Use of Analytical Modeling and Limited Data for Prediction of Mesoscale Eddy Properties

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ABSTRACT

The use of analytical modeling in the study of oceanic eddies is considered. Limited observational data, in combination with eddy models, are used to obtain analytical approximations to environmental effects (including current and temperature perturbations) throughout the eddy. Techniques which efficiently use discrete measurements are presented for accurate specification of any given analytical model, containing an arbitrary number of parameters, to an observed eddy. Questions of unique parameter specification and data sufficiency are considered for various data types and amounts, using a previously derived eddy model. Examples with bathythermograph data are presented in which eddy size, strength and center position are to be determined. AXBT data are emphasized, and an investigation is made of the influence of the number of such instruments on the accuracy of parameter estimates. It is then shown how data obtained from oceanographic moorings might be utilized to specify eddy drift speed and direction. In both the bathythermograph and mooring examples, it is demonstrated that even when the type of data available leads to non-unique parameter specification, significant information can be obtained about the observed eddy. Results in this paper suggest possible efficiencies in data utilization and in the design of subsequent experiments.

1. Introduction

Mesoscale eddies have been observed to be responsible for significant variations in the ocean. Investigations of individual eddies have typically employed extensive deep bathythermograph readings, to depths of several thousand meters at frequent horizontal locations (see, e.g., Fuglister, 1972). Other studies have also incorporated current data obtained from moorings (e.g., Koshlyakov and Grachev, 1973). In such experiments, the resulting discrete readings are used to approximate the continuum of eddy environmental effects by either simple smoothing of the data, or the more sophisticated techniques of objective analysis (Bretherton *et al.*, 1976). However, these procedures typically require a high density of readings throughout the oceanic region containing the eddy, requiring fairly extensive ship time and equipment. Further, results from these procedures are restricted to regions within the eddy where data were acquired.

Efforts to explain and describe eddy structure have produced several analytical eddy models. Stern (1975) considered variational principles for equilibrium of barotropic eddies, with possible extensions to the baroclinic case. Andrews and Scully-Power (1976) proposed a model for the velocity fields and dynamic heights of an anticyclonic eddy. Flierl discussed the use of both nonlinear dynamics to study eddies linked to a barotropic shear flow (Flierl, 1976) and linear dynamics to study the evolution and translation of eddy-like disturbances (Flierl, 1977). The above studies did not attempt to compare theoretical structures, such as those of temperature and current, to those of observed eddies. However, a recent model of several of the authors (Henrick *et al.*, 1977) was shown to give satisfactory agreement with current and temperature perturbations observed in one Gulf Stream ring.

We remark that any eddy model will inevitably reflect the ultimate objectives of the modeler. No one model could be expected to appropriately describe all eddies, nor could it fully describe one particular eddy from its nascent to terminal stages. Thus, a model can only be viewed as a dynamical approximation, valid to some specified degree of accuracy.

We consider here the problem of utilizing analytical models and limited data in the description of properties of an observed eddy. It is possible that such an approach might require less extensive meas-

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urements than presently used, while simultaneously avoiding the known hazards of ocean undersampling. In this paper, we make the important assumption that enough evidence is available to warrant application of the particular model chosen. Our major purposes are to specify the types and amounts of data necessary for determination of model parameters, and to present some techniques for parameter specification. Another study (McWilliams and Flierl, 1976) used a linear wave model to describe a field of eddies, and thus was concerned with larger scales and more parameters than models for a single eddy.

In Section 2, a brief review of a previously derived eddy model is presented. The assumptions implicit in this model are outlined and the model parameters (including eddy size, depth, direction and speed of rotation, and drift trajectory) are presented. This model will be used in subsequent sections only to provide illustrative examples for the parameter-estimation techniques presented, with no implication for its general usefulness in describing eddies. The problem of using observational data to specify the parameters of an arbitrary parametric eddy model is discussed in Section 3. The question of the minimum amount of required data, a necessary prerequisite for accurate parameter specification, is considered. Then, a technique for accurately specifying model parameters, using more than the minimal data, is presented. In Section 4 the use and processing of temperature and current data in selecting model parameters is discussed. Several distinct varieties of such data are then used to specify model parameters, for the model of Section 2 as an example. The parameter-fitting technique is first applied in Section 5 to time-independent data obtained from bathythermographs. It is shown that if data through the eddy are taken in a path lying in a vertical plane, ambiguity in the position of the eddy center can result, while inclusion of nonplanar data leads to unique parameter values. In Section 6 an investigation is made on the use of oceanographic moorings to study both eddy structure and translation. Non-uniqueness of parameter values results if only a single thermistor string is used, but inclusion of additional thermistor strings or current meters results in unique parameter specification. The principal results of the paper are summarized in Section 7.

2. One eddy model

In this section, a previously derived analytical model (Henrick *et al.* 1977) for one class of quasi-geostrophic eddies is summarized. This model has been shown to agree fairly well with observed features of one Gulf Stream ring, so it will be used as an example in subsequent sections. However, we stress that this model does not admit such possibly significant features as radial asymmetry, salinity variation or near-surface effects. Of course, these limitations do not affect the general objectives of this paper, since any eddy model which contains an arbitrary number of parameters may be employed in the procedures to be discussed.

We consider the dynamical fluid equations for an ocean of depth D, with a Cartesian coordinate system centered at some latitude ϕ. The x axis is oriented positively to the east, the y axis to the north, and the depth z is measured positively downward from the ocean surface. The equations are simplified through appropriate scaling, after which solutions for the density and pressure fields are sought in terms of static states plus motion-induced perturbations:

$$\rho = \rho_s(z) + \tilde{\rho}(x,y,z,t), \tag{1a}$$

$$P = P_s(z) + \tilde{P}(x,y,z,t). \tag{1b}$$

The depth-dependent static pressure $P_s(z)$ and density $\rho_s(z)$ are comprised of isentropic components plus small variations modeled by an algebraic potential density. The motion-induced density and pressure perturbations $\tilde{\rho}$ and $\tilde{P}$ are functions of both positive (x,y,z) and time t. The resulting horizontal current $\mathbf{V} = (u,v)$ can be written in terms of a streamfunction ψ as

$$u = -\psi_y, \tag{2a}$$

$$v = \psi_x, \tag{2b}$$

while density and pressure perturbations are given in terms of ψ by

$$\tilde{\rho} = (2\Omega\rho_0 g^{-1} \sin\phi)\psi_z, \tag{2c}$$

$$\tilde{P} = (2\rho_0\Omega \sin\phi)\psi, \tag{2d}$$

where Ω represents the earth's angular velocity, ρ_0 the surface density and g gravitational acceleration. The streamfunction is constrained to follow a time-independent quasi-geostrophic potential vorticity equation. Quasi-geostrophic balance has been used by other eddy modelers (see, e.g., Flierl, 1977), and was tested in the MODE experiment (Bryden *et al.*, 1975) although the results were inconclusive.

