli, F. & De Renzi, E. (1993). Primary dyscalculia after a medial frontal lesion of the left hemisphere. Journal of Neurology, Neurosurgery, and Psychiatry, 56, 304-307.
Martini, L., Domahs, F., Benke, T., & Delazer, M. (2003). Everyday numerical abilities in Alzheimer's disease. Journal of the International Neuropsychological Society, 9, 871-878.
McCloskey, M. (1992). Cognitive mechanisms in numerical processing: evidence from acquired dyscalculia. Cognition, 44, 107-157.
McCloskey, M., Aliminosa, D., & Sokol, S. M. (1991). Facts, rules, and procedures in normal calculation: evidence from multiple single-patient studies of impaired arithmetic fact retrieval. Brain and Cognition, 17, 154-203.
McCloskey, M., Caramazza, A., & Basili, A. (1985). Cognitive mechanisms in number processing and calculation: evidence from dyscalculia. Brain and Cognition, 4, 171-196.
McCloskey, M., Harley, W., & Sokol, S. M. (1991). Models of arithmetic fact retrieval: an evaluation in light of findings from normal and brain-damaged subjects. Journal of Experimental Psychology: Learning, Memory and Cognition, 17, 377-397.
McCloskey, M. & Lindemann, M. (1992). Mathnet: Preliminary results from a distributed model of arithmetic fact retrieval. In J. I. Campbell, The Nature and Origins of Mathematical Skills Amsterdam: North Holland / Elsevier.
McCloskey, M., Sokol, S. M., & Goodman, R. A. (1986). Cognitive processes in verbal-number production: inferences from the performance of brain-damaged subjects. Journal of Experimental Psychology: General, 115, 307-330.
McNeil, J. E. & Warrington, E. K. (1994). A dissociation between addition and subtraction with written calculation. Neuropsychologia, 32, 717-728.
Miller, K. F. & Parades, D. R. (1990). Starting to add worse: Effects of learning to multiply on children's addition. Cognition, 37, 213-242.
Miller, K. F., Perlmutter, M., & Keating, D. (1984). Cognitive arithmetic: Comparison of operations. Journal of Experimental Psychology: Learning, Memory and Cognition, 10, 46-60.
Noël, M.-P. & Seron, X. (1993). Arabic number reading deficit: A single case study or when 236 is read (2306) and judged superior than 1258. Cognitive Neuropsychology, 10, 317-339.
Noël, M. P. & Seron, X. (1995). Lexicalization errors in writing arabic numerals: a single-case study. Brain and Cognition, 29, 151-179.
Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). Street Mathematics and School Mathematics. Cambridge University Press.
Parkman, J. M. (1972). Temporal aspects of simple multiplication and comparison. Journal of Experimental Psychology, 95, 437-444.
Pesenti, M., Depoorter, N., & Seron, X. (2000). Noncommutability of the N + 0 arithmetical rule: a case study of dissociated impairment. Cortex, 36, 445-454.
Pesenti, M., Seron, X., & Van der Linden, M. (1994). Selective impairment as evidence for mental organisation of arithmetical facts: BB, a case of preserved subtraction? Cortex, 30, 661-671.
Rickard, T. C. & Bourne, L. E. (1996). Some tests of an identical elements model of basic arithmetic skills. Journal of Experimental Psychology: Learning, Memory and Cognition, 22, 1281-1295.
Rickard, T. C., Healy, A. F., & Bourne, L. E. (1994). On the Cognitive Structure of Basic Arithmetic Skills: Operation, Order, and Symbol Transfer Effects. Journal of Experimental Psychology: Learning, Memory and Cognition, 20, 1139-1153.
Rosselli, M. & Ardila, A. (1989). Calculation deficits in patients with right and left hemisphere damage. Neuropsychologia, 27, 607-617.

Rossor, M. N., Warrington, E. K., & Cipolotti, L. (1995). The isolation of calculation skills. Journal of Neurology, 242, 78-81.

Rubinstein, J. S., Meyer, D. E., & Evans, J. E. (2001). Executive Control of Cognitive Processes in Task Switching. Journal of Experimental Psychology: Human Perception and Performance, 27, 763-797.

Siegler, R. S. (1988). Strategy Choice Procedures and the Development of Multiplication Skill. Journal of Experimental Psychology: General, 117, 258-275.

Singer, H. D. & Low, A. A. (1933). Acalculia (Henschen): A clinical study. Archives of Neurology and Psychiatry, 29, 476-498.

Sokol, S. M., McCloskey, M., Cohen, N. J., & Aliminosa, D. (1991). Cognitive representations and processes in arithmetic: inferences from the performance of brain-damaged subjects. Journal of Experimental Psychology: Learning, Memory and Cognition, 17, 355-376.

Spelke, E. S. & Tsivkin, S. (2001). Language and number: a bilingual training study. Cognition, 78, 45-88.

Stazyk, E. H., Ashcraft, M. H., & Hamann, M. S. (1982). A network approach to simple multiplication. Journal of Experimental Psychology: Learning, Memory and Cognition, 8, 320-335.

Temple, C. (1991). Procedural dyscalculia and number fact dyscalculia: Double dissociation in developmental dyscalculia. Cognitive Neuropsychology, 8, 155-176.

van Harskamp, N. J. & Cipolotti, L. (2001). Selective impairments for addition, subtraction and multiplication. Iimplications for the organisation of arithmetical facts. Cortex, 37, 363-388.

van Lehn, K. (1986). Arithmetic procedures are induced from examples. In J. Hiebert (Ed.) Conceptual and Procedural Knowledge: The Case of Mathematics. Hillsdale, NJ: Erlbaum.

Verguts, T. & Fias, W. (in press). Interacting neighbours: A connectionist model of retrieval in single-digit multiplication. to appear in Memory and Cognition.

Warrington, E. K. (1982). The fractionation of arithmetical skills: a single case study. Quarterly Journal of Experimental Psychology A, 34, 31-51.

Whalen, J., McCloskey, M., Lindemann, M., & Bouton, G. (2002). Representing arithmetic table facts in memory: Evidence from acquired impairments. Cognitive Neuropsychology, 19, 505-522.

Winkelman, J. & Schmidt, J. (1974). Associative confusions in mental arithmetic. Journal of Experimental Psychology, 102, 734-736.