花瓣 Fibonacci 数的生存函数研究

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摘要　花瓣与 Fibonacci 数有着密切关系,根据 Fibonacci 数与 Lucas 数的递归关系,给出了关于 Fibonacci 数的生存函数 $F(r,x)$ 和 $S(r,n,x)$ 的定义,得到了关于 Fibonacci 数的生存函数,揭示了 Fibonacci 数的内在联系。

关键词　花瓣;Fibonacci 数;Lucas 数;生存函数
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Study on Generating Function of Fibonaccinumbers in Petals

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Abstract　Petals were closely related to the Fibonacci numbers. The generating function about Fibonacci numbers $F(r,x)$ and $S(r,n,x)$ were defined, some generating function about Fibonacci numbers were obtained based on Fibonacci numbers and Lucas numbers recursive relations. The inherent law of Fibonacci numbers was revealed.

Key words　Petal;Fibonacci number;Lucas number;Generating function

最常见的花瓣数目为 5 枚,如梅花、桃花、李花、樱花、杏花、梨花等。常见的花瓣数还有:3 枚,如鸢尾花、百合花;8 枚,如飞燕草;13 枚,如瓜叶菊;向日葵的花瓣为 21 枚或 34 枚;雏菊的花瓣为 34、55 或 89。这些花瓣数目即 Fibonacci 数。所谓 Fibonacci 数 $\{F_n\}$ 由如下初值和递推关系给出[1-4]:$F_0=0,F_1=1,F_n=F_{n-1}+F_{n-2},n=2,3,4,5\cdots$。向日葵的种子排列组成两组相嵌在一起的螺旋线,一组是顺时针方向,一组是逆时针方向。虽然不同品种的向日葵会有所不同,但是这两组螺旋线的数目一般是 34 和 55、55 和 89 或 89 和 144,其中,前一组数字是顺时针螺旋线数目,后一组数字是逆时针螺旋线数目,而每组数字都是 Fibonacci 数中相邻的两个数。在植物的花、叶、枝条、果实、种子中均可发现 Fibonacci 数,即生物学上著名的"鲁德维格定律"[4-8]。笔者研究了 Fibonacci 数的生存函数,揭示了 Fibonacci 数的内在联系,旨在探讨花的生长规律。

1　基本概念及相关记号

与 Fibonacci 数 $\{F_n\}$ 有密切关系的 Lucas 数 $\{L_n\}$ 满足如下初值和递推关系[1-3]:

$L_0=2,L_1=1,L_n=L_{n-1}+L_{n-2},n=2,3,4,5\cdots$。分别用 α 和 β 表示特征方程 $x^2-x-1=0$ 的 2 个特征根,即:$\frac{1+\sqrt{5}}{2}$ 和 $\frac{1-\sqrt{5}}{2}$,则 $\alpha\beta=-1,\alpha-\beta=\sqrt{5}$,且易知:

$$F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}},L_n=\alpha^n+\beta^n$$

设 r 是正整数,记:

$$F(r,x)=\sum_{i=0}^{\infty}F_i^r x^i \tag{1}$$

$$S(r,n,x)=\sum_{i=0}^{n}F_i^r x^i \tag{2}$$

2　Fibonacci 数的生存函数

定理 1　设 r 是非负整数,则:

$$F(2r+1,x)=\left(\frac{1}{\sqrt{5}}\right)^{r}\sum_{k=0}^{r}\binom{2r+1}{k}\frac{F_{2r-2k+1}x}{1-(-1)^kL_{2r-2k+1}x-x^2} \tag{3}$$

$$F(2r,x)=\left(\frac{1}{\sqrt{5}}\right)^{r}\left[\sum_{k=0}^{r-1}(-1)^k\binom{2r}{k}\frac{2-(-1)^kL_{2r-2k}x}{1-(-1)^kL_{2r-2k}x+x^2}+\binom{2r}{r}\frac{(-1)^r}{1-(-1)^rx}\right] \tag{4}$$

证明　由 $F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$ 可知:

$$F(r,x)=\sum_{i=0}^{\infty}\left[\sum_{k=0}^{r}\binom{r}{k}(\alpha^i)^k(-\beta^i)^{r-k}\right]x^i$$

$$=\left(\frac{1}{\sqrt{5}}\right)^{r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\sum_{i=0}^{\infty}(\alpha^k\beta^{r-k}x)^i$$

$$=\left(\frac{1}{\sqrt{5}}\right)^{r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\frac{1}{1-\alpha^{r-k}\beta^kx}$$