Solutions that exhibit eddy-like characteristics are sought to the potential vorticity equation. Since eddies are often observed to be nearly circular features drifting through the ocean (see, e.g., Fuglister, 1972), approximate solutions are considered which are radially symmetric about a central axis and which translate with time along some assumed path $[x_0(t),y_0(t)]$. Hence the variable

$$r^2 = [x - x_0(t)]^2 + [y - y_0(t)]^2 \tag{3a}$$

is introduced, and the streamfunction is written as

$$\psi(x,y,z,t) = \chi(r,z). \quad (3b)$$

A reasonable set of boundary conditions can be postulated from observed eddy features. For example, eddy currents and perturbation density and pressure are required to vanish at some finite distance r_0 from the eddy center and at some depth z_0 ($\leqslant D$), below which eddy effects are assumed negligible. An approximate solution to this boundary value problem is obtained by superimposing the vertical barotropic and first baroclinic modal solutions:

$$\chi(r,z) = K_1 U_0 r_0 [J_0(\alpha_1 r/r_0) - J_0(\alpha_1)][F(z) - F(z_0)], \quad (4a)$$

where

$$K_1 = \pm Dg\{2\alpha_1\alpha_2 c_0 \Omega \sin\phi[F(0) - F(z_0)]\}^{-1}, \quad (4b)$$

$$F(z) = (1 + Bz)^{-1/2}\{\cos[(\gamma/2)\ln(1 + Bz)] + (1/\gamma)\sin[(\gamma/2)\ln(1 + Bz)]\}, \quad (4c)$$

$$\gamma = 2\pi/\ln(1 + Bz_0). \quad (4d)$$

In (4), J_0 is the Bessel function of the first kind and order zero, $\alpha_1 = 3.83$ is the first zero of the Bessel function J_1 of the first kind and order one, and $\alpha_2 = 0.582$ is the maximum of J_1. The symbol U_0 is the maximum surface current speed, $c_0 = 1540$ m s^{-1} is the assumed surface sound speed, and $B = 2.312$ km^{-1} is a stratification parameter from the potential density. In (4b) the plus(minus) sign corresponds to an anticyclonic (cyclonic) eddy with clockwise (counterclockwise) rotational velocity. Below z_0, we assume that both the streamfunction $\chi(r,z)$ and its vertical gradient, representing eddy environmental effects, are zero. We note that imposition of these conditions implies the vanishing of both baroclinic and barotropic vertical solutions for $z_0 < z < D$, and that the resulting vertical eigenvalue (4d) approximates that obtained when barotropic effects extend to D. The separability of each of the individual baroclinic and barotropic modes and their superposition have been exploited in other eddy models (e.g., Flierl, 1976).

The resulting velocity, density and pressure fields are determined through (2)–(4). Temperature effects are then approximated from pressure and density values, assuming a constant salinity of 35‰, from either a table of values or a suitable state equation. In order to determine eddy-environmental effects, model parameters characterizing an eddy must be specified. In the particular model described above, we need to choose the radius r_0, depth of influence z_0, maximum current speed U_0, direction of rotation [the sign in (4b)], and the translational trajectory described by $x_0(t)$ and $y_0(t)$.

3. Parameter determination

Using observational data to specify values of parameters in an appropriate eddy model requires matching observed perturbations with eddy effects predicted by the model. We now consider the problem of using eddy-induced environmental deviations of general types for accurate selection of model parameters.

Any model solution that specifies eddy-environmental effects depends on position and time, and is assumed to include n parameters $\mathbf{p} = (p_1, \ldots, p_n)$. For fixed location and time, the eddy model determines eddy effects as nonlinear functions of the n parameters. Suppose an environmental deviation Δ is observed at a particular position and time (x_1,y_1,z_1,t_1). A corresponding relation of the form

$$M = f(x_1,y_1,z_1,t_1; \mathbf{p}) \quad (5)$$

for the deviation M predicted by the model can be obtained. It is tempting to suppose that given a sufficient number ($\geqslant n$) of observations, a unique choice of the n parameters chould be determined by solving Eqs. (5) with each M replaced by the corresponding Δ at n distinct positions or times. However, it is possible that the functional forms of (5) may prohibit these measurements from being taken at arbitrary points in space or time. Moreover, unique parameter values may not result, even if a proper sampling is performed.

We suppose here that the model equations at any (x,y,z,t) are separable into a product of vertical and horizontal functions, each with distinct parameters:

$$M = f_H(x,y,t; \mathbf{p}_H) f_V(z,t; \mathbf{p}_V). \quad (6)$$

In (6), $\mathbf{p}_H$ represents the vector of n_H parameters $p_H^{(i)}$, $1 \leqslant i \leqslant n_H$, associated with the description of the horizontal structure and translation of the eddy. Similarly, $\mathbf{p}_V$ is the vector of n_V parameters $p_V^{(i)}$, $1 \leqslant i \leqslant n_V$, which describe the eddy vertical structure. Such separability would be characteristic of many eddy models, and appears justifiable from observations of near-uniform decay of eddies with depth. However, if such separability were not permitted in the model, it would be necessary to choose all the parameters at once.

Suppose that deviations in the ocean's static state $\Delta_1^{(j)}$ are observed at a particular fixed horizontal position and time (x_1,y_1,t_1) and at $n_V + 1$ different depths $z_1^{(j)}$, $1 \leqslant j \leqslant n_V + 1$. Equating observed and predicted environmental deviations for each j yields

$$\Delta_1^{(j)} = f_H(x_1,y_1,t_1; \mathbf{p}_H) f_V(z_1^{(j)},t_1; \mathbf{p}_V), \quad 1 \leqslant j \leqslant n_V + 1. \quad (7)$$

The horizontal function f_H can then be eliminated from these equations by solving for f_H at a single depth and substituting into the remaining n_V

equations:

$$\Delta_1^{(j)} f_V(z_1^{(1)},t_1;\, \mathbf{p}_V) - \Delta_1^{(1)} f_V(z_1^{(j)},t_1;\, \mathbf{p}_V) = 0, \quad 2 \leq j \leq n_V + 1, \quad (8)$$

where, without loss of generality, we have chosen the first equation to eliminate horizontal dependence.

To obtain a more accurate analytical approximation to the vertical eddy structure, it is very desirable to utilize more than the minimum required data set. The problem of generating more accurate choices of model parameters can be viewed as overspecifying system (8) by using more points (and hence more equations) than parameters. The choice of these parameters can be made by minimizing the sum of the squares of the relative errors of the fit thereby avoiding biasing of the larger near surface observations:

$$\underset{\mathbf{p}_V,\bar{M}}{\text{Min}} \sum_{j=1}^{m_V} \{[\bar{M} f_V(z_1^{(j)},t_1;\, \mathbf{p}_V) - \Delta_1^{(j)}]^2/(\Delta_1^{(j)})^2\}, \quad (9)$$

where $\bar{M} = f_H(x_1,y_1,t_1;\, \mathbf{p}_H)$. In (9), the magnitude of the horizontal structure $\bar{M}$ is varied, since elimination of this quantity, using an inaccurate measurement as in (8), can lead to significant error. The vertical structure is then specified by solving (9) by numerical techniques such as a modified Levenburg-Marquardt algorithm (Brown and Davis, 1972) or statistical nonlinear regression techniques. For accurate specification, we have found from numerical experience that it is necessary to utilize readings to a depth of 750 m or more. If such deep measurements are available at several horizontal locations, a better fit can be obtained by solving problem (9) at each location and averaging the selected parameter values.

Knowledge of the vertical structure can be used to eliminate vertical dependence of measurements at any point by defining the vertically scaled environmental deviation

$$\bar{\Delta}_1^{(j)} = \Delta_1^{(j)}/f_V(z_1^{(j)},t_1;\, \mathbf{p}_V), \quad 1 \leq j \leq m. \quad (10)$$

Of course, care must be taken in utilizing measurements where f_V is small, to avoid magnification of errors in the data. Typically, measurements would be available at m depths $z_1^{(j)}$, $1 \leq j \leq m$. The magnitude of eddy effects at (x_1,y_1,t_1) can then be determined by defining the average vertically scaled environmental deviation

$$\bar{\Delta}_1 = (1/m) \sum_{j=1}^{m} \bar{\Delta}_1^{(j)}. \quad (11)$$

This averaging procedure avoids propagation of errors in data at any one depth. We note that if the vertical model structure fits the observed vertical structure exactly, then the terms in sum (11) are identical.

The horizontal parameters can then be chosen using the vertically independent measurements from (11). Given observations at n_H distinct horizontal positions and/or times (x_i,y_i,t_i), $1 \leq i \leq n_H$, $\bar{\Delta}_i$ can be determined at each position as in (11) and equated with the vertically scaled deviation predicted by the model, resulting in n_H equations in the form

$$\bar{\Delta}_i = f_H(x_i,y_i,t_i;\, \mathbf{p}_H). \quad (12)$$

Again, more accurate values for the horizontal parameters can be obtained by overconditioning (12), resulting in the horizontal problem:

$$\underset{\mathbf{p}_H}{\text{Min}} \sum_{i=1}^{m_H} \{[\bar{\Delta}_i - f_H(x_i,y_i,t_i;\, \mathbf{p}_H)]^2/(\bar{\Delta}_i)^2\}, \quad (13)$$

where relative least squares is again advantageous to avoid biasing larger terms. More accurate parameter fits will be obtained by using a wide distribution of points throughout the eddy, rather than a narrow grouping of points in one area, such as near the eddy edge. Further, because of the ordinarily large length scales of eddies, the points should be separated by at least 10 km to ensure distinct readings; equivalently, if data are taken at the same horizontal location, but at different times, intervals of several days between points used in (13) are sufficient.

We conclude this section by specializing the above discussion to the particular model of Section 2. The parameters of this eddy model are the direction of rotation, depth of influence z_0, radius r_0, maximum rotational current speed U_0, and n_D parameters for the form assumed for the eddy trajectory. For example, if a constant drift velocity is assumed, the position of the eddy center at time t is given by

$$[x_0(t),y_0(t)] = [X_0 + u_D t, Y_0 + v_D t], \quad (14)$$

where (X_0,Y_0) is the position of the eddy center at time $t = 0$, and $\mathbf{V}_D = (u_D,v_D)$ is the horizontal drift velocity. Thus, there are $n_D = 4$ drift parameters in this case. We note that $n_D \geq 2$ always, since two parameters representing the initial position of the eddy center will always be present. Since the direction of rotation can be chosen unambiguously from the sign of the perturbation temperature, there are $n_H = n_D + 2$ horizontal parameters. Moreover, $n_V = 1$ since the only vertical parameter is z_0. For any parameter values, deviations in pressure and density structure, and current velocities, are obtained from (2)–(4) as

$$\bar{P} = (2\rho_0\Omega \sin\phi)U_0 r_0 K_1[J_0(\alpha_1 r/r_0) - J_0(\alpha_1)][F(z) - F(z_0)], \quad (15a)$$

$$\bar{\rho} = (2\rho_0\Omega g^{-1} \sin\phi)U_0 r_0 K_1[J_0(\alpha_1 r/r_0) - J_0(\alpha_1)]F'(z), \quad (15b)$$

$$(u,v) = U_0 K_1 \alpha_1 J_1(\alpha_1 r/r_0)[F(z) - F(z_0)] \times [-y + y_0(t), x - x_0(t)]/r. \quad (15c)$$

4. Typical observational data

The most commonly observed environmental effect of oceanic eddies is their large temperature perturbations. In the process of constructing an eddy model, however, temperature is typically not a primary quantity. Solutions to the fluid equations are usually in terms of pressure, density and current. To effectively utilize the parameter-specification techniques of the previous section, temperature measurements should be preprocessed.

Eddy temperature effects result from perturbations in the static density, salinity and pressure profiles. However, assuming salinity variations across an eddy can be ignored (Fuglister, 1972), a Boussinesq argument by Spiegel and Veronis (1960) shows that density and temperature are insensitive to pressure perturbations. Hence, we may approximate the observed density by

$$\rho \approx E[T,P_s(z)], \tag{16}$$

where (16) represents either a form of a state equation with constant salinity (see, e.g., Baer and Jacobson, 1974) or a tabulated density value.

The static state of an ocean area under consideration is typically known from long-term observations or archival sources. Eddy density perturbations $\Delta\rho$ may then be approximated from the observed temperature perturbations ΔT by the relation

$$\Delta\rho = E(T_s + \Delta T,P_s) - E(T_s,P_s), \tag{17}$$

where T_s is the known static temperature. Theoretical density predictions from the eddy model can then be used in comparison with those of (17) in the procedure of Section 3.

Current measurements also permit acquisition of significant information about eddies. A single current meter provides both the magnitude $|\mathbf{V}|$ and the direction θ of the observed current, where we take the angle θ to be measured positively counterclockwise from an eastward latitudinal parallel. Observations of current magnitude can be used with predictions of current speeds from the model equations directly in the procedure of Section 3. Additionally, current-direction readings provide more information. How this information is exploited depends on the eddy model. In a model with assumed circular streamlines, such as in Section 2, the unit tangent vector $(\cos\theta, \sin\theta)$ to a streamline at a given point will be perpendicular to a line through the eddy center, i.e., $x_0(t)$ and $y_0(t)$. Thus, if a current direction θ_i is observed at the horizontal position (x_i,y_i) at time t_i, the eddy center is known to lie on the line

$$[y_0(t_i) - y_i] = [x_0(t_i) - x_i] \tan(\pi/2 - \theta_i). \tag{18}$$

Eq. (18) is a relation between the coordinates $x_0(t_i)$, $y_0(t_i)$ of the eddy center at the measurement time.

If current direction is measured at two or more points simultaneously, and if the points do not lie on the same eddy diameter, then $x_0(t_i)$ and $y_0(t_i)$ are uniquely determined by the intersection of lines in the x_0, y_0 plane of the form (18). Of course, serious numerical errors may arise if the eddy is not nearly circular, if the constructed lines are nearly parallel, or if significant errors are present in the current measurements. Effects of the last two errors can be avoided by using several widely spaced current meters, and avoiding readings where $|\mathbf{V}|$ is small.

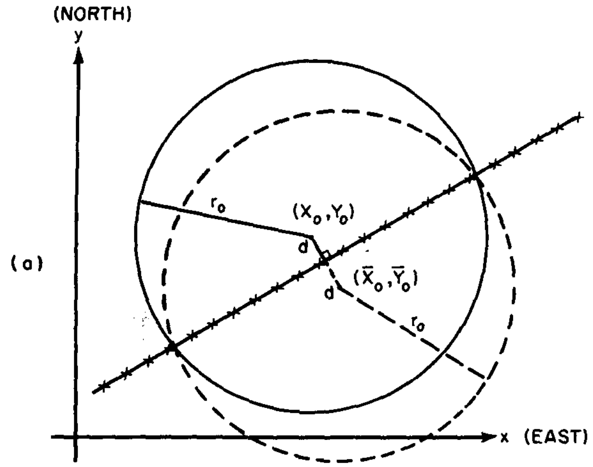
5. Time-independent problems

Large numbers of eddy observations consist of temperature measurements, taken over sufficiently short time intervals so as to be considered time-independent. Several eddy experiments have made use of ship-dropped bathythermographs (BT's) to get continuous temperature readings to depths below that of significant eddy influence. For example, much data from the MODE, POLYGON, and POLYMODE experiments are available (see, e.g., Volkmann, 1977) in the form of temperature sections taken in linear traversals of an ocean area, often obtained from expendable bathythermographs (XBT's). Also, sections are obtained from airborne expendable bathythermographs (AXBT's), which are convenient and require lower support costs.