再根据 $L_n=\alpha^n+\beta^n$,则:

$$F(2r+1,x)=\left(\frac{1}{\sqrt{5}}\right)^{2r+1}\sum_{k=0}^{r}(-1)^k\binom{2r+1}{k}\left(\frac{1}{1-\alpha^{2r-k+1}\beta^kx}-\frac{1}{1-\alpha^k\beta^{2r-k+1}x}\right)$$

$$=\left(\frac{1}{\sqrt{5}}\right)^{2r+1}\sum_{k=0}^{r}(-1)^k\binom{2r+1}{k}\frac{(\alpha^{2r-k+1}\beta^k-\alpha^k\beta^{2r-k+1})}{1-(\alpha^{2r-k+1}\beta^k+\alpha^k\beta^{2r-k+1})x+(\alpha^{2r+1}\beta^{2r+1})x^2}$$

$$=\left(\frac{1}{\sqrt{5}}\right)^{r}\sum_{k=0}^{r}\binom{2r+1}{k}\frac{F_{2r-2k+1}x}{1-(-1)^kL_{2r-2k+1}x-x^2}$$

同样地有,

$$F(2r,x)=\left(\frac{1}{\sqrt{5}}\right)^{r}\sum_{k=0}^{r-1}(-1)^k\binom{2r}{k}\left(\frac{1}{1-\alpha^{2r-k}\beta^kx}+\frac{1}{1-\alpha^k\beta^{2r-k}x}\right)+\binom{2r}{r}\frac{(-1)^r(\frac{1}{\sqrt{5}})^{2r}}{1-(-1)^rx}$$

$$=\left(\frac{1}{\sqrt{5}}\right)^{r}\left[\sum_{k=0}^{r-1}(-1)^k\binom{2r}{k}\right.$$

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$$\frac{2-(\alpha^{2r-k}\beta^{k}+\alpha^{k}\beta^{2r-k})x}{1-(\alpha^{2r-k}\beta^{k}+\alpha^{k}\beta^{2r-k})x+(\alpha^{2r}\beta^{2r})x^{2}}+\binom{2r}{r}\left(\frac{(-1)^{r}}{1-(-1)^{r}x}\right)$$

$$=\left(\frac{1}{\sqrt{5}}\right)^{r}\left(\sum_{k=0}^{r-1}(-1)^{k}\binom{2r}{k}\right.$$

$$\left.\frac{2-(-1)^{k}L_{2r-2k}x}{1-(-1)^{k}L_{2r-2k}x+x^{2}}+\binom{2r}{r}\left(\frac{(-1)^{r}}{1-(-1)^{r}x}\right)\right)$$

因此,(2)、(3)两公式成立。

把 $r=0$ 代入公式(3),同样地可得到已知的结果[2],即:

$$F(1,x)=\sum_{i=0}^{\infty}F_{i}x^{i}=\frac{x}{1-x-x^{2}}$$

在公式(3)、(4)中分别取 $r=1$,可得如下的推论:

推论 1 $\sum_{i=0}^{\infty}F_{i}^{2}x^{i}=\frac{x(1-x)}{(1+x)(1-3x+x^{2})}$ (5)

$$\sum_{i=0}^{\infty}F_{i}^{3}x^{i}=\frac{25x(1-2x-x^{2})}{(1-4x-x^{2})(1+x-x^{2})} \quad (6)$$

推论 2 $\sum_{i=0}^{\infty}F_{i}F_{i+1}x^{i}=\frac{x}{(1+x)(1-3x+x^{2})}$ (7)

证明　由 $F_{0}=0,F_{i+2}=F_{i+1}+F_{i}$,可得:

$$(1-x^{2})\sum_{i=0}^{\infty}F_{i}F_{i+1}x^{i}=\sum_{i=0}^{\infty}F_{i}F_{i+1}x^{i}-\sum_{i=0}^{\infty}F_{i}F_{i+1}x^{i+2}$$

$$=\sum_{i=1}^{\infty}F_{i}(F_{i}+F_{i-1})x^{n}-\sum_{i=0}^{\infty}F_{i+1}(F_{i+2}-F_{i+1})x^{n+2}$$

$$=\sum_{i=1}^{\infty}F_{i}^{2}x^{i}+\sum_{i=1}^{\infty}F_{i}F_{i-1}x^{i}-\sum_{i=0}^{\infty}F_{i+1}F_{i+2}x^{i+2}+\sum_{i=0}^{\infty}F_{i+1}^{2}x^{i+2}$$

$$=(1+x)F(2,x)$$

利用公式(5),可得到:

$$\sum_{i=0}^{\infty}F_{i}F_{i+1}x^{i}=\frac{1+x}{1-x^{2}}\cdot\frac{x(1-x)}{(1+x)(1-3x+x^{2})}$$

$$=\frac{x}{(1+x)(1-3x+x^{2})}$$

定理 2　设 r 是非负整数,则:

$$S(2r+1,n,x)=\left(\frac{1}{\sqrt{5}}\right)^{r}x\sum_{k=0}^{r}\binom{2r+1}{k}\frac{F_{2r-2k+1}-(-1)^{kn}F_{(2r-2k+1)(n+1)}x^{n}-(-1)^{k(n-1)}F_{(2r-2k+1)n}x^{n+1}}{1-(-1)^{k}L_{2r-2k+1}x-x^{2}} \quad (8)$$

$$S(2r,n,x)=\left(\frac{1}{\sqrt{5}}\right)^{r}\left(\sum_{k=0}^{r-1}(-1)^{k}\binom{2r}{k}\frac{2-(-1)^{k}L_{2r-2k}x-(-1)^{k(n+1)}L_{(2r-2k)(n+1)}x^{n+1}+(-1)^{kn}L_{(2r-2k)n}x^{n+2}}{1-(-1)^{k}L_{2r-2k}x+x^{2}}+(-1)^{r}\binom{2r}{r}\frac{(-1)^{r(n+1)}x^{n+1}-1}{(-1)^{r}x-1}\right) \quad (9)$$