We divide this section into two analytically similar, but experimentally distinct, parts. First, we consider data from a vertical planar cross section through an eddy, as might be obtained by a ship linearly traversing an eddy or by a plane dropping AXBT's along a linear trajectory. Then, data taken throughout the eddy in a nonplanar fashion are analyzed. The procedures of Sections 3 and 4 are used and the validity of the model of Section 2 is assumed in approximating the corresponding eddy environmental effects. In both the planar and nonplanar cases, the data are assumed to be taken over sufficiently short time intervals that eddy drift may be neglected. Thus, the coordinates of the center, $x_0(t) = X_0$, $y_0(t) = Y_0$, are constant. These, together with the radius r_0 and maximum current speed U_0, result in four horizontal parameters ($n_H = 4$), with depth of influence z_0 as the sole vertical parameter ($n_V = 1$).

a. Planar temperature data

Sampling of oceanic temperature may be obtained along straight-line traversals of large ocean areas. Large cold- or warm-temperature anomalies may be observed, suggesting the presence of a mesoscale eddy. Discrete drops along an eddy chord are illustrated by the crosses in Fig. 1a. A portion of an



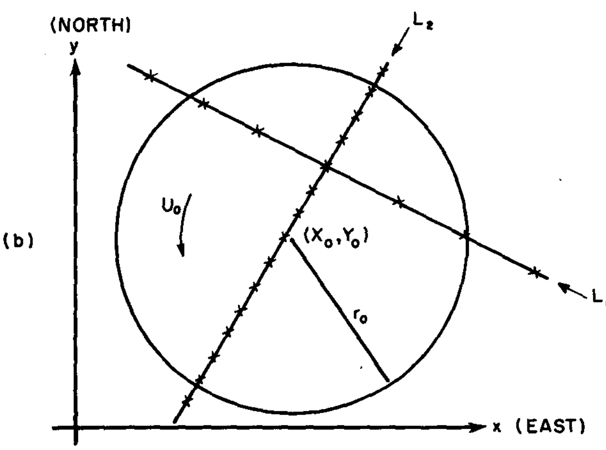


FIG. 1. (a) Chord through eddy, showing ambiguity in eddy center locations; (b) one scheme for obtaining non-planar eddy temperature data.

XBT section through the Atlantic, taken by Seaver (1975) as part of the MODE program, is shown in Fig. 2, indicating the presence of a large cyclonic eddy.

As proposed in Section 3, z_0 is selected first, using temperature measurements at the same horizontal position (x_1, y_1) at m_V (>2) depths $z_1^{(j)}$, $1 \leq j \leq m_V$. We assume here the availability of only a single deep measurement of the eddy. If additional readings are available, the following procedures should be carried out for each reading, and an average value of z_0 used. Perturbation density $\Delta\rho_1^{(j)}$, calculated at each depth as in (17), is used as the environmental deviation in (9), with the theoretical perturbation density obtained from (15). The solution of the resulting problem,

$$\underset{z_0, \bar{M}}{\mathrm{Min}} \sum_{j=1}^{m_V} \{[\Delta\rho_1^{(j)} - \bar{M}F'(z_1^{(j)})]^2/(\Delta\rho_1^{(j)})^2\}, \tag{19}$$

then specifies the vertical eddy structure. At each of the m_H (>4) horizontal positions (x_i, y_i), perturbation density is calculated at each depth $z_i^{(j)}$. Then the average vertically scaled perturbation density is computed from (11) and (15) as

$$\overline{\Delta\rho_i} = \sum_{j=1}^{m_V} [\Delta\rho_i^{(j)}/F'(z_i^{(j)})]/m_V. \tag{20}$$

Determination of the remaining horizontal parameters follows from (13), using $\overline{\Delta\rho_i}$ from (20) as the average vertically scaled deviation, resulting in

$$\underset{U_0, r_0, X_0, Y_0}{\mathrm{Min}} \sum_{i=1}^{m_H} \{\overline{\Delta\rho_i} - (2\rho_0\Omega g^{-1} \sin\phi)U_0 r_0 K_1 \times [J_0(\alpha_1 r_i/r_0) - J_0(\alpha_1)]^2/(\overline{\Delta\rho_i})^2\}, \tag{21a}$$

where

$$r_i = [(x_i - X_0)^2 + (y_i - Y_0)^2]^{1/2}. \tag{21b}$$

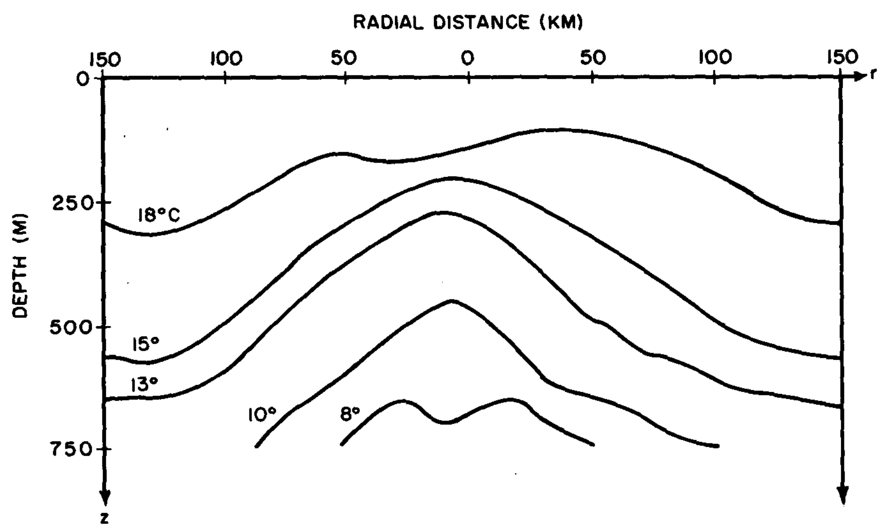


FIG. 2. Experimental isotherms (interpolated) through a large cyclonic eddy (adapted from Seaver, 1975).

TABLE 1. Summary of parameter-specification results for various data types, using eddy model of Section 2.

Data type	Parameter: Depth of influence z_0	Eddy radius r_0	Maximum surface current speed U_0	Eddy center $x_0(t),y_0(t)$	Drift speed $S(t)$
Planar temperature data (Fig. 1a)	U	U	U	A,I	
Nonplanar temperature data (Fig. 1b)	U	U	U	U,I	
One thermistor string (Fig. 7a)	U	F	F	F,L	F,L
Two or more thermistor strings	U	U	U	U,L	U,L
One current-temperature mooring	U	U	U	U,L	U,L
Two or more current-temperature moorings (Fig. 7b)	U	U	U	U	U

U, uniquely determined.
A, ambiguity between two values.
I, time-independent specification.
F, restricted to a family of parameters.
L, linear drift assumed.

Unique determination of the eddy center is not possible here. As indicated in Fig. 1a, two eddies can exist, each having identical size, strength and distance d from the chord, but located on opposite sides. Although differentiation between the two centers is not possible, values for the other horizontal parameters can be obtained. The results of the planar temperature data problem are summarized in the first line of Table 1. In this example, eddy depth of influence z_0, radius r_0 and maximum current speed U_0 can be uniquely determined. An ambiguity exists in the time-independent position of the eddy center, and no information is available concerning the drift of the eddy.

To illustrate our procedure, we use data from Fig. 2, taken every 20 km. At each horizontal location, the temperature was noted at seven equally spaced depths from 150 to 750 m. Utilizing an XBT from near the point of maximum temperature variation, a value of $z_0 = 2640$ m was obtained by solving (19) numerically. We note that use of an XBT at a different position should give a similar value, but use of the maximum temperature change minimizes the effects of observational inaccuracies. Eddy size and strength were determined by solving (21) numerically, resulting in $r_0 = 150$ km and $U_0 = 110$ cm s^{-1}. The distance of closest approach of the chord to the eddy center was found to be 10 km, so that the ambiguity in the center location is small in this example. For an acceptable horizontal fit, at least eight readings were necessary to obtain physically reasonable parameter values. Addition of more points led to better agreement in the shape and elevation of the isotherms, although inclusion of more than 20 points led to insignificant changes in parameter values. Resulting isotherms are shown in Fig. 3. Satisfactory agreement is noted between depths and overall shapes of the observed and predicted isotherms in Figs. 2 and 3.

b. *Non-planar temperature data*

We consider now the use of certain non-planar temperature data, obtained from experiments designed to study a particular oceanic eddy. Previous experimental studies have required deep temperature data to determine the full vertical structure and depths of significant eddy influence. However, if high accuracy at large depths is not required, then shallower data can be used to obtain the approximate vertical eddy structure. We shall focus on using AXBT's, although the results will be valid also for BT's or XBT's.

To determine the vertical structure, we have noted that measurements to depths of 750 m are necessary. Although limited to depths of about 350 m at present, the possible development of deeper AXBT's is under investigation. Thus, estimation of z_0 by solving (19) would require the use of either one or more such AXBT's, or another type of vertical-profile measurement.

With the eddy vertical structure specified, AXBT's are dropped at m_H (>4) horizontal positions (x_i,y_i), and readings are obtained at depths $z_i^{(j)}$. We illustrate in Fig. 1b one scheme for obtaining nonplanar data. The scheme was chosen for its simplicity; it is not intended to be optimum and it presupposes a nearly circular eddy. A linear path L_1, with relatively widely spaced AXBT drops (e.g., every 25 km), is followed until several temperature anomalies are found. Then, readings near the eddy center are obtained by making more closely spaced drops, say every 10 km, following a path L_2. The second path, perpendicular to L_1 and passing

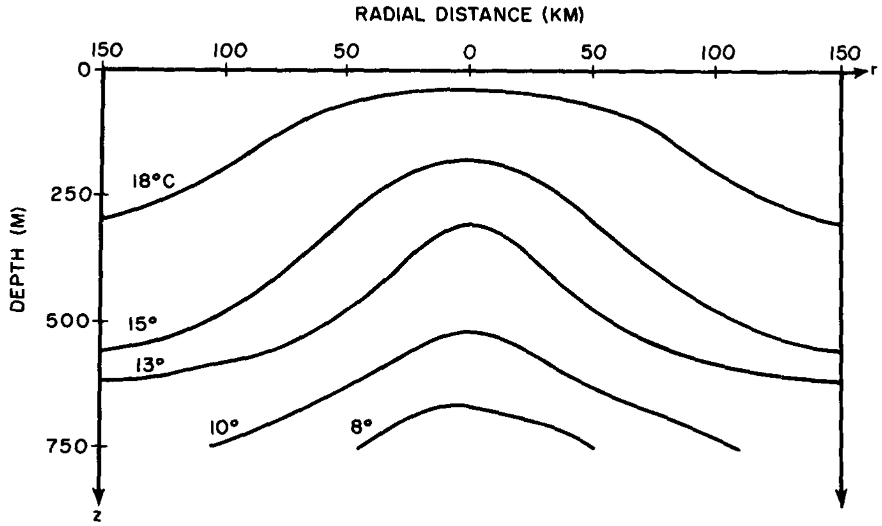


FIG. 3. Theoretical isotherms for an eddy with radius 150 km, maximum current speed 110 cm s^{-1} and depth of influence 2640 m.

through the position of maximum perturbed temperature obtained previously along L_1, thus approximates a diameter. Information from both paths permits an accurate determination of U_0, r_0, X_0 and Y_0, by first computing the average vertically scaled perturbation density as in (17) and (20) at each drop and then solving problem (21). Moreover, ambiguity in eddy center position no longer exists. Solution of the horizontal problem does not require data below 350 m and hence AXBT's are ideal in providing data througout the eddy, after the vertical structure is specified. We summarize these results in the second line of Table 1. Here, eddy depth of influence, radius, maximum current speed and position of the eddy center can be determined at the time of the measurements. However, nothing can be inferred concerning the translation direction or speed of the eddy.

As an illustrative example, we use BT data from a large cyclonic eddy studied by Khedouri and Gemill (1974). The eddy was observed to have significant environmental effects to depths of over 2 km, with a radius of ~150 km, and a maximum current of 110 cm s^{-1}. The observed temperature data in a vertical cross section through the eddy center was rotated to generate an axisymmetric eddy. A single temperature measurement (to 800 m) near the center was used to determine the depth of influence z_0. The resulting value of 2600 m agrees with observations of the vertical extent of the eddy. The equivalent of AXBT data can be constructed by sampling the data only to a depth of about 350 m. Examples of typical temperature deviations at several distances from the eddy center are shown in Fig. 4a. The temperature perturbations increase rapidly with depth, reaching maxima near 500 m. A small surface expression of the eddy, seen in the observational data, and presumably due to near-surfacing mixing, is not included in our model of Section 2.

Using the scheme of Fig. 1b, 23 AXBT's were simulated, with readings taken at depths of 150, 250 and 350 m, at the positions illustrated in Fig. 5a. We obtained from the solution of (21) the values U_0 = 120 cm s^{-1} and r_0 = 150 km. In addition, the eddy center was located within 6 km of the observed position. Thus, good agreement between theory and observations was obtained. Theoretical temperature perturbations are shown in Fig. 4b. They exhibit features and magnitudes quite similar to the AXBT data in Fig. 4a, as well as deeper structure not used in the fit, indicated by dashed lines. In addition, the model specified all eddy characteristics at all locations. For example, Fig. 6 shows the vertical decay of the maximum rotational currents compared with maximum geostrophically computed currents (Khedouri and Gemill, 1974) shown as crosses. At any depth, these maximum currents occur about half way between the eddy center and edge and decay rapidly with depth.

The number of points necessary for accurate parameter specification was tested by solving the horizontal problem (21) using data sets of increasing size, taken from the eddy of Fig. 4a. Initially, three AXBT drops were made on L_1 (Fig. 5a) and two on L_2, and the resulting horizontal problem solved. Additional AXBT drops were made, until all 23 AXBT's were included. After each drop, U_0, r_0, x_0 and y_0 were determined by solving (21). Resulting values of U_0 and r_0 are plotted in Fig. 5b as a function of the number of AXBT's utilized in the horizontal fit. In this example, as few as eight AXBT's led to good estimates of parameter values. However,

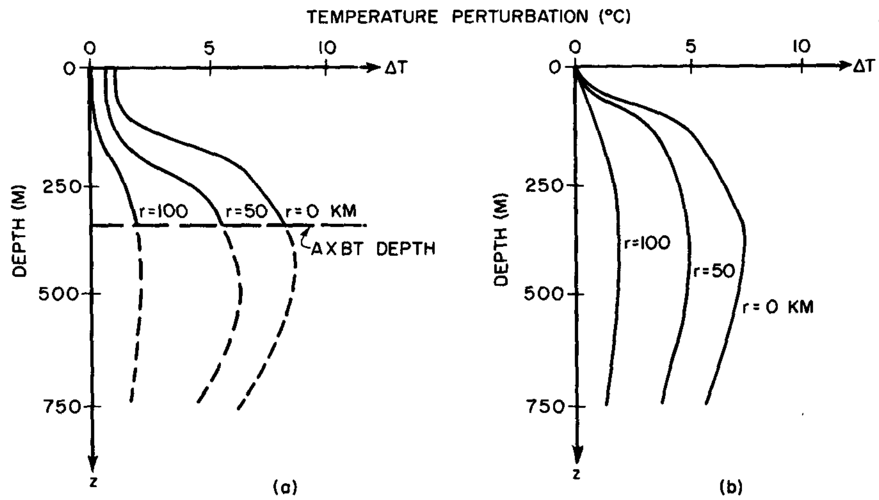


FIG. 4. Eddy temperature perturbations: (a) "observational" AXBT data, (b) theoretical results.