证明　由于

$$S(r,n,x)=\sum_{i=0}^{n}\left(\sum_{k=0}^{r}\binom{r}{k}(\alpha^{i})^{k}(-\beta^{i})^{r-k}\right)x^{i}$$

$$=\left(\frac{1}{\sqrt{5}}\right)\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\sum_{i=0}^{n}(\alpha^{k}\beta^{r-k}x)^{i}$$

$$=\left(\frac{1}{\sqrt{5}}\right)\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\frac{(\alpha^{k}\beta^{r-k}x)^{n+1}-1}{\alpha^{k}\beta^{r-k}x-1}$$

根据 $F_{n}=\frac{\alpha^{n}-\beta^{n}}{\sqrt{5}},L_{n}=\alpha^{n}+\beta^{n}$,分别讨论 r 为奇数和偶数两种情况,再运用“定理 1”的证明方法,同样可得到公式(8)、(9)。

3　结语

利用 Fibonacci 数与 Lucas 数的递归关系与性质,得到了关于 Fibonacci 数的生存函数,即定理 1 与定理 2,在定理 1 与定理 2 中分别取 r,n,x 为一些特殊的值,如取 $r=0,1,2;n=4m,8m;x=1,-1$ 可以得到若干个关于 Fibonacci 数的恒等式,建立许多的关于 Fibonacci 数的数学模型,从而进一步探索花的生长奥秘。

参考文献

[1] DICKSON L E. History of the theory of numbers[M]. New York:Chelsea,1966.

[2] HOGGATT V E. Fibonacci and lucas numbers[M]. New York: Houghton Mifflin Company,1969.

[3] STANLEY RABINOWITZ. Algorithmic summation of reciprocals of rroducts of fibonacci numbers[J]. The Fibonacci Quarterly,1999,37(2):122-127.

[4] 吴振奎. 斐波那契序列[M]. 沈阳:辽宁教育出版社,1987.

[5] WARDLAW C W. Phyllotaxis and organogenesis in fems [J]. Nature,1949,164:167-169.

[6] 翁羽翔. 美丽是可以描述的——描述花卉形态的数理方程[J]. 物理,2005,34(4):254-261.

[7] 曹则贤. 花叶序研究拾趣[J]. 自然杂志,2008,30(2):106-108.

[8] 申芳芳,张万里,李德志. 植物叶序研究的源流与发展[J]. 东北林业大学学报,2006,34(5):83-86.

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[26] MATA G,MURRIETA H D M,IGLESIAS A L G. Changes in lignocellulolytic enzyme activites in six *Pleurotus* spp. strains cultivated on coffee pulp in confrontation with *Trichoderma* spp[J]. World Journal of Microbiology & Biotechnology,2005,21:143-150.

[27] HONDA S,NISHIMURA Y,CHIBA H,et al. Determination of carbohydrates by condensation with 3-methyl-2-benzothiazolinonehydazone[J]. Analytica Chimica Acta,1981,131:293-296.

[28] ANTHON G E,BARRETT D M. Determination of reducing sugars with 3-Methyl-2-benzothiazohnonehydrazone[J]. Analytical Biochemistry,2002,305:287-289.

[29] WIRTH S J,WOLF G A. Dye-labelled substrates for the assay and detection of chitinase and lysozyme activity[J]. Journal of Microbiological Methods,1990,12(3/4):197-205.

[30] WIRTH S J,WOLF G A. Micro-plate colourimetric assay for Endo-acting cellulase,xylanase,chitinase,1,3-β-glucanase and amylase extracted from forest soil horizons[J]. Soil Biology and Biochemistry,1992,24(6):511-519.

[31] GE Y,ANTOULINAKIS E G,GEE K R,et al. An ultrasensitive,continuous assay for xylanase using the fluorogenic substrate 6,8-difluoro-4-methylumbelliferyl β-D-xylobioside[J]. Analytical Biochemistry,2007,362:63-68

[32] 王迎春,李春华,刘晓军,等. 茶叶中 β-葡萄糖苷酶研究进展[J]. 安徽农业科学,2008,36(10):4151-4153.

[33] QAQISH R B,AMIJI M M. Synthesis of fluorescent chitosan derivative and its application for the study of chitosan- mucin interaction[J]. Carbohydr Polym,1999,38:99-107.

[34] TIKHONOV V E,LOPEZ-LLORCA L V,SALINAS J,et al. Endochitinase activity determination using N-fluorescein-labeled chitin[J]. Journal of Biochemical and Biophysical Methods,2004,60(1):29-38.