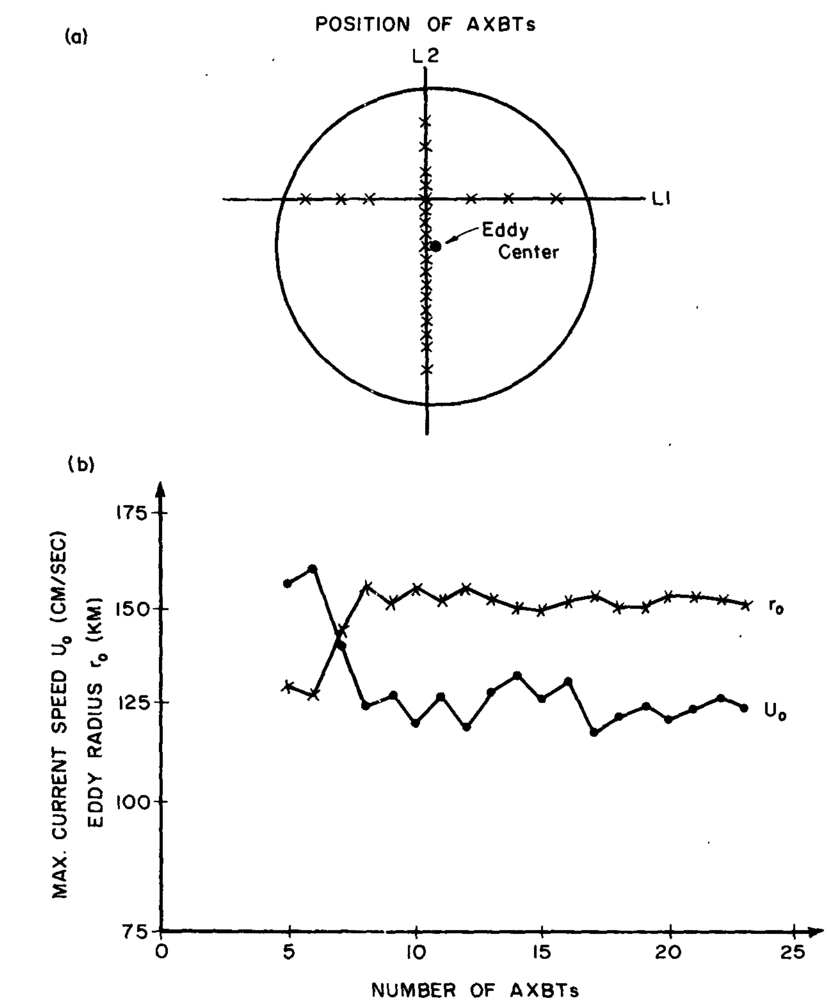


FIG. 5. (a) Position of AXBT drops in eddy of Fig. 4 and (b) variation of eddy radius and maximum current speed with increasing numbers of AXBT's.

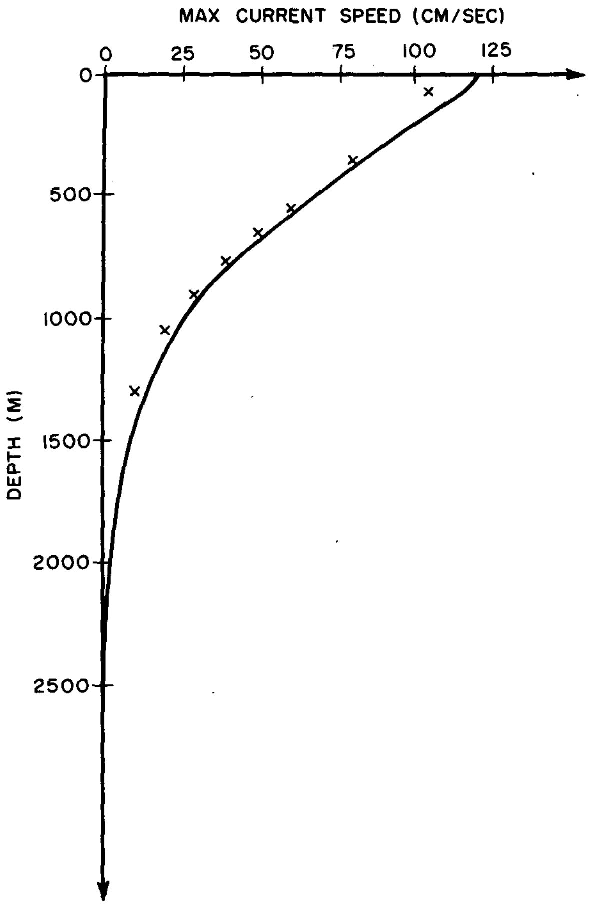


FIG. 6. Theoretical vertical decay of maximum rotational current (solid line) versus maximum current from observations (crosses).

variations of as much as 10% in parameter values from the "final" values of $U_0 = 120$ cm s^{-1} and $r_0 = 150$ km are noted if fewer than 20 AXBT's are utilized. Similar results are observed in the location of the eddy center. Of course, the accuracy of the parameter specifications varies not only with the number of AXBT's used, but also with the positions of the AXBT's, the eddy under consideration and the accuracy of the model approximation. We have conducted similar studies on other observed eddies and with other horizontal locations of the readings. From these studies, we have found that a minimum of eight AXBT's, dropped near both the eddy edge and center, are ordinarily necessary for good estimates of parameter values. Variation of parameter values decreased with increasing density of AXBT's, until inclusion of more than 20 AXBT's led to insignificant parameter variation, as in Fig. 5b.

6. Time-dependent problems

We examine here the analysis of eddy observations from fixed moorings, containing instrument packages at discrete fixed depths. In contrast to BT's, the time series of data from moorings can provide information on eddy translational rates and directions, as well as on eddy size and strength. We first consider utilization of only temperature data obtained from thermistor strings, and then include current observations obtained from current-temperature moorings, using the model of Section 2.

Consider first the use of data from a single thermistor string, with thermistors located at m_V (>2) depths. At a fixed time t_1, eddy vertical structure may be determined by solving problem (19). At each time t_i the average vertically scaled perturbation density $\Delta\rho_i$ is then determined by (20). To further describe the eddy, we would like to specify the translation of the eddy center $x_0(t)$ and $y_0(t)$. However, using a single thermistor string, it will be possible to determine only the speed by which the eddy drifts, not the direction of translation. That is, only the eddy drift speed,

$$S(t) \equiv |\mathbf{V}_D(t)| \equiv \{[x_0'(t)]^2 + [y_0'(t)]^2\}^{1/2}, \qquad (22)$$

can be found. We shall assume here that this quantity is constant, as in (14), so that

$$S = (u_D{}^2 + v_D{}^2)^{1/2} \qquad (23)$$

is one horizontal parameter in our problem. This assumption may be valid if the eddy is observed only over a sufficiently short time interval or if the eddy is not significantly influenced by other ocean currents and by topographical features. Moreover, any eddy of a fixed strength and at the same radial distance r from the thermistor string will produce identical temperature effects. Hence, it is not possible to determine the exact initial position (X_0,Y_0) but only the initial distance R from the string. Thus

$$R = (X_0{}^2 + Y_0{}^2)^{1/2} \qquad (24)$$

is a second horizontal parameter. Further, only the distance d of closest approach of the eddy center to the string can be determined, giving a third horizontal parameter. The eddy radius r_0 and maximum current speed U_0 comprise the remaining horizontal parameters, so that $n_H = 5$.

The vertically scaled perturbation density $\overline{\Delta\rho_i}$ is computed at m_H (>5) times t_i, $i = 1, \ldots, m_H$. To determine the predicted effects through (15b), the radial distance r must be computed at each time. We define a moving coordinate system (x',y'), with origin at the eddy center. The x' axis is parallel to the direction of eddy drift, with decreasing x' in the direction of $\mathbf{V}_D$, as illustrated in Fig. 7a. To an observer fixed in the (x',y') system, the thermistor string will appear to translate in the positive x' direction with speed S. Initially, the eddy center will be located at

$$(X_0',Y_0') = [-(R^2 - d^2)^{1/2}, -d]. \qquad (25a)$$

From elementary geometry, the desired radial distance $r(t)$ at any time t will then be

$$r(t) = [(X_0' + St)^2 + d^2]^{1/2}. \quad (25b)$$

Eq. (25) and (21) comprise the problem for the five horizontal parameters. Examination of this problem shows that it can be solved only for the parameter combinations U_0r_0, R/r_0, d/r_0 and S/r_0. That is, unique solutions for individual parameter values are not possible, since U_0, R, d, and S can be determined only as (simple) functions of r_0. Thus, each member of a one-parameter family of eddies is a possible fit to the observed temperature-perturbation data. The non-unique parameter specification is summarized in the third line of Table 1. Eddy depth of influence can be determined uniquely, but eddy radius, maximum current speed and the center position and trajectory can only be determined to within a one-parameter family of values, assuming a linear drift form.

To illustrate this non-uniqueness, we present simulated thermistor string data in Fig. 8 for a large cyclonic eddy with linear drift. The data for the resulting cold-core eddy was generated using the model of Section 2, assuming a drift speed $S = 5$ km day^{-1}, radius $r_0 = 125$ km, maximum current speed $U_0 = 150$ cm s^{-1}, closest approach distance $d = 20$ km, and depth of influence $z_0 = 2100$ m. At the initial time $t = 0$, the thermistor was chosen to be on the eddy edge, so that $R = r_0$. Applying the procedure of this section, readings taken at the depths shown on Fig. 8 every 5 days reproduce the exact (inputed) value of $z_0 = 2100$ m, and yield constant values for U_0r_0, d/r_0 and S/r_0. The parameters U_0, d and S are illustrated as functions of r_0 in Fig. 9. We note that for any assigned value of the eddy radius, S, d and U_0 are uniquely specified, as indicated by the dashed line corresponding to the exact radius $r_0 = 125$ km. To generate curves such as those in Fig. 9, a minimum of four readings at different times is required; for accurate specification, we have found that at least eight readings are necessary, each separated by several days.

Addition of more than one thermistor string at different horizontal positions supplies sufficient information to determine unique parameter values, and to specify drift direction as well as speed. The vertical parameter z_0 is again chosen first by using data from any one string at a fixed time. If we assume the linear trajectory (14), the radial distance $r_i(t)$ from the thermistor string T_i to the eddy center at time t is clearly

$$r_i(t) = \{[x_i - (X_0 + u_Dt)]^2 + [y_i - (Y_0 + v_Dt)]^2\}^{1/2}. \quad (26)$$

To determine the horizontal parameters U_0, r_0, X_0, Y_0, u_D and v_D, m_H (>6) readings are required.

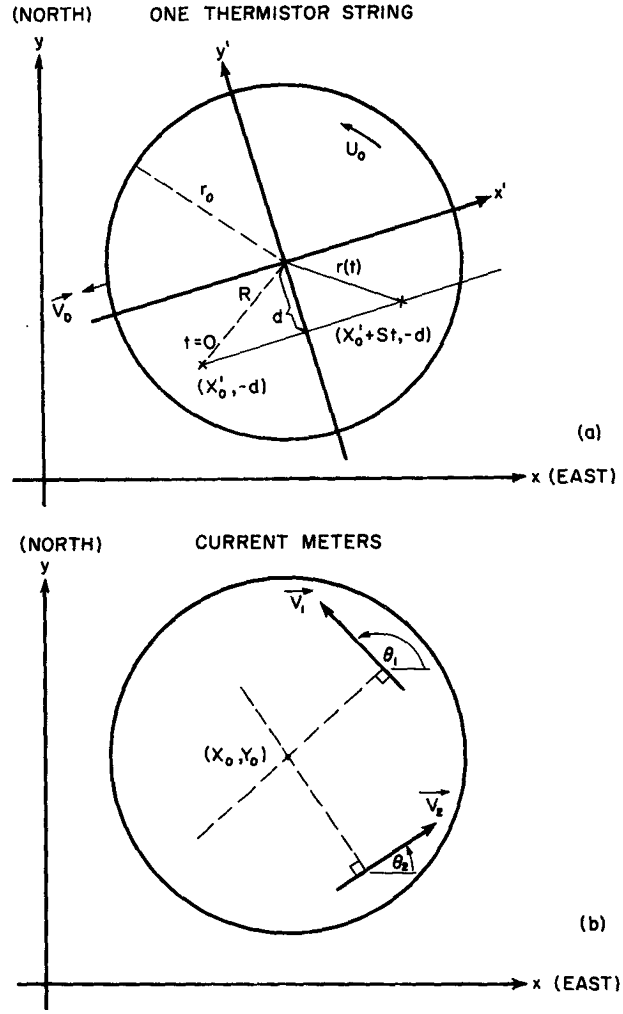


FIG. 7. (a) Geometry of one-thermistor string and (b) location of eddy center using two current meters.

Using measurements from at least two thermistor strings, a unique solution can be obtained, since parameters now occur in (21a) and (26) in the six distinct combinations U_0r_0, $1/r_0$, X_0/r_0, Y_0/r_0, u_D/r_0 and v_D/r_0. To accurately specify model parameters, a significant overlap in time (say, 15 days) from data taken from different strings must be present, and separation between thermistor strings should not be small in comparison to the size of the eddy (e.g., 50 km). Thus, by the inclusion of additional thermistor strings, the indeterminacy of the one-string case is avoided, and unique parameter specification is possible, as summarized in the fourth line of Table 1. Eddy depth of influence, radius, maximum current speed, center position and drift, assuming linear drift, are uniquely determined. More complex forms for the eddy translation $x_0(t)$ and $y_0(t)$ may be assumed, leading to additional parameters, which can also be uniquely determined using two or more thermistor strings.

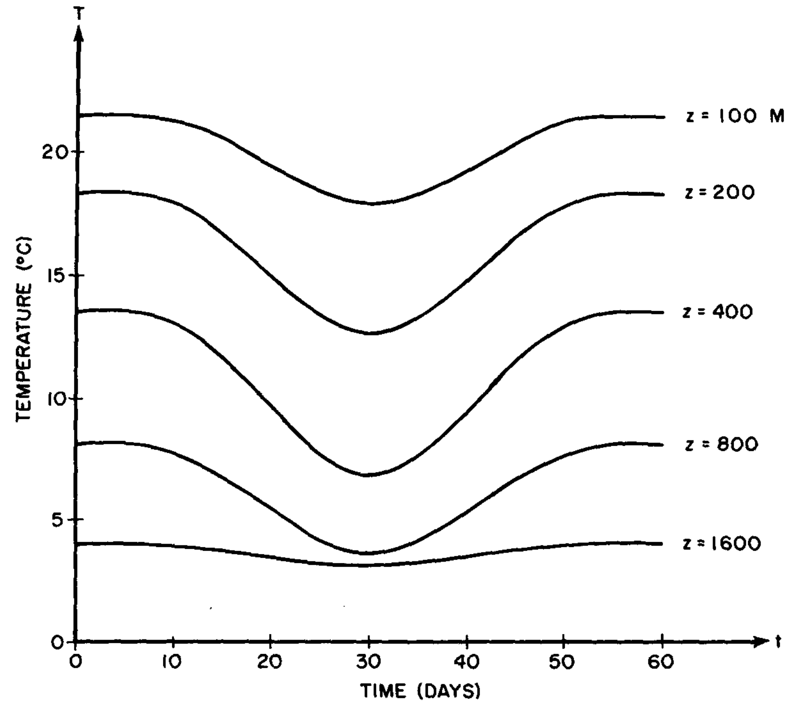


FIG. 8. Model-simulated thermistor string data from eddy with linear drift speed 5 km day^{-1}, radius 125 km, maximum current speed 150 cm s^{-1}, closest-approach distance 25 km and depth of influence 2100 m.

Addition of current meters to a single thermistor string also provides more information, both on eddy strength through current speed observations and on the location of the center via current direction. Current speed can be used in both the vertical and horizontal problems, by utilizing the predicted current obtained from (15c) and including the error between observed and predicted currents in

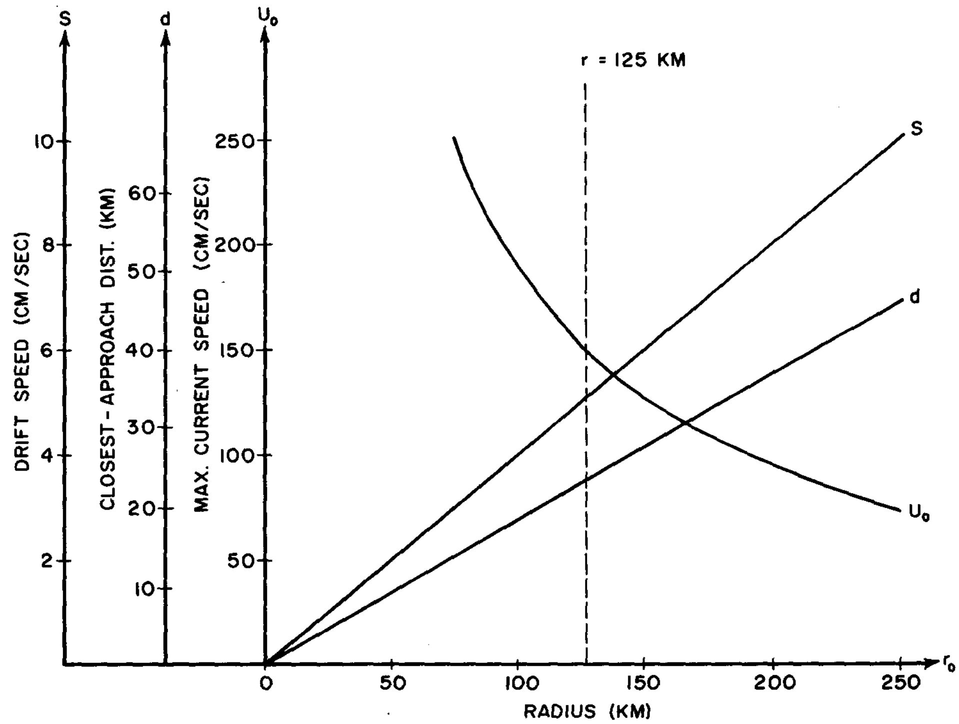


FIG. 9. Family of parameter values for eddy of Fig. 8.

problems similar to (19) and (21). We caution against using readings where current magnitudes are less than some minimum speed (say, 20 cm s^{-1}), as errors induced by eddy drift and other environmental effects might become significant. Further, if at time t_i a current direction Θ_i is observed, relationship (18) gives a line of possible centers, expressing $y_0(t_i)$ as a function of $x_0(t_i)$, and hence eliminating the parameters implicit in $y_0(t)$. Assuming again a linear drift for $x_0(t)$, there are four horizontal parameters, X_0, u_D, r_0 and U_0, and the sole vertical parameter z_0. From (18), the radial distance from the eddy center at time t_i is given by

$$r(t_i) = |X_0 + u_D t_i|[1 + \tan^2(\pi/2 - \theta_i)]^{1/2}. \quad (27)$$

We note U_0 now occurs distinctly in the theoretical current term (15c), and hence unique solutions can be found, as summarized in line 5 of Table 1. This situation contrasts with the non-uniqueness for temperature data only (in line 3).

Suppose now that n (>1) current-temperature moorings are present at any time t. From the discussion of Section 4, the position of the eddy center $[x_0(t), y_0(t)]$ can be determined by the intersection of perpendiculars to the average vertically scaled current at each mooring, computed as in (18). This is indicated in Fig. 7b for the case of two current meters. The position of the eddy center is known whenever two or more current meters measure significant currents. For accurate specification of the center position, readings should be taken from more than two moorings at several times. Of course, because of observational inaccuracies, if $n > 2$, the $(n-1)!$ intersection points would not coincide. They would be expected to be close enough so that a reasonable estimate of the center could be made if the guidelines of Section 4 are followed; otherwise, the radially symmetric model may not be applicable to the observed eddy. The only remaining horizontal parameters are then eddy radius r_0 and maximum current speed U_0. Results for this case are summarized in the last line of Table 1. Eddy depth of influence, radius, maximum surface current speed, and eddy center position and drift can all be determined, independent of any assumptions concerning the form of the drift trajectory.

7. Summary

The major purpose of this paper is to describe the use of analytic models to determine approximate environmental effects of an oceanic eddy using limited observational data. The question of the amount and types of data necessary for such model specification is discussed, and techniques for accurate model parameter specification presented.

In order to provide illustrative examples of the ideas in this paper, a previously derived model is briefly reviewed, but any model with an arbitrary number of parameters could be utilized. Thus, parameter specification for a general model is discussed. Separable models are considered in which parameters associated with the vertical and horizontal structure are determined separately. The minimum amount of data required to specify an eddy model (a necessary prerequisite to accurate model approximation) is analyzed. Then an efficient strategy is developed for accurate determination of model parameters, employing successive minimization problems for the vertical and horizontal parameters.

The use of oceanographic data in parameter-specification schemes is discussed. Density anomalies induced by an eddy obtained from temperature measurements can be used to accurately determine model parameters. Utilization of current measurements is discussed also. The procedures are then applied to several typical experimental situations, using our particular circular eddy model for illustration. First, temperature data acquired on a linear path through the eddy is considered. Assuming that the data are taken over small enough intervals to be time-independent, it is shown that the size and strength of a particular eddy can be determined uniquely. However, an ambiguity results in the position of the eddy center, so that either of two possible eddies could be responsible for the observed temperature perturbations. Next, we study nonplanar time-independent data, as might be obtained during an extensive study of an eddy. It is shown that unique determination of size, strength and position is possible. Examples are presented from actual eddy observations, with size and strength determined to within 5% of observed values, and center position accurate to within 10 km. Guidelines are given for placement of bathythermographs for accurate parameter specification, and numerical sensitivity to the amount and position of data is considered.

Two distinct types of time-series data are examined, the first being only temperature readings that might be obtained from one or more thermistor strings, and the second including both temperature and current readings from one or more moorings. For a single thermistor string, it is shown that the model parameters cannot be determined uniquely, but that possibilities are restricted to a one-parameter family of eddies. However, addition of more strings does lead to unique parameter values, as well as specification of drift direction. Addition of current meters to a single thermistor string leads to complete eddy specification when linear drift is assumed. Finally, data from two or more such moorings can be used to determine arbitrary drift as well as eddy depth, radius and maximum rotational current speed.

Analytical eddy models specify the amounts and types of data required to determine model parameters and permit efficient approximation of eddy environmental effects. However, additional work is needed in the areas of model improvement and further sensitivity analyses. In such uses of eddy modeling, as with any dynamical modeling, caution must be exercised. However, proper use of refined models and techniques will hopefully lead to the economical description of ocean eddies.

